Select Topics in Graph Theory - II

These notes arise from the scribing efforts of the students of MA 5109: Graph theory (this is not the entire course!) that I offered in the Fall semester. The presentation (as it is scribed here) largely reflects the way I taught these topics.
There is also likely 'irregularity' in the way the different topics are presented here, and that would be owing to the fact that the scribes brought in their individual perspectives into their writing. Some of the students have also added additional notes and that has improved the quality of the notes a great deal. Their enthusiasm in this regard is largely responsible for the quality of the output. I however take responsibility for any mistakes/errors that remain amongst these notes.

I thank all the students for participating in this project wholeheartedly, and for their proactive interest in this little project.

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Graph Theory	2023
Dense Graph Limits	
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The theory of graph limits was developed by Lovász and his collaborators in a series of works starting around 2003. Here is the central objects in the theory of dense graph limits.

Definition 1. *A Graphon is a symmetric measurable function $W : [0,1]^2 \to \mathbb{R}$. Here symmetric means $W(x,y) = W(y,x)$ for all x, y. The set of all graphons is denoted by $\mathcal{W}$.*

$$\mathcal{W}_0 := \{W : [0,1]^2 \to [0,1] \text{ : } W(x,y) = W(y,x),\ W \text{ is lebesgue measure}\}$$
$$\mathcal{W}_1 := \{W : [0,1]^2 \to [-1,1] \text{ : } W(x,y) = W(y,x),\ W \text{ is lebesgue measure}\}$$

What is the limiting behavior for a sequence of graphs?

Definition 2. *Say that a sequence (G_n) of graphs converges if $t(H, G_n)$ converges for all H finite, equivalently, $d(H, G_n)$ converges for all H finite.*

Definition 3. *For a graph G and another graph H, the Homomorphism density of H in G is*

$$t(H,G) := \frac{|Hom(H,G)|}{|v(G)|^{|v(H)|}}$$

Where, $Hom(H,G) = \{\varphi : H \to G$ is a graph Homomorphism.$\}$. $\varphi : V(H) \to V(G)$ is a graph Homomorphism if $\{\varphi(u), \varphi(v)\} \in E(G)$ whenever $\{u, v\} \in E(H)$. This is also the probability that a uniformly random map $V(H) \to V(G)$ induces a graph homomorphism from H to G. Note that as $|V(G)| \to \infty$ for G. Fixed H,

$$t(H,G) = \mathbb{P}(\textit{a random map density a labelled copy of } H) + o_n(1)$$

Example 4. *Following Graphs are convergent.*

- $G_n = K_n$
- $G_n = K_{n,n}$
- $G_n = K_{\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor}$ *with* $\alpha + \beta = 1$
- $G_n = T_r(n)$

Alternates formulation:

$$d(H,G) = \mathbb{P}(\text{a random subset of } |H| \text{ vertices induces a copy of } H \text{ in } G)$$

Here, $d(H,G)$ is called isomorphism density of H in G. One question arises in our mind, what can (G_n) converge to? If $W \in \mathbb{W}_0$, could we ask, $G_n \to W$?

Digression

Suppose, $\mathcal{H}$ is a (real) Hilbert Space. $\sqrt{\langle x, x\rangle} := \|x\|$ is a norm. $\mathcal{H} \to \mathcal{H}$ (usual topology is a norm topology). (x_n) converges weakly if for any $T : \mathcal{H} \to \mathbb{R}$ bounded linear functional, (Tx_n) converges.

Definition 5. *Suppose, $W \in \mathcal{W}$ and $V(H) = \{1, 2, ..., h\}$,*

$$t(H, W) := \int_{[0,1]^h} \left(\prod_{ij \in E(H)} W(x_i, x_j) \right) dx_1....dx_h$$

$$d(H, W) := \frac{|H|!}{Aut(H)} \int_{[0,1]^h} \left(\prod_{ij \in E(H)} W(x_i, x_j). \prod_{ij \neq E(H)} (1 - W(x_i, x_j)) \right) dx_1....dx_h$$

For a Graphon $W \in \mathcal{W}_0$, we can define a W-random graph $G_W^{(n)}$ of order n as:-
Let W be a graphon. The n-vertex W-random graph $G_W^{(n)}$ denotes the n-vertex random graph (with vertices labeled $1, ..., n$) obtained by first picking $x_1, ..., x_n$ uniformly at random from $[0, 1]$, and then putting an edge between vertices i and j with probability $W(x_i, x_j)$, independently for all $1 \leq i < j \leq n$.
The term $|H|!/Aut(H)$ appears since $d(H, W)$ only counts induced copies that are not labelled.

Definition 6. *Given two symmetric measurable functions $U, W : [0, 1]^2 \to \mathbb{R}$, we define their cut distance (or cut metric) to be*

$$d_\square(U, W) := \sup_{a,b:[0,1] \to [-1,1]} \left| \int_{[o,1]^2} a(x)b(y)(W(x, y) - U(x, y))dxdy \right|$$

Here a, b to be measurable.It is easy to show that $d_\square(U, W)$ is a metric on $\mathcal{W}_0$.It is possible for two different graphons to have cut distance zero. For example, they could differ on a measure-zero set, or they could be related via measure preserving maps.

Theorem 7. *For any graph H, and $U, W \in \mathcal{W}_0$,*

$$|t(H, W) - t(H, U)| \leq e(H).d_\square(U, W)$$

So,$t(H, .) : \mathcal{W}_0 \to [0, 1]$ are continuous on $\mathcal{W}_0$ with respect to $d_\square(,)$.

Proof. Suppose, $V(H) = \{1, 2, ..., h\}$. Let, $\{1, 2\} \in E(H)$ (Without loss of generality). Fix, $x_3, ..., x_h \in [0, 1]$ and define,

$$a(x) := \prod_{i \geq 3; 1, i \in E(H)} W(x, x_i),$$

$$b(x) := \prod_{i \geq 3; 1, i \in E(G)} W(x, x_i),$$

$$c := \prod_{i \geq 3; i, j \in E(H)} W(x_i, x_j)$$

Then,

$$\begin{aligned}
&\left| \int_{[0,1]^2} (W(x_1,x_2) - U(x_1,x_2). \prod_{ij\in E(G);\{i,j\}\neq\{1,2\}} W(x_i,x_j) dx_1 dx_2) \right| \\
&= \left| c \int_{[0,1]^2} (W(x_1,x_2) - U(x_1,x_2) a(x_1) b(x_2) dx_1 dx_2) \right| \\
&\leq \left| \int_{[0,1]^2} (W(x_1,x_2) - U(x_1,x_2) a(x_1) b(x_2) dx_1 dx_2) \right| \\
&\leq d_\square(W,U)
\end{aligned}$$

□

Theorem 8. *Given $W \in \mathcal{W}_0$, there is a sequence (G_n) of graphs such that $d(H,G_n) \to d(H,W)$,$\forall$ finite H. In this case, we say that the sequence (G_n) has W as a limit.*
More precisely, suppose G_n is a W- random graph of order n, then with probability 1 , $d(H,G_n) \to d(H,W)$ for all H.

We first recall what graphons are.

Definition 9. *A measurable function $W : [0,1]^2 \mapsto \mathbb{R}$ such that $W(x,y) = W(y,x)$ for every $x, y \in [0,1]$, is called a graphon.*

We also define

$$\mathcal{W} := \{W : W \text{ is a graphon}\}$$

$$\mathcal{W}_0 := \{W : [0,1]^2 \mapsto [0,1] : W \text{ is a graphon}\}$$

$$\mathcal{W}_1 := \{W : [0,1]^2 \mapsto [-1,1] : W \text{ is a graphon}\}$$

For $W \in \mathcal{W}_0$, we can define a W-random graph G_n of order n as follows: Pick $x_1, \ldots, x_n$ independently and uniformly from $[0,1]$. Let the vertex set of G_n be $[n] := \{1, 2, \ldots, n\}$. Then for any $i, j \in [n], i \neq j$, $\{i,j\}$ is an edge of G_n with probability $W(x_i, x_j)$. Note that if $W \equiv p$ for some $p \in (0,1)$, then a W-random graph is just a Erdős-Rényi random graph with parameter p.
Finally, we define the *isomorphism density* (also known as *induced homomorphism density*) of a fixed graph H w.r.t graphs and graphons as follows:

1. If G is a graph of order n, then

$$\begin{aligned} d(H,G) &:= \mathbb{P}\left(\text{A random subset of } |H| \text{ vertices induces a copy of } H \text{ in } G\right) \\ &= \frac{\text{Number of induced copies of } H \text{ in } G}{n^h} \end{aligned}$$

2. If W is a graphon, then

$$\begin{aligned} d(H,W) &:= \mathbb{P}(\text{A uniformly sampled point from } [0,1]^h \text{ induces } H \text{ as a } W\text{-random graph}) \\ &= \frac{h!}{|\operatorname{Aut}(H)|} \int_{[0,1]^h} \prod_{ij \in E(H)} W(x_i, x_j) \prod_{ij \notin E(H)} (1 - W(x_i, x_j)) dx_1 \cdots dx_h \end{aligned}$$

where $\operatorname{Aut}(H)$ is the set of automorphisms of H, and $h = |V(H)|$.

We can now state the main theorem of this lecture.

Theorem 10. *[9] Given any $W \in \mathcal{W}_0$, there exists a sequence of graphs $\{G_n\}_{n\in\mathbb{N}}$ such that G_n converges to W as $n \to \infty$. More precisely, for any fixed H,*

$$\mathbb{P}\left(\lim_{n\to\infty} d(H, G_n) = d(H, W)\right) = 1$$

where G_n is a W-random graph of order n.

We shall follow [9] in the proof. But before that, we'll need some Martingale theory. Thus we digress a bit to cover some basics of Martingale theory.

A Brief Introduction to Martingales

Let $X_1, \ldots, X_n$ be random variables. We call $(X_1, \ldots, X_n)$ a *finite martingale* if $\mathbb{E}\left[X_i | X_1, \ldots, X_{i-1}\right] = X_{i-1}$ for every $i \in \{2, \ldots, n\}$. The following observation, due to J.L. Doob, gives us a general template to construct martingales.

Observation 11 (Doob's Martingale)**.** *Consider a probability triple* $(\Omega, \mathcal{B}, \mathbb{P})$ *defining a random variable* X*, and consider a* filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n$ *of sub-*σ*-algebras of* $\mathcal{B}$*. Then* $(X_1, \ldots, X_n)$ *is a martingale, where* $X_i := \mathbb{E}\left[X | \mathcal{F}_i\right]$*. Note that if* $\mathcal{F}_n = \mathcal{B}$*, then* $X_n = X$*.*

Also, we shall need a very famous concentration inequality involving martingales.

Lemma 12 (Azuma's Inequality)**.** *Suppose* $(X_0, X_1, \ldots, X_n)$ *is a finite martingale such that* $|X_i - X_{i-1}| \leq c_i$ *with probability 1, for every* $i \in [n]$*, where* $c_1, \ldots, c_n$ *are some real numbers. Then*

$$\mathbb{P}\left(|X_n - \mathbb{E}\left[X_n\right]| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right)$$

Proof of Theorem ??: We shall use the *vertex exposure* martingale. Namely, let G_n be a W-random graph. Let X be the number of induced copies of H in G_n. Let $\mathcal{F}_i$ be the σ-algebra generated by "exposing" the statuses of all edges incident upon vertices in $[i]$, where $i \in [n] \cup \{0\}$. Clearly, $\mathcal{F}_i$'s form a filtration, and thus we can construct the Doob Martingale $(X_0, \ldots, X_n)$. Furthermore, notice that $|X_i - X_{i-1}|$ measures the increase in the number of induced copies of H when the status of the edges incident on i is revealed. Consequently, $|X_i - X_{i-1}|$ can be at most the number of labeled subsets of size $h-1$ of $[i-1]$, which equals $h!\binom{i-1}{h-1} \leq n^{h-1}$.
Also, note that

$$\mathbb{E}\left[X\right] = \sum_{S \in \binom{[n]}{h}} \mathbb{P}(G_n[S] \simeq H) = \binom{n}{h} d(H, W)$$

Finally, note that $X_n = X$, and thus, setting $c_i = n^{h-1}$ for $i \in [n]$, and $t = \varepsilon n^h$ for some $\varepsilon > 0$, and applying Azuma's inequality on the above martingale yields,

$$\mathbb{P}\left(\left|X_n - \binom{n}{h} d(H, W)\right| \geq \varepsilon n^h\right) \leq 2 \exp\left(-\frac{\varepsilon^2 n^{2h}}{2n^{2h-1}}\right) = 2 \exp\left(-\frac{n\varepsilon^2}{2}\right)$$

On the other hand, note that X_n is the number of induced copies of H in G_n, and consequently, $X_n = \binom{n}{h} d(H, W)$. Thus $\mathbb{P}\left(|d(H, G_n) - d(H, W)| \geq \frac{\varepsilon n^h}{\binom{n}{h}}\right) \leq 2 \exp\left(-\frac{n\varepsilon^2}{2}\right)$. Note that $\frac{n^h}{\binom{n}{h}} = h! \frac{n^h}{n(n-1)\cdots(n-h+1)} \leq h! 2^h$ for $n \geq 2h$, and consequently, for large enough n,

$$\underbrace{\mathbb{P}\left(|d(H, G_n) - d(H, W)| \geq h! 2^h \varepsilon\right)}_{=: p_n} \leq 2 \exp\left(-\frac{n\varepsilon^2}{2}\right)$$

In particular, $\sum_{n \geq 2h} p_n < \infty$, implying that $\sum_n p_n < \infty$. We can now invoke the *Borel-Cantelli lemma*, which we state below:

Lemma 13 (Borel-Cantelli Lemma)**.** *Suppose* $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ *be a sequence of events in the probability space* $(\Omega, \mathcal{B}, \mathbb{P})$*. Then:*

1. *$\sum_{n\in\mathbb{N}} \mathbb{P}(\mathcal{E}_n) < \infty$ implies that $\mathbb{P}\left(\limsup_{n\to\infty} \mathcal{E}_n\right) = 0$, where $\limsup_{n\to\infty} \mathcal{E}_n$ is the set of $\omega \in \Omega$ for which there are infinitely many events $\mathcal{E}_n$ such that $\omega \in \mathcal{E}_n$.*

2. *If $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ are all independent and $\sum_{n\in\mathbb{N}} \mathbb{P}(\mathcal{E}_n) = \infty$, then $\mathbb{P}\left(\limsup_{n\to\infty} \mathcal{E}_n\right) = 1$.*

We shall only need the first part of the Borel-Cantelli Lemma [1]. Indeed, let $\mathcal{E}_{n,h}$ denote the event $|d(H, G_n) - d(H, W)| \geq h!2^h\varepsilon$. Then by the Borel-Cantelli Lemma, almost surely $\lim_{n\to\infty} d(H, G_n)$ is within $h!2^h\varepsilon$ of $d(H, W)$, for every $\varepsilon > 0$. Set up a sequence $\varepsilon_m := \frac{1}{m}$, and note that

$$\left\{\lim_{n\to\infty} d(H, G_n) = d(H, W)\right\} = \bigcap_{m\in\mathbb{N}} \left\{\left|\lim_{n\to\infty} d(H, G_n) - d(H, W)\right| \leq h!2^h\varepsilon_m\right\}$$

Since we're taking a countable intersection of almost sure events, the resulting intersection is almost sure too, and that concludes the proof of the theorem.

[1] which has a very easy proof: Indeed, let $N := |\{n \in \mathbb{N} : \mathcal{E}_n \text{ occurs}\}|$. Then N is a $\mathbb{N} \cup \{0, \infty\}$-valued random variable, and $\mathbb{E}[N] = \sum_n \mathbb{P}(\mathcal{E}_n) < \infty$. Since the expectation of N is finite, $\mathbb{P}(N = \infty) = 0$.

Graph Theory — 3rd October, 2023

Norms and Distances for Graphs

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We continue our search for a metric that we can use to compare graphs on the same vertex set. Recall the “cut norm” which was defined as follows:

$$||W||_{\square} := \sup_{f,g:[0,1]\to[0,1]} \left| \int_{[0,1]^2} f(x)g(y)W(x,y)\,dx\,dy \right| ; f, g \text{ measurable}$$

We had also seen that $d_{\square}(U,W) = ||W-U||_{\square}$ is a distance and that the homomorphism density $t(H,\cdot)$ is continuous with respect to this metric.

FACT: The cut norm also satisfies

$$||W||_{\square} = \sup_{S,T\subseteq[0,1],\text{measurable}} \left| \int_{S\times T} W(x,y)\;dx\,dy \right|$$

Example 14. *Consider the following graphs and their corresponding graphons:*

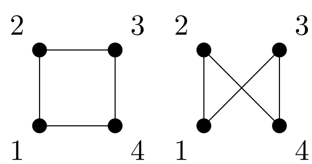

Figure 1: Graphs G_1 and G_2

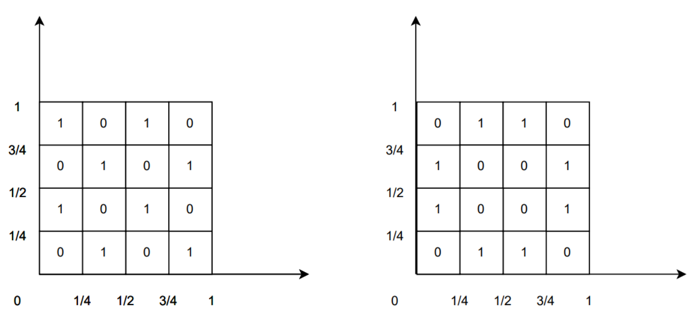

Figure 2: Graphons W_{G_1} and W_{G_2}

The measurable, symmetric functions $W_{G_1}, W_{G_2} : [0,1]^2 \to [0,1]$ as defined in fig. 2 are the graphons obtained by rotating the adjacency matrices of the graphs G_1 and G_2 by $3\pi/2$.

These are indeed graphons, and as the graphs are "the same", we would like the distance between the two to be zero with respect to the cut norm, which is not the case.

Since our aim is to find a suitable metric to measure the distances between graphs on the same vertex set, the above example indicates that the distance given by the cut norm does not suit our purpose. So, we would need to make a few changes before we achieve a desirable metric.

Definition 15. *A measure preserving bijection (abbreviated to MBP) $\varphi : [0,1] \to [0,1]$ is a measurable function whose inverse is also measurable and for all $A \in \mathscr{B}_{[0,1]}$,*

$$m(A) = m(\varphi(A)) = m(\varphi^{-1}(A))$$

For integrable graphons $U, W \in \mathcal{W}$, define the *pseudo*-distance (the 'pseudo' will be justified soon) between them as follows:

$$\delta_\square(U, W) := \inf_{\varphi,\psi \text{ MBP}} ||U^\varphi - W^\psi||_\square$$

where $U^\varphi(x, y) := U(\varphi(x), \varphi(y))$.
Unfortunately, the above defined $\delta_\square$ is **NOT** a metric as we can have $\delta_\square(U, W) = 0$ even when $U \neq W$. So, even this does not serve our purpose, and yet again, we look for a fix.

Reminder: We say that two graphons U and W are equal if the set $\{(x, y) \mid U(x, y) \neq W(x, y)\}$ has measure zero.

Define a relation as follows:

$$U \sim W \text{ if } \delta_\square(U, W) = 0 \tag{1}$$

Claim: The relation defined above is an equivalence relation.

Proof. It is clear that the relation given in eq. (1) is symmetric and reflexive. All that remains to be shown is the transitivity of the relation. Let $U \sim V$ and $V \sim W$. From these relations, we have that for $\varepsilon > 0$ and any $f, g : [0,1] \to [0,1]$, measurable,

$$\left| \int_{[0,1]} \int_{[0,1]} f(x)g(y)(U^\varphi(x, y) - V(x, y))\, dx\, dy \right| \leq ||U - V||_\square < \varepsilon \tag{2}$$

and

$$\left| \int_{[0,1]} \int_{[0,1]} f(x)g(y)(V(x, y) - W^\psi(x, y))\, dx\, dy \right| \leq ||V - W||_\square < \varepsilon \tag{3}$$

Adding the above two equations gives us

$$\left| \int_{[0,1]} \int_{[0,1]} f(x)g(y)(U^\varphi(x, y) - W^\psi(x, y))\, dx\, dy \right| < 2\varepsilon \tag{4}$$

that is, $\delta_\square(U, W) = 0$ which implies that $U \sim W$. $\square$

Recall our notation $\mathcal{W}_0$ for the set of graphons $W : [0,1]^2 \to [0,1]$. Let

$$\widetilde{\mathcal{W}_0} = \mathcal{W}_0 / \sim$$

This solves the problem we had before by identifying all graphons whose separating distance is zero as a single graphon.

Theorem 16 (**Lovász - Szegedy, 2007**)**.** *$(\widetilde{\mathcal{W}_0}, \widetilde{\delta_\square})$ is a compact metric space.*

Before proving the theorem, we present an Analytic version of Szemerédi's Regularity Lemma for a general Hilbert Space.

The Hilbert Space Regularity Lemma

Definition 17. *A Hilbert Space is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.*

Theorem 18 (Hilbert Space Regularity Lemma)**.** *Suppose $\mathcal{H}$ is a real Hilbert Space and $K_1, K_2, \ldots$ are non-empty subspaces of $\mathcal{H}$. Given $\varepsilon > 0$ and $f \in \mathcal{H}$, there exists $m \le 1/\varepsilon^2$ and $\gamma_i \in \mathbb{R}$ and $f_i \in K_i$ for $1 \le i \le m$ such that $\forall\, g \in K_{m+1}$,*

$$\left|\langle g, f - \sum_{i=1}^{m} \gamma_i f_i \rangle\right| \le \varepsilon \|f\| \cdot \|g\|$$

Proof. For each $m \in \mathbb{N}$, define

$$\eta_m := \inf_{\{\gamma_i\},\{f_i\}} \left|\left| f - \sum_{i=1}^{m-1} \gamma_i f_i \right|\right|$$

It is easy to observe that as m increases, the linear combination of the f_is approximates f more accurately, that is,

$$||f||^2 \ge \eta_1^2 \ge \eta_2^2 \ge \cdots > 0$$

So, there exists an m such that $\eta_m^2 \le \eta_{m+1}^2 + \varepsilon^2 ||f||^2$. That is to say, there exist $\gamma_i \in \mathbb{R}$ and $f_i \in K_i$ for $1 \le i \le m$ such that

$$\left|\left| f - \sum_{i=1}^{m} \gamma_i f_i \right|\right|^2 \le \eta_{m+1}^2 + \varepsilon^2 ||f||^2 \tag{5}$$

In particular, for any $\lambda \in \mathbb{R}$ and any $g \in K_{m+1}$,

$$\left|\left| f - \sum_{i=1}^{m} \gamma_i f_i - \lambda g \right|\right|^2 \;\ge\; \eta_{m+1}^2 \tag{6}$$

Let f^* denote $\sum_{i=1}^{m} \gamma_i f_i$. Then, eq. (5) and eq. (6) give us

$$||f - f^*||^2 \;\le\; \eta_{m+1}^2 + \varepsilon^2 ||f||^2 \;\le\; ||f - (f^* + \lambda g)||^2 + \varepsilon^2 ||f||^2$$

The above inequality reduces to

$$||g||^2 \lambda^2 - 2\lambda < f - f^*, g > + \varepsilon^2 ||f||^2 \ge 0 \text{ for all real values of } \lambda. \tag{7}$$

The above equation is a quadratic in λ and the discriminant must be at most 0. Thus,

$$\left|\langle g, f - \sum_{i=1}^{m} \gamma_i f_i \rangle\right| \le \varepsilon \|f\| \cdot \|g\|$$

This concludes the proof. $\square$

1 A consequence of the Hilbert Space Regularity Lemma

Let $\mathcal{H} = \mathcal{L}^2([0,1]^2)$ and let K_i be the set of all indicator functions of the form $\mathbb{1}_{S\times S}$ for some measurable $S \subseteq [0,1]$. Let $W \in \mathcal{W}_0$ be a graphon.

As per our prior notation, let $W = f$ and let $\mathbb{1}_{S\times S} = g$. Then, by the Hilbert Space Regularity Lemma,

$$\left| \int_{S\times S} W - W^* \right| \leq \varepsilon \text{ for all } S \subseteq [0,1] \tag{8}$$

where $W^* = \sum_{i=1}^{m} \gamma_i \mathbb{1}_{S_i\times S_i}$ and $m \leq 1/\varepsilon^2$.

One can also verify that for $S, T \subseteq [0,1]$,

$$\left| \int_{S\times T} W - W^* \right| \leq 2\varepsilon$$

2 The Weak Regularity Lemma

Let $\mathscr{P} = \{S_1, S_2, \ldots, S_m\}$ be a partition of $[0,1]$.

Definition 19. *A step function U is a function of the form*

$$U(u,v) := \sum_{i,j} u_{i,j} \mathbb{1}_{S_i\times S_j}(u,v)$$

where $S_i \times S_j$ is the unique part in $\mathscr{P}$ that contains the point (u,v).

Definition 20. *Given an integrable graphon W and a partition $\mathscr{P}$ of $[0,1]$, the stepping of W by $\mathscr{P}$ is the step function with*

$$u_{i,j} = \begin{cases} \frac{1}{\lambda(S_i)\cdot\lambda(S_j)} \int_{S_i\times S_j} W(x,y)\, dxdy & \text{if } \lambda(S_i)\lambda(S_j) > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda(S_i)$ denotes the Lebesgue measure of S_i.

The above definition says that we get $W_{\mathscr{P}}$ by averaging W over the steps (rectangles in the partition $\mathscr{P}$ of $[0,1]^2$).

FACT: If $\mathscr{P}$ is a partition, then the stepping operator is a contraction with respect to the cut norm, that is,

$$||W_{\mathscr{P}}||_\square \leq ||W||_\square$$

This is also true with respect to the $\mathcal{L}^1$ and the $\mathcal{L}^2$ norms.

Lemma 21 (Weak Regularity Lemma (Analytic Version))**.** *Given $W \in \mathcal{W}_0$, there exists a step function W^* with at most $2^{O(1/\varepsilon^2)}$ steps such that*

$$||W - W^*||_\square \leq 2\varepsilon$$

Let $W_{\mathscr{P}}$ be the stepping of W by the partition $\mathscr{P}$ that is described by W^*. Then,

$$||W - W_{\mathscr{P}}||_{\square} \leq ||W - W^*||_{\square} + ||W^* - W_{\mathscr{P}}||_{\square}$$

Observing that stepping W^* with respect to $\mathscr{P}$ has no effect, we have

$$||W - W_{\mathscr{P}}||_{\square} \leq 4\varepsilon$$

If $W = W_{G_n}$ for some graph G_n, then

$$\left| \int_{S \times T} W - W_{\mathscr{P}} \right| \leq 4\varepsilon \tag{9}$$

If we take K_is (as in Hilbert Space Regularity Lemma) as the algebra generated by intervals of the form $(i/n, (i+1)/n) \times (j/n, (j+1)/n)$ with $0 \leq i, j \leq n-1$, then the Weak Regularity Lemma (Analytic Version) is equivalent to the following:

Lemma 22 (Weak Regularity Lemma)**.** *Given a graph G on n vertices, there is an equitable partition of $V(G)$ into $m \leq 2^{O(1/\varepsilon^2)}$ parts $\{V_1, V_2, \ldots, V_m\}$ such that for any $S, T \subseteq V(G)$,*

$$\left| e(S,T) - \sum_{i,j} |S \cap V_i| \cdot |T \cap V_j| \right| \leq \varepsilon n^2 \tag{10}$$

Notice the similarity between equations eq. (9) and eq. (10)!

As a recap, we had seen the following theorem last time:

Theorem 23. *$(\widetilde{\mathcal{W}_0}, \widetilde{\delta_\square})$ is a compact metric space.*

We study some consequences of the above result.

Proposition 24. *Given any $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that: For any $W \in \mathcal{W}_0$, there exists a graph G with N vertices such that $\delta_\square(W, G) < \epsilon$.*

Before we prove this, we need the following lemma:

Lemma 25. *Graphs are dense in $\mathcal{W}_0$, w.r.t. $\delta_\square$ norm.*

Proof. Given any $\epsilon > 0$ and $W \in \mathcal{W}_0$, by weak regularity we can find a partition $\mathcal{P}$ such that $||W - W_\mathcal{P}||_\square < \epsilon$. By composing with a measure-preserving bijection, we can assume the parts in $\mathcal{P}$ are intervals; in that case $\delta_\square(W, W_\mathcal{P}) < \epsilon$. Now take a further subdivision of $\mathcal{P}$ into equal parts to get a graph that is close enough to W. □

Now we return to the proof of the proposition.

Proof of Proposition 2

Let $\mathcal{F}$ be the set of all finite graphs. Consider the collection $\{B_\epsilon(G) \mid G \in \mathcal{F}\}$, where $B_\epsilon(G) = \{W \in \widetilde{\mathcal{W}_0} \mid \widetilde{\delta_\square}(W, G) < \epsilon\}$. By the above lemma, this is an open cover of $\widetilde{\mathcal{W}_0}$.
Hence, there is a finite subcover $B_\epsilon(G_1), \dots, B_\epsilon(G_m)$ for some $m = m(\epsilon)$.
$\implies$ For any $W \in \widetilde{\mathcal{W}_0}$, there exists a graph G_i with at most $|V(G_m)|$ vertices such that $\widetilde{\delta_\square}(W, G_i) < \epsilon$.
Let $N = \mathrm{lcm}(|G_i| \mid i = 1, 2, \dots m) = N(\epsilon)$. Then each G_i can be subdivided into a graph with N vertices $G_N^{(i)}$ such that $G_i = G_N^{(i)}$ as graphons, for all $i \le m$. These $G_N^{(i)}$ uniformly approximate the graphons. ■

Digression about functional analysis

Given $f \in L^2[0, 1]$ and $W \in \mathcal{W}_0$, we can define the linear operator

$$T_W f(x) = \int_{[0,1]} f(y) W(x, y) dy$$

This T_W turn out to be a self-adjoint compact operator on $L^2[0, 1]$. By spectral theorem, T_W has a discrete spectrum, and we call the eigenvalues of T_W as eigenvalues of W itself. This definition matches the one for eigenvalues of graphs, when they are depicted as graphons. Further, if some sequence of graphs $G_n \to W$ in $\delta_\square$, then for any positive integer k, the k largest eigenvalues of G_n tend to the k largest eigenvalues of W (as a vector).

The Strong Regularity Lemma

Theorem 26 (Strong Regularity Lemma, Analytic Version). *Given a sequence* $\underset{\sim}{\epsilon} = (\epsilon_1, \epsilon_2, \dots) \in (0,1)^\omega$*, there exists a positive integer* $M = M(\underset{\sim}{\epsilon})$ *such that: For any* $W \in \mathcal{W}_0$*, we can write* $W = W_{step} + W_{pseudo} + W_{small}$*, where:*

1. W_{step} *is a step function with* $k \leq M$ *steps.*
2. $||W_{pseudo}||_\square \leq \epsilon_k$
3. $||W_{small}||_\square \leq \epsilon_1$

Proof. Given a $W \in \mathcal{W}_0$, using a standard measure theory result, there exists a step function $U \in \mathcal{W}_0$ such that $||W - U||_1 < \epsilon_1$. For any W, define

$$k(W) = \min\{k \in \mathbb{N} \mid \text{There exists a } k\text{-step graphon } U \text{ with } ||W - U||_1 < \epsilon_1\}$$

Consider the collection $\{B_{\epsilon_{k(W)}}(W)\}_{W \in \widetilde{\mathcal{W}_0}}$; this is an open cover of $\widetilde{\mathcal{W}_0}$. $\implies$ By compactness, there is a finite subcover

$$\{B_{\epsilon_{k(W_1)}}(W_1), B_{\epsilon_{k(W_2)}}(W_2), \dots, B_{\epsilon_{k(W_l)}}(W_l)\}.$$

Let $M = \max\{k(W_1), k(W_2), \dots, k(W_l)\}$. Note that M only depends on $\underset{\sim}{\epsilon}$. Further, for any $W \in \widetilde{\mathcal{W}_0}$, there exists a $W' \in \widetilde{\mathcal{W}_0}$ and a step function $U \in \mathcal{W}_0$ with $k = k(W') \leq M$ parts such that $||W' - U||_1 < \epsilon_1$ and $\widetilde{\delta_\square}(W, W') < \epsilon_k$. The latter inequality implies the existence of a measure-preserving bijection φ such that $||W - (W')^\varphi||_\square < \epsilon_k$. Further, U^φ is still a step function, and

$$||(W')^\varphi - U^\varphi||_1 = ||W' - U||_1 < \epsilon_1$$

because φ is measure preserving. Thus we can write

$$W = U^\varphi + (W - (W')^\varphi) + ((W')^\varphi - U^\varphi)$$

and we can let $W_{\text{step}} = U^\varphi$, $W_{\text{pseudo}} = W - (W')^\varphi$, and $W_{\text{small}} = (W')^\varphi - U^\varphi$. □

There is also a quantitative version of strong regularity:

Theorem 27 (Strong Regularity Lemma, Quantitative Version). *Given* $\epsilon > 0$ *and* $W \in \mathcal{W}_0$*, there exists a positive integer* $M = M(\epsilon)$ *and a partition* $\mathcal{P} = \{S_1, S_2, \dots, S_m\}$ *into* $m \leq M$ *measurable parts such that:*

1. $m(S_i) = m(S_j)$ *for all* $i, j \leq m$*.*
2. $\left|\int_R (W - W_{\mathcal{P}})\right| < \epsilon$ *where* R *is union of any subset of rectangles in* $[0,1]$ *induced by* $\mathcal{P}$*.*

Proof. Omitted from the course. □

A version of strong regularity also exists for graphs:

Theorem 28 (Strong Regularity Lemma, Graph Version). *Given a sequence* $\underset{\sim}{\epsilon} = (\epsilon_1, \epsilon_2, \dots) \in (0,1)^\omega$*, there exists a positive integer* $M = M(\underset{\sim}{\epsilon})$ *such that: For any graph* G*, there exist partitions* $\mathcal{P}$ *and* $\mathcal{Q}$ *of* $V(G)$ *satisfying:*

1. $\mathcal{Q}$ *is a refinement of* $\mathcal{P}$.

2. $|\mathcal{Q}| \leq M$.

3. $\mathcal{Q}$ *is* $\epsilon_{|\mathcal{P}|}$*-regular.*

4. $q(\mathcal{Q}) \leq q(\mathcal{P}) + \epsilon_1$

Proof. We use the normal regularity lemma repeatedly. Start with partition $\mathcal{P}_0 = \{V(G)\}$. If we have partition $\mathcal{P}_i$, using the normal regularity lemma we can get a partition $\mathcal{P}_{i+1}$ with size at most $|\mathcal{P}_i|M'(\epsilon_{|\mathcal{P}_i|})$ that is $\epsilon_{|\mathcal{P}_i|}$-regular (where M' is the function coming from the normal regularity lemma). Note that

$$0 \leq q(\mathcal{P}_0) \leq q(\mathcal{P}_1) \leq \cdots \leq 1$$

Therefore there exists an $i \leq \frac{1}{\epsilon_1}$ such that $q(\mathcal{P}_{i+1}) - q(\mathcal{P}_i) < \epsilon_1$. Now just take $\mathcal{P} = \mathcal{P}_i$ and $\mathcal{Q} = \mathcal{P}_{i+1}$. $\square$

Some further consequences of compactness

We shall now look into some more applications of compactness of Graphon space wrt the cut metric.
Notation: $\mathcal{F}$ is the set of finite graphs.

Lemma 29 (Moments' Lemma)**.** *Suppose* $W, U \in \widetilde{\mathcal{W}_0}$ *such that* $t(H, W) = t(H, U)$ *for all graphs(finite)* H*. Then* $\delta_\square(W, U) = 0$*. In other words, the sequence* $\{t(H, W)\}_{H\in\mathcal{F}}$ *uniquely determines* W*.*

Proof. Omitted from the course. $\square$

Lemma 30 (Inverse Counting Lemma)**.** $W_n \to W$ *in* $t(H, .)_{H\in\mathcal{F}}$ *iff* $\widetilde{\delta_\square}(W_n, W) \to 0$

Proof. One implication is obvious from the counting lemma:

$$|t(H, U) - t(H, W)| \leq |E(H)|||U - W||_\square \leq |V(H)|^2 \cdot \delta_\square(U, W),$$

which proves the reverse direction.
Now, conversely consider $\Phi : \widetilde{\mathcal{W}_0} \to [0, 1]^{\mathcal{F}}$ given by $W \to (t(H, W))_{H\in\mathcal{F}}$. Moments' lemma implies that Φ is injective. Also, this map is continuous since each component $t(H, .)$ is continuous. Since $[0, 1]^{\mathcal{F}}$ is compact and Hausdorff, $Im(\Phi)$ is Hausdorff, $\widetilde{\mathcal{W}_0}$ is compact. This implies $\Phi : \widetilde{\mathcal{W}_0} \to Im(\Phi)$ is a continuous bijective map from compact and Hausdorff, thus it's a homeomorphism. This implies that its inverse is continuous, as required. $\square$

An immediate corollary follows:

Corollary 31. *Given* $\epsilon > 0, \exists k \in \mathbb{N}, \eta > 0$ *such that, if* $|t(H, U) - t(H, W)| < \eta$ *for all* $H \in \mathcal{F}$ *with* $|V(H)| \leq k$*, then* $\widetilde{\delta_\square}(U, W) < \epsilon$*.*

A stronger Inverse counting lemma due to Borgs, Chayes and Lovasz is as follows:

Theorem 32 (Stronger Inverse Counting)**.** *Let* $U, W \in \widetilde{\mathcal{W}_0}$*. If for all* $H \in \mathcal{F}$ *with* $|V(H)| \leq k$*, we have* $|t(H, U) - t(H, W)| < 2^{-k^2}$*, then* $\widetilde{\delta_\square}(U, W) \leq O\left(\frac{1}{\sqrt{\log k}}\right)$

Proof. Omitted from the course. $\square$

Property Testing

Given a graph G, we want a randomized algorithm to determine if G has a triangle or if G is 'ϵ-far' from being triangle-free, i.e., it is enough to delete $\leq \epsilon n^2$ edges to remove all triangles in G.

Algorithm 1 Randomized algorithm for triangle-free checking

```
step ← 1
tfree ← True
while step ≤ M do
    Select random triples u, v, w ∈ V(G)
    if u, v, w form a triangle then
        declare G has the triangle u, v, w
        tfree ← False
        break
    end if
    step ← step + 1
end while
if tfree then
    declare G is ϵ-far from being triangle-free
end if
```

Analysis of correctness

We want $\mathbb{P}(\text{error in algo}) \leq \frac{1}{3}$. Error occurs when $\geq \epsilon n^2$ edges are needed to be deleted, but no triangle detected on random triplet sampling. Triangle Removal Lemma implies that, there are $\geq \delta \frac{n^3}{6}$ triangles in G for some $\delta = \delta(\epsilon) > 0$. So, $\mathbb{P}(\text{error}) \leq (1-\delta)^M \leq e^{-\delta M}$. This is $< \frac{1}{3}$ if $M > \frac{1}{\delta} \log 3$. It follows that, if we want $\mathbb{P}(\text{error}) < \eta$, it's an $O(\log \frac{1}{\eta})$ time algorithm.

One last connection

Recall that, if $p > 0$ is a given constant and if $t(K_2, G_n) = p + o_n(1)$ and $t(C_4, G_n) = p^4 + o_n(1)$, then $\forall H \in \mathcal{F}$, $t(H, G_n) = p^{e(H)} + o_n(1)$. We have the following equivalence for graphons:

Theorem 33. *Suppose $0 < p < 1$ and $W \in \mathcal{W}_0$. If $t(K_2, W) = p$ and $t(C_4, W) = p^4$, then $W = p$ almost everywhere.*

Proof. Let $\omega(z) := \int_{[0,1]} W(x,z)dx$ (we can think of this as degree of z). Then, $\int_{[0,1]} \omega(z)dz = p$.

Also, $p^2 = (\int_{[0,1]} \omega(z)dz)^2 \leq \int_{[0,1]} \omega(z)^2 dz$ by Cauchy-Schwarz. Now, consider:

$$
\begin{aligned}
0 &\leq \int_{[0,1]^2} \left(\int_{[0,1]} (W(x,z)W(y,z) - p^2)dz \right)^2 dxdy \qquad (1)\\
&= \int_{[0,1]^2} \left(\int_{[0,1]^2} (W(x,z)W(y,z) - p^2)(W(x,z')W(y,z') - p^2)dzdz' \right) dxdy \\
&= \int_{[0,1]^4} W(x,z)W(y,z)W(x,z')W(y,z')dzdz'dxdy \quad \ldots \text{[note, this equals } t(C_4, W) = p^4] \\
&- 2p^2 \int_{[0,1]^3} W(x,z)W(y,z)dzdxdy + p^4 \quad \ldots \text{[note, first term equals } \int_{[0,1]} \omega(z)^2 dz] \\
&= 2p^2 \left(p^2 - \int_{[0,1]} \omega(z)^2 dz \right) \leq 0 \;\; \text{as shown above}
\end{aligned}
$$

This implies $\int_{[0,1]} \omega(z)^2 dz \leq p^2 \leq \int_{[0,1]} \omega(z)^2 dz$. Thus, equality holds in Cauchy-Schwarz and thus, $\omega(z) = p$ a.e. Further, equality holds in (1), so $\int_{[0,1]} W(x,z)W(y,z)dz = p^2$ for almost all x, y. From this, we can show that $\int_{[0,1]} W(x,z)^2 dz = p^2$ for almost all x (the proof of this is relegated to the exercises). Now, $\omega(z) = p$ a.e. and symmetricity of graphons implies $\int_{[0,1]} W(x,z)dz = p$ for almost all x. Combining the above two facts, we have:

$$
\begin{aligned}
&\int_{[0,1]} (W(x,z) - p)^2 dz \\
&= \int_{[0,1]} W(x,z)^2 dz - 2p \int_{[0,1]} W(x,z)dz + p^2 \\
&= 0 \;\; \text{a.e on } x \in [0,1]
\end{aligned}
$$

From this, we get that $W(x,z) = p$ for almost all x, z, i.e., $W = p$ a.e. $\square$

Graph Theory	2023
An Erdős Theorem and Edge Colorings	
Lecturer: Niranjan Balachandran	Scribe: Rohinee

In this part, we prove the Erdős Theorem which tells us that usual graph colors can also be quite contrary to simple heuristic suggestions. Then we look at a beautiful proof of Vizing's theorem.

Theorem 34. *(Erdős) Given $k, g \in \mathbb{N}$, there exists a graph G such that the length of the smallest cycle in the graph is g and $\chi(G) > k$.*

Proof. We will pick a random graph in which the edges are chosen with probability p. Note that, p can't be an absolute constant, otherwise, the graph will be 'super sparse', and thus the chromatic number can't be large. Let $p = \frac{f(n)}{n}$, and the function $f(n)$ is to be determined. Let N be the number of cycles in G of size less than or equal to g. Then

$$\begin{aligned}
\mathbb{E}N &= \sum_{(v_1, v_2, \ldots, v_t) 3 \leq t \leq g} \mathbb{P}(v_i v_{i+1} \in E(G) \forall i) \\
&= p^t \\
&= \sum_{t=3}^{g} \frac{N(N-1)\ldots(N-t+1) \cdot p^t}{2t} \\
&\geq \frac{g(np)^g}{6} \\
&= \frac{g(f(n))^g}{6}
\end{aligned}$$

The first equality follows from the definition. For the second equality, note that the edges are chosen independently, each with probability p. The numerator of the third equality calculates the number of t-tuples, and we divide by the factor of $2t$ because cyclic permutations and computing in reverse order give the same cycle. The last equality indicates that $g \cdot f(n)^g$ should be small. Choose l such that $\chi \geq \frac{n}{\alpha(G)} \geq \frac{n}{l}$. Note that, $\mathbb{P}(\alpha(G) \geq l) = \mathbb{P}$ (there exists an l-subset of V(G) with no edges in it). Thus,

$$\begin{aligned}
\mathbb{P}(\alpha(G) \geq l) &\leq \binom{n}{l} \cdot (1-p)^{\binom{l}{2}} \\
&< \frac{n^l \cdot e^{-p \cdot \binom{l}{2}}}{l!} \\
&< \frac{e^{\frac{-pl^2}{6} + l \cdot \log n}}{l!}
\end{aligned}$$

In order to make $\frac{-pl}{3} + \log n$ negative, choose $l = \frac{n}{2k}, p = \frac{12k \log n}{n}$. Then $\mathbb{P}(\alpha(G) \geq l) = o_n(1)$ as $n \to \infty$. Using the Markov's inequality, we get

$$\mathbb{E}N < O_{g,k} \cdot (\log n)^g \implies \mathbb{P}(N > C_{g,k} \cdot (\log n)^g) < \frac{1}{2}.$$

In particular, with positive probability we have G with $\alpha(G) \leq \frac{n}{2k}$ and $N > C_{g,k} \cdot (\log n)^g$. Throw away a vertex from every cycle of size less than or equal to g in this graph to obtain a new graph G^*. Note that, in G^* the length of the smallest cycle is greater than g, and $\alpha(G^*) \leq \alpha(G) \leq \frac{n}{2l}$. So $\chi(G^*) \geq \frac{0.9 \cdot n \cdot 2k}{n} = 1.8k$. This completes the proof. $\square$

The following theorem by Erdős states that the chromatic number of a graph can be made arbitrarily large, even though the chromatic number of "some" induced subgraphs is extremely small.

Theorem 35 (Erdős)**.** *Given $k \in \mathbb{N}$, there exists $\epsilon_0 = \epsilon_0(k)$ such that there exist graphs G_n (for $n >> 0$) such that $\chi(G) > k$ and $\chi(G[S]) \leq 3$ for every $S \subseteq V$ with $|S| \leq \epsilon_0 \cdot n$.*

Graph colorings are indeed very surprising. We will now see a result that is a bit more assuring about graph colorings to repose some faith in the readers that it is not too hopeless!

Edge Colorings

We say that a graph G is k edge colorable if the edges of G can be colored using $\{1, 2, \dots, k\}$ such that no two adjacent edges have the same color. The edge chromatic number is denoted by $\chi'(G)$. Note that, the edge coloring of a graph G is the same as the vertex coloring of the corresponding line graph $L(G)$. It is easy to see that $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G. The following theorem gives a neat upper bound on the edge chromatic number.

Theorem 36 (Vizing)**.** *In a graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Proof. For the contrary, suppose there exist counter-examples to Vizings upper bound. Let G be a counter-example of minimal size that is, if one edge of G is removed, then the graph satisfies the upper bound. Let $e = uv_1$ be the edge- if removed, satisfies the Vizing's bound. We color one edge at a time $uv_1, uv_2, uv_3, \dots$ using $\Delta(G)+1$ colors $a_0, a_1, a_2, a_3, \dots$ and adjust the coloring such that the coloring stays proper.

Note that each vertex misses at least one color. Let a_i be a color absent at v_i. Color uv_{i+1} using a_i. The chain stops at $k \in \mathbb{N}$ when either a_k is a color absent at u, or a_k is already used on uv_j for some $j < k$. If a_k is absent at u, then we can reassign colors a_i to uv_i for each i, and we get a proper coloring. So now suppose a_k is not absent at u. Let a_0 be a color absent at u. Then recolor uv_i for $i \leq j-1$, and remove the color a_k from uv_j. Now we should find a way to color uv_j. Note that a_k is absent at both v_j and v_k.

Consider the following cases,
Case 1: If a_k is absent at u, then color uv_j with a_k.
Case 2: If a_0 is absent at v_j, then color uv_j with a_0.
Case 3: If a_0 is absent at v_k, then shift one color along the cycle and recolor uv_i for $j \leq i < k$ and color uv_k with a_0. The coloring is proper because none of the uv_i for $j \leq i < k$ are colored with a_0 or with a_k.

If none of these conditions hold, then consider the subgraph G' of G consisting only of edges colored with a_0 or a_k (and their corresponding vertices). Note that G' is a disjoint union of paths and cycles. Since none of the above conditions hold, $\{u, v_j, v_k\}$ must be the endpoints of the paths, so all of them can't be in one connected component. In the component containing exactly one of these vertices, switch a_0 with a_k, and apply either case 2 or case 3. Thus, we get a proper coloring of G using at most $\Delta(G)+1$ colors.

□

Remark: Even though $\chi'(G)$ is narrowed down so much, determining whether $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ is an NP-complete problem.
We will look at some properties of graphs with $\chi'(G) = \Delta(G)$ in the next lecture.

The König-Egerváry Theorem

Lecturer: Niranjan Balachandran Scribe: Anish

In this part, we provide a non-constructive proof of the König-Egerváry, which characterizes the relationship between the size of the min. vertex cover and max. matching in a bipartite graph G. Furthermore, we use this result to directly show that the complement of a bipartite graph is *perfect* (no Lovász's theorem required!)

Definition 37. *A **vertex cover** of $G = (V, E)$ is a subset $U \subseteq V$ such that $\forall\, e \in E$, e has at least one end in U.*

A *minimum vertex cover* is a vertex cover of least size.

Definition 38. *A **matching** of $G = (V, E)$ is a subset $M \subseteq E$ such that $\forall\, v \in V$, v covers (i.e. is the end point of) atmost one edge in M.*

A *maximum matching* is a matching of maximum size.

Observation 39. *Let's suppose W is a minimum vertex cover and M is a maximum matching. Then, since each $v \in C$ can cover atmost one $e \in M$, $|W| \geq |M|$.*

Let us denote the size of the min. vertex cover in any graph G as $\nu(G)$ and the size of the maximum matching as $\mu(G)$. Then $\nu(G) \geq \mu(G)$.

Theorem 40 (Knig-Egevry). *Let $G = (V, E)$ be a bipartite graph. Then, $\nu(G) = \mu(G)$.*

Proof by contradiction. Suppose there exists a counterexample to the above. Consider the set of counterexamples with minimal $|V|$ (say $|V| = n$). Let G be the graph in this set with minimal $|E|$. Clearly, G is connected. If $d(v) \leq 2\ \forall\, v \in V$, then G would be either a path or an even cycle (since G is bipartite). In such a case $\mu(G) = \nu(G) = \lfloor \frac{n}{2} \rfloor$. So, $\exists\, u \in V$ such that $d(u) \geq 3$.

Consider $v \in N(u)$. Since G was minimal, the graph $G' = G \backslash \{v\}$ has a min. vertex cover W' such that $|W'| = \nu(G') = \mu(G')$. If $\mu(G') < \mu(G)$ then the set $W' \cup \{v\}$ would cover G and have size $\leq \mu(G)$. Hence, $\mu(G') = \mu(G)$ and so there exists a maximum matching M of G not containing v. If no edge in M was incident on u then we could add (u, v) to M, so u has an edge in M incident on it. Since $d(u) \geq 3$, however, $\exists\, e'$ incident on u, not incident on v and not part of M.

Consider $G'' = G \backslash \{e'\}$. Since G was minimal, G'' has a min. vertex cover W'' such that $|W''| = \nu(G'') = \mu(G'') = \mu(G)$, since M is maximum in G and is present in G''. Every $w \in W''$ covers atmost one edge in M, but $|W''| = |M|$ so every $w \in W''$ must cover exactly one edge in M. Since no $e \in M$ is incident on v, $v \notin W''$, but since $(u, v) \in E(G'')$ must be covered by W'', $u \in W''$. However, this means that the edge e' will be covered by W'' when re-introduced in G'', and so W'' covers G.

Existence of $W'' \implies \nu(G) \leq \mu(G)$. By 39, $\nu(G) = \mu(G)$.

Now, let us focus on the question of coloring: specifically, proper colorings of $\overline{G}$. First, some observations for bipartite G:

Observation 41. *For a maximum matching M, let $\mu_M(G)$ be the set of vertices in G unmatched by M. Then, $n = 2\mu(g) + \mu_M(G)$.*

Observation 42. *$W \subseteq V$ is a vertex cover of G*
$\iff$ the subgraph induced by $V \backslash W$ is empty.
$\iff$ $V \backslash W$ is an independent set.

Now, 41 leads us to the following idea of coloring G using $\mu(G) + \mu_M(G)$ colors as follows:
$\forall\ e \in M$, assign its two endpoints a color unique to e. $\forall$ unmatch $v \in V$, assign a color unique to v. We observe that any two non-adjacent $u, v \in V$ will not have an edge $e = (u, v)$ present in M, so they will be colored differently.

However, $(u, v) \notin E(G) \iff (u, v) \in E(\overline{G})$, meaning the above coloring is proper for $\overline{G}$ and so

$$\chi(\overline{G}) \leq \mu(G) + \mu_M(G) = n - \mu(G) \tag{11}$$

Finally, from 42 we see that a min. vertex cover correlates to a max. independent set; in particular,

$$n = \nu(G) + \alpha(G) \tag{12}$$

Since independent sets in G correspond to cliques in $\overline{G}$,

$$\alpha(G) = \omega(\overline{G}) \tag{13}$$

Also, $\nu(G) = \mu(G)$ by 40. Thus,

$$n = \mu(G) + \omega(\overline{G}) \tag{14}$$

Substituting this in the first equation gives us $\chi(\overline{G}) \leq \omega(\overline{G})$. However, we know that $\chi(H)$ for any graph H is bounded from below by the clique number. So, we obtain

$$\chi(\overline{G}) = \omega(\overline{G}) \tag{15}$$

and have thus proven that $\overline{G}$ is perfect for any bipartite G.

Perfect Graphs

Lecturer: Niranjan Balachandran Scribe: Om Swostik

If G is a line graph, then $w(G) \leq \chi(G) \leq w(G)+1$ ($w(G)$ = clique number of G).
This follows as a corollary of Vizing's theorem.
Natural question: When is $w(G)=\chi(G)$? In other words, can we characterize graphs for which $w(G)=\chi(G)$?
Consider vertex-disjoint K_k and Erdos graph with $\chi = k$ and large girth.
The above illustrates that the question is poorly posed.
Better Question: Given a graph G, is $\chi(H)=w(H)$, $\forall$ induced subgraphs H of G?

Definition 43 (C.Berge, late 60s)**.** *A graph G satisfying $\chi(H)=w(H)$, $\forall\, H \subseteq G$, where H must be an induced subgraph, is called perfect.*

Here are some examples of perfect graphs:

- K_n
- Bipartite graphs
- Line graphs of bipartite graphs

Theorem 44. *If G is bipartite, then $\chi'(G)=\Delta(G)$.*

Proof. WLOG assume G is Δ-regular (else embed G into a bipartite graph that is Δ-regular). If $G(A,B,E)$ is Δ-regular, then note that $|A|=|B|$. It suffices to show: G has a perfect matching i.e. there is a set of edges $\mathcal{M}=\{(a_i,b_i)\}$ that are pair-wise disjoint and cover $V(G)$ (every vertex is adjacent to exactly one edge). To establish this, we need a classical theorem:

Theorem 45 (Hall)**.** *$G(A,B,E)$ admits a matching that saturates A i.e. a matching in which every $a \in A$ is incident with a matching edge iff for every $S \subseteq A$, $|N(S)| \geq |S|$ where $N(S)=\cup_{a\in S}N(s)$.*

Fix $S \subseteq A$. Since $G(A,B,E)$ is Δ-regular, we have, $e(S,N(S))=\Delta|S|$.

$$\Delta|N(S)| \geq e(N(S),N(N(S))) \geq e(S,N(S)) \geq \Delta|S| \Rightarrow |N(S)| \geq |S|$$

From Hall's theorem, the result follows. $\square$

From Theorem 44, it follows that line graphs of bipartite graphs are perfect.

Definition 46 (Vertex Cover)**.** *A set of vertices that (together) touch every edge.*

Theorem 47 (Konig)**.** *If G is bipartite, then size of a minimum vertex cover is the same as the size of a maximum matching.*

Using Theorem 47, we can establish that complement of bipartite graphs are also perfect. From the next theorem, it will follow that, complement of $L(G)$, where G is bipartite is also perfect.

Conjecture 48 (Weak Perfect Graph Conjecture)**.** *G is perfect $\Leftrightarrow$ $\overline{G}$ is perfect.*

Theorem 49 (Lovasz, mid 70's)**.** *G is perfect $\Leftrightarrow$ $\overline{G}$ is perfect.*

We'll see a proof this due to Gasparian (1996).

Definition 50 (Minimally imperfect graphs)**.** *Call a graph G minimally imperfect if G is not perfect, but every proper induced subgraph of G is perfect.*

For example, all odd cycles are minimally imperfect. We need the following observations:

Observation 51. *If G is minimally imperfect, then $\chi(G) = w(G) + 1$. Note that for any vertex x of G, $w(G \setminus \{x\}) = w(G)$.*

Observation 52. *Call a clique $\mathcal{C}$ large if $|\mathcal{C}| = w(G)$. Then for any non-empty, independent set $I \subseteq G$, there exists some large clique $\mathcal{C}$ of G such that $I \cap \mathcal{C} = \Phi$.*

Suppose I meets every large clique. Note that, $|I \cap \mathcal{C}| \in \{0, 1\}$, $\forall\, I \neq \Phi$. We have, $\chi(G \setminus I) = w(G \setminus I) \leq w(G) - 1$. So $G \setminus I$ can be $w(G) - 1$ colored $\Rightarrow G$ can be $w(G)$ colored. Contradiction. Write $\alpha = \alpha(G)$, $w = w(G)$. Let $I_0 = \{v_1, v_2 \dots v_\alpha\}$ be a maximum independent set.

$$V \setminus \{v_i\} \ = \ I_1^{(i)} \uplus I_2^{(i)} \uplus \cdots \uplus I_w^{(i)}$$

since $\chi(V \setminus \{i\}) = w(G)$.

$$\mathcal{I} = \{I_0, I_j^{(i)} \ : \ 1 \leq i \leq \alpha,\ 1 \leq j \leq w\}$$

where $\mathcal{I}$ is the set of all independent sets. Let $N = |\mathcal{I}| = 1 + \alpha w$.

Claim 53. *If $\mathcal{C}$ is a large clique in G, then $\mathcal{C}$ is disjoint from ≤ 1 of the sets of $\mathcal{I}$.*

Proof. Suppose $\mathcal{C} \cap I_0 = \Phi$, then $\mathcal{C} \cap I_j^{(i)} \neq \Phi$, $\forall\, i, j$. To show this, notice that

$$\mathcal{C} \subseteq V \setminus \{v_i\} \ = \ I_1^{(i)} \uplus I_2^{(i)} \uplus \cdots \uplus I_w^{(i)}$$

Since $|\mathcal{C}| = w$, it must touch each $I_j^{(i)}$, $\forall\, i, j$. Now, suppose $\mathcal{C} \cap I_j^{(i)} = \Phi$, for some i, j. This means, following a similar line of argument as before, $\mathcal{C} \cap I_0 = \{v_i\}$. We have, $\mathcal{C} \cap I_k^{(l)} \neq \Phi$, $\forall k, l$ with $l \neq i$. Further, since $|\mathcal{C}| = w$, we have $\mathcal{C} \cap I_k^{(i)} \neq \Phi$, $\forall k$ with $k \neq j$. The claim follows. $\square$

Let $C_1, C_2 \dots, C_N$ be large cliques such that each C_i is disjoint with exactly one of the independent sets of $\mathcal{I}$. Let $\mathcal{C}$ be the collection of sets $\{C_i\}$. Reorder the sets in $\mathcal{C}$ such that C_i is disjoint with I_i in $\mathcal{I}$. Let A be an $N \times n$ adjacency matrix such that $A(i, j) = 1$ iff I_i contains vertex j, otherwise 0. Similarly, define B as an $N \times n$ adjacency matrix with the rows labelled by cliques (in order) and columns labelled by vertices of G. Since I_i misses C_i, $I_i \cap C_i = \Phi$, but $|I_j \cap C_i| = 1$ for all $i \neq j$ $\Rightarrow (AB^T)_{n \times n} = J - I$. Since $J - I$ is invertible, we have (by rank arguments),

$$N = 1 + \alpha w \leq n.$$

Note that $n \leq \alpha(G)\chi(G)$, for any graph G. In particular,

$$\chi(G \setminus v) = w(G) \ \text{and} \ \alpha(G \setminus v) \leq \alpha(G).$$

So,

$$\begin{aligned}
&n - 1 \leq \alpha(G \setminus v)\chi(G \setminus v) \leq \alpha(G)w(G) = \alpha w \\
\Rightarrow\ &n \leq 1 + \alpha w = N \\
\Rightarrow\ &N = n \\
\Rightarrow\ &n = N = 1 + \alpha(G)w(G) = 1 + \alpha(\overline{G})w(\overline{G}).
\end{aligned}$$

In particular, if G is minimally imperfect, $\overline{G}$ cannot be perfect either. This is because, if $\overline{G}$ is perfect, then

$$\begin{aligned}
&w(\overline{G}) = \chi(\overline{G}) \\
\Rightarrow\ &n \leq \alpha(\overline{G})\chi(\overline{G}) = \alpha(\overline{G})w(\overline{G}) < n.
\end{aligned}$$

which is a contradiction.
We have proved so far that G being minimally imperfect $\Rightarrow$ $\overline{G}$ is not perfect. Say G is perfect and $\overline{G}$ is not perfect. There exists a minimally imperfect induced subgraph $\overline{H}$ of $\overline{G}$. But $H = \overline{\overline{H}}$ is not perfect, which is a contradiction! Theorem 49 follows.
For an account of Lovasz's proof, see the book Modern Graph Theory by Bollobas.

Definition 54 (Odd Hole)**.** *An odd hole is an induced odd cycle of size* ≥ 5*.*

Definition 55 (Anti-Hole)**.** *An anti-hole is a complement of a hole.*

Conjecture 56 (Strong Perfect Graph Conjecture, Berge, 70s)**.** *G is perfect iff G has no odd holes/antiholes.*

This became a theorem in 2006.

Theorem 57 (Chudnovsky, Robertson, Seymour, Thomas, 2006)**.** *The conjecture is true.*

See [5] for the (150-page) proof!

Graph Theory	2023
List Colourings and a result of Alon	
Lecturer: Niranjan Balachandran	Scribe: Krishna Agaram

In this lecture we first prove Brooks' theorem for list colourings, and then prove a result of Alon'00 that establishes lower bounds on the list chromatic number in terms of the minimum degree of the graph.

Definition 58 (List Colouring)**.** *Suppose $\mathbb{C}$ is a nonempty set (of colours) and for each $v \in V$ we are given a set $\mathcal{L}(v) \subseteq \mathbb{C}$. The collection $\mathcal{L}$ of these subsets is called a* List-Assignment *for G. We say that G is $\mathcal{L}$-choosable if there is a function $\varphi : V \to \mathbb{C}$ such that for every $v \in V$, $\varphi(v) \in \mathcal{L}(v)$ and for every edge $uv \in E$, $\varphi(u) \neq \varphi(v)$.*

Definition 59 (List chromatic number)**.** *Given graph G, the list chromatic number $\chi_l(G)$ is the smallest k such that every list assignment $\mathcal{L}$ satisfying $|\mathcal{L}(v)| \geq k$ for each $v \in V$ is $\mathcal{L}$-choosable.*

Definition 60 (Degeneracy number)**.** *The degeneracy number of graph G is defined by*

$$d(G) := \max_{H \subseteq_{ind} G} \delta(H).$$

It is easy to see that $\chi_l(G) \leq d(G) + 1 \leq \Delta(G) + 1$ using a straightforward greedy colouring algorithm. Also, we have $\chi(G) \leq \chi_l(G)$ (if $\chi_l(G) = k$, then the list assignment assigning $[k]$ to each vertex is choosable, so $\chi(G) \leq k$).

3 Brooks' theorem

Theorem 61 (Brooks)**.** *Let $\Delta \geq 3$. Suppose G is connected with maximum degree at most Δ but is not a clique. Then $\chi_l(G) \leq \Delta$.*

Proof. We present a proof due to *M. Krivelevich* (2022) [8]. The proof proceeds by induction on $n = |V(G)|$. The base case $n \leq \Delta$ is obvious given any list assignment L for $V(G)$, one can just choose distinct colours for all vertices of G. We may now assume that G is Δ-regular: if there is a vertex v with degree $< \Delta$, colour $G \backslash v$ first, and we will have a colour left for v.
Fix a list assignment $\mathcal{L}$ with $|\mathcal{L}(v)| = \Delta$ for each $v \in V$. Since G is connected and not a clique, we can find vertices v_1, v_2, v_3 such that $v_1v_2, v_2v_3 \in E$ and $v_1v_3 \notin E$ (for example, consider a pair of vertices x, y with $xy \notin E$. Since G is connected, there is a shortest path between x and y. The last three vertices on the path serve as v_1, v_2, v_3). Consider now a longest path that begins in v_1, v_2, v_3, say $v_1, \ldots, v_k$.

- **Case 1:** $k = n$. We aim to find a colouring φ for vertices v_1 and v_3 that satisfies

$$|\mathcal{L}(v_2) \cap \{\varphi(v_1), \varphi(v_3)\}| \leq 1. \tag{$\star$}$$

  To this end, suppose that $\mathcal{L}(v_1) \neq \mathcal{L}(v_2)$. Choose $\varphi(v_1) \in \mathcal{L}(v_1) \backslash \mathcal{L}(v_2)$ and $\varphi(v_3) \in \mathcal{L}(v_3)$ arbitrarily. Proceed similarly if $\mathcal{L}(v_3) \neq \mathcal{L}(v_2)$. Otherwise, $\mathcal{L}(v_1) = \mathcal{L}(v_2) = \mathcal{L}(v_3)$, in which case simply choose $c \in \mathcal{L}(v_1)$ arbitrarily and set $\varphi(v_1) = \varphi(v_3) = c$. Note that there are at most $\Delta - 2$ neighbours of v_2 not coloured yet, so by ($\star$), v_2 can be coloured even after all its neighbours are coloured. Suppose that v_j ($j > 3$) is adjacent to v_2 (since

$\Delta \geq 3$ and G is Δ-regular). Colour the rest of the graph greedily in the following order: $(v_1, v_3, v_4, \ldots, v_{j-1}, v_n, v_{n-1}, \ldots, v_j, v_2)$. Note that we will never lack a colour for v_i $(i \neq 2)$ because there is a neighbour of v_i that follows it in the ordering.

- **Case 2:** $k < n$. Note that all neighbours of v_k lie on the path. Suppose that the smallest $i \geq 1$ for which v_i is adjacent to v_k is j. Then $v_j, v_{j+1}, \ldots, v_k$ form a cycle C such that $N(v_k) \subseteq V(C)$. Let us relabel $v_k \to v$. Colour the graph $G \backslash C$ inductively, producing colouring φ. Pick arbitrary $w \in G \backslash C$. Since G is connected, there is a path from v to w, say the first vertex after v on the path is $u \in C$. As before, we try to colour v so that

$$|\mathcal{L}(u) \cap \{\varphi(v), \varphi(w)\}| \leq 1.$$

If $\varphi(w) \notin \mathcal{L}(u)$, set $\varphi(v) \in \mathcal{L}(v)$ arbitrarily. Otherwise, if $\mathcal{L}(v) \neq L(u)$, pick $\varphi(v) \in \mathcal{L}(v) \backslash \mathcal{L}(u)$. Else we must have $\varphi(w) \in \mathcal{L}(u) = \mathcal{L}(v)$, set $\varphi(v) = \varphi(w)$ in this case. We may now colour u after all its neighbours. Label the vertices of C following the cycle as $v_1, v_2, \ldots, v_{|C|}$ where $v = v_1$ and $u = v_{|C|}$. Finally, extend the colouring φ to C in the order $(v_1, v_2, \ldots, v_{|C|})$.

The proof follows by induction. □

We also get for free Brooks' theorem for the chromatic number:

Corollary 62. *Suppose G is connected, not a clique and satisfies $\Delta(G) \geq 3$. Then $\chi(G) \leq \Delta(G)$.*

4 Degrees and the list chromatic number

In this section, we look at a striking dissimilarity between the chromatic and list chromatic numbers. The theorem below asserts that *every* graph G with minimum degree at least d must have a large list chromatic number - at least nearly $\frac{1}{2}\log d$. But for bipartite graphs of arbitrarily large minimum degree, the chromatic number stays at 2. Alon actually proved a slightly weaker result a few years previous:

Theorem 63 (Alon, 90s)**.** *Let G have minimum degree at least d. Then*

$$\chi_l(G) \geq \Omega\left(\frac{\log d}{\log\log d}\right)$$

Proof. This is Theorem 5.1 in [1]. □

Definition 64 (Transversal)**.** *Let S be a set and $\mathscr{F}$ be a collection of subsets of S. A subset $T \subseteq S$ is a transversal for $\mathscr{F}$ if its intersection with every element of $\mathscr{F}$ is nonempty.*

Suppose that list assignment Ł has a colouring φ on some subset A of vertices. Then the set $\varphi(A)$ is a transversal of the family $\mathcal{L}(A) := \{\mathcal{L}(v) \mid v \in A\}$.

Theorem 65 (Alon, 2000 [2])**.** *Let G have minimum degree at least d. Then*

$$\chi_l(G) \geq \left(\frac{1}{2} - o_d(1)\right)\log_2 d$$

where $o_d(1) \xrightarrow{d\to\infty} 0$.

Proof. [The big idea] Suppose $\mathbb{C} = [L]$. We would like to exhibit a list assignment $\mathcal{L}$ with $|\mathcal{L}(v)| = s$ for each v that is *not* choosable. The idea is to (probabilistically) construct a small subset $B \subseteq V(G)$ and a (disjoint from B) large subset GOOD satisfying: when lists for B are chosen uniformly at random, then with positive probability, for every $v \in$ GOOD and every $T \subseteq C$ of size $L/2$ there is a neighbour $b \in B$ of v with $\mathcal{L}(b) \subseteq T$. Suppose that B is coloured by φ. For $a \in$ GOOD, the transversal $\varphi(N_B(a))$ intersects every $L/2$-subset of $[L]$, which means it cannot have size less than $L/2$ (otherwise it would miss an $L/2$-sized subset contained in its complement in $[L]$). This essentially means that $L/2$ colours are ruled out for a: a can only be coloured if its s-sized list had a nonempty intersection with the complement of the transversal - which (we will show) has probability at most $1 - 1/2^{s+1}$. Since GOOD is much larger than 2^{s+1} we can make the probability of successfully extending φ to GOOD $< 1/s^{|B|}$, which means that when lists are assigned to vertices of GOOD uniformly at random, the probability that there is a colouring of B that extends to GOOD is < 1, which means that some list assignment for G exists where this is not possible, establishing $\chi_l(G) > s$.

[The details] We choose the vertices of B randomly: pick $v \in V$ independently with probability p. For small enough p, this will give us a small set with high probability, which is what we are looking for. Having done so, we assign s-sized list assignments to the vertices of B uniformly at random from $\mathbb{C}$. Call a vertex *good* if it is not in B and satisfies that for every subset $T \subseteq C$ of size $L/2$, there is a neighbour $b \in B$ of v with $L(b) \subseteq T$. These *good* vertices will form our set GOOD. Fix a vertex $v \in V$.

$$\begin{aligned}
\mathop{\mathbb{P}}_{\substack{B\sim G(n,p)\\ \mathcal{L}(u\in B)\sim\mathcal{U}(\binom{[L]}{s})}}[v \in V \text{ is not good}] &= p + (1-p)\,\mathbb{P}\left[v \in V \text{is not good} \,|\, v \notin B\right] \\
&= p + (1-p)\,\mathbb{P}\left[\exists T \subseteq \binom{[L]}{L/2} : \forall b \in N(v) : b \notin B \text{ or } L(b) \not\subseteq T\right] \\
&\le p + (1-p)\binom{L}{L/2}\left(1 - p\frac{\binom{L/2}{s}}{\binom{L}{s}}\right)^d \\
&\le p + \frac{2^L}{4}\left(1 - p\frac{\binom{L/2}{s}}{\binom{L}{s}}\right)^d
\end{aligned}$$

where the last step is true for $L \ge 9$ (which will correspond to $s \ge 3$). Now

$$\frac{\binom{L/2}{s}}{\binom{L}{s}} = \frac{1}{2^s}\prod_{i=1}^{s-1}\left(1 - \frac{i}{L-i}\right) \ge \frac{1}{2^s}\prod_{i=1}^{s-1}\left(1 - \frac{i}{L-s}\right) \ge \frac{1}{2^s}\left(1 - \frac{\sum_{i=1}^{s-1} i}{L-s}\right) \ge \frac{1}{2^{s+1}}$$

where we choose $L = s^2$ in the last step. This means

$$\mathbb{P}\left[v \text{ is not good}\right] \le p + \frac{1}{4}\exp\left(s^2 \log 2 - \frac{pd}{2^{s+1}}\right)$$

We would like for the $\exp(\cdot)$ term to be $< 1/2$ or $pd > 2^{s+1}(s^2+1)\log 2$. We also set $p = 1/\sqrt{d}$ (the reason for this being that later we will have another constraint that looks like $1/p > 2^s$, to satisfy both in one go we set $pd = 1/p$ or $p = 1/\sqrt{d}$). Thus for $d > 2^{2s+2}(s^2+1)^2\log^2(2)$, $\mathbb{P}[v \text{ is not good}] < 1/8 + 1/8 = 1/4$. Thus, the expected size of GOOD is $\mathbb{E}\left[\sum_{v\in V} \mathbb{1}_{\{v \text{ is good}\}}\right] \geq 3n/4$. We have by Markov's inequality

$$\mathbb{P}[|\text{GOOD}| \leq n/2] = \mathbb{P}[|!\text{GOOD}| > n/2] \leq \frac{\mathbb{E}\,|!\text{GOOD}|}{n/2} < 1/2$$

and similarly

$$\mathbb{P}[|B| \geq 2np] \leq \frac{\mathbb{E}\,|B|}{2np} = \frac{np}{2np} = 1/2$$

so with positive probability, $|\text{GOOD}| > n/2$ and $|B| \leq 2np$ (the probability of this is $> 1/2+1/2-1 = 0$). We now *condition* on this event (it has non-zero probability), equivalently we fix a B and a list assignment for B satisfying the event. There are at most $s^{|B|}$ list colourings φ on B. Assign lists to the elements of GOOD randomly, and fix a colouring φ on B. Also fix $a \in A$. We have seen that φ can extend to a if and only if the list assigned to a is not contained in $\varphi(N_B(a))$ which is very large - it has size at least $L/2$. The probability of φ extending to all of GOOD is then (since each $a \in A$ is assigned its list independently)

$$\mathbb{P}[\varphi \text{ extends to GOOD}] \leq \left(1 - \frac{\binom{L/2}{s}}{\binom{L}{s}}\right)^{|\text{GOOD}|} \leq \left(1 - \frac{1}{2^{s+1}}\right)^{\frac{n}{2}} \leq e^{-\frac{n}{2^{s+2}}}$$

The probability that some colouring of B will extend to GOOD is thus

$$\mathbb{P}[\text{some } \varphi \text{ on } B \text{ extends to GOOD}] \leq s^{|B|}e^{-\frac{n}{2^{s+2}}} = \exp\left(n\left(2p\log s - \frac{1}{2^{s+2}}\right)\right)$$

To finish, we need $2p\log s < 1/2^{s+2}$, or $\sqrt{d} = 1/p > 2^{s+3}\log s$ which is satisfied since $\sqrt{d} > 2^{s+1}(s^2+1)\log 2$. Thus there a list assignment to vertices in GOOD that no colouring of B can extend to (and so G is not choosable for this choice of lists), so $\chi_l(G) > s$.
Finally, $d > 2^{2s+2}(s^2+1)^2\log^2(2)$ implies that $s < (\frac{1}{2} - o_d(1))\log_2(d)$. Since any s satisfying this will do, we have

$$\chi_l(G) \geq \left(\frac{1}{2} - o_d(1)\right)\log_2 d$$

as required. $\square$

Graph Theory	2023
Vector 3 colorable graphs with large chromatic number	
Lecturer: Niranjan Balachandran	Scribe: Kunal Kundwani

In this writeup, we will show the existence of a family of graphs which admit a vector 3 coloring but have arbitrarily large chromatic number. In fact, we will show that certain Kneser graphs have this property!
Lets start with the definition of Kneser graphs.

Definition 66 (Kneser Graphs)**.** *The Kneser graph $K(n, k)$ is the graph whose vertex set is the set of all k sized subsets of $[n]$, and two subsets share an edge in it iff they are disjoint.*

We will first show that for certain (infinitely many) pairs (n, k), $K(n, k)$ is vector 3-colorable. For that, consider the following embedding f of sets $A \in \binom{[n]}{k}$ into S^n:

$$f(A)_i := \begin{cases} \frac{1}{\sqrt{n}} & \text{if } i \in A \\ \frac{-1}{\sqrt{n}} & \text{otherwise} \end{cases} \tag{16}$$

where $f(A)_i$ means the ith coordinate of $f(A)$. To make above embedding satisfy the vector 3 colorability condition, we would want:

$$\langle f(A), f(B)\rangle \leq \frac{-1}{2} \ \forall A, B \in \binom{[n]}{k} \text{ satisfying } A \cap B = \emptyset$$

It is easy to verify that if $A \cap B = \emptyset$ and A, B have size k, then $\langle f(A), f(B)\rangle = \frac{n-4k}{n}$. To make this fraction $\leq \frac{-1}{2}$, we would want $2n - 8k \leq -n \implies k \geq \frac{3n}{8}$. As $\frac{3n}{8} < \frac{n}{2}$, non trivial and arbitrarily large vector 3 colorable Kneser graphs exist.
Next, we will prove the following result concerning the chromatic number of $K(n, k)$:

Theorem 67. $\chi(K(n, k)) > n - 2k + 1$

The proof we will outline, is from [3].
To prove the above, we will need two important results.

Theorem 68 (Borsuk's Theorem [4])**.** *If S^t is the union of $t + 1$ sets which are open in S^t. Then one of these sets contains a pair of antipodal points.*

(x and y are said to be antipodal here iff $x = -y$)

Definition 69. *For any point $a \in S^t$, define $H(a) = \{x \in S^t : \langle x, a\rangle > 0\}$*

Theorem 70 (Gale's Theorem [6])**.** *For all $p, q \in \mathbb{N}$, there exists a set $V \subseteq S^p$ of size $2q + p$ such that $\forall a \in S^p, |H(a) \cap V| \geq q$.*

Now, back to the proof of theorem 2:

Proof. Let $t = n - 2k$. Assume to the contrary that $K(n,k)$ can indeed be colored with $t+1$ colors.Let the color set be $[t+1]$ and any one of such valid colorings be c (i.e. the set A is colored with the color $c(A)$).
Now applying Gale's theorem with $p = t, q = k$ we get a set V of size $t + 2k = n$ such that $\forall a \in S^t, |H(a) \cap V| \geq k$. Let $V = \{v_1, v_2, \ldots, v_n\}$.Now, for $i \in [t+1]$ define $U_i := \{x \in S^t : \exists A \in \binom{[n]}{k}$ such that $c(A) = i$ and $v_j \in H(x) \forall j \in A\}$. Less formally, we put x in U_i iff $H(x) \cap V$ has an i colored subset. (w.r.t. the correspondence $v_j \to j$).
Now, observe that U_i must be open in S^t for all $i \in [t+1]$ since:

$$U_i = \bigcup_{\substack{A \in \binom{[n]}{k} \\ c(A)=i}} \bigcap_{j \in A} H(v_j)$$

. Since, $H(x)$ is open for all $x \in S^t$. (not very hard to verify), the above just involves finitely many unions and intersections of sets open in S^t and thus U_i is indeed open in S^t. Now, as for every $x \in S^t$ $H(x)$ has atleast k elements of V, there will be some $A \in \binom{[n]}{k}$ wholly inside $H(x)$, and thus x will belong to $U_{c(A)}$. Hence, every $x \in S^t$ belongs to some U_i. Thus we have written S^t as a union of $t+1$ sets(the U_i's) open in S^t. Hence by Borsuk's theorem one of those sets will have an antipodal pair!. Thus for some $i \in [t+1]$ and some a, both a and $-a$ belong to U_i which means that both $H(a)$ and $H(-a)$ contain i colored subsets. Let one of those sets be A and B respectively. Now observe that $H(a) \cap H(-a) = \emptyset$ since $\langle x, a \rangle > 0 \iff \langle x, -a \rangle < 0$. This would imply that $A \cap B = \emptyset$ and so A, B must have an edge between them in $K(n,k)$. However c assigns the same color to both A and B and so, c can't be a valid coloring, which contradicts our starting assumption about c.

Hence we can't color $K(n,k)$ with less than $n - 2k + 2$ colors and so $\chi(K(n,k)) > n - 2k + 1$. $\square$

Now, if $n = 8d$ and $k = 3d$ then $k \geq \frac{3n}{8}$ and so $K(n,k)$ is vector 3 colorable and $\chi(K(n,k)) > 2d+1$ which can be made arbitrarily large by choosing d arbitrarily large, which concludes our objective!

Construction of an extremal case

Theorem 71. *There exist graphs that admit vector 3-vector coloring but with $\chi(G) > n^\delta$ for some fixed $\delta > 0$.*

For this we look at Johnson Graphs.

Definition 72. *Johnson Graphs $J(n,r,s)$ are defined as follows-*

- $V(J(n,r,s)) = \binom{[n]}{r}$
- $A, B \in \binom{[n]}{r}$ *are adjacent iff* $|A \cap B| = s$.

A special family is when s=0 called Kneser graphs denoted by $K(n,r)$. We assume $n \geq 2r+1$

Example 73. *Petersen Graph is $K(5,2)$.*

Theorem 74. *(Erdős-Ko-Rado) Suppose $n \geq 2r+1$ size of maximum intersecting family of r-sets in [n] is $\binom{n-1}{r-1}$. Further the maximal families are $STARS$ i.e contain a particular element.*

In Kneser Graphs edges are between non intersecting sets. In an independent set of Kneser graphs all sets are Intersecting. By Erdős-Ko-Rado $\alpha(G) = \binom{n-1}{r-1}$.

A natural representation of a member of J(r,n,s) is as follows with 1 for elements which belong to A and -1 which belong to A^c

$$A \leftrightarrow \frac{1}{\sqrt{n}}(1\ 1...1| - 1 - 1... - 1) = v_A$$

$$\begin{aligned}
\langle v_A, v_B \rangle &= \frac{1}{n}(|A \cap B| - |A \cap B^c| - |A^c \cap B| + |A^c \cap B^c|) \\
&= \frac{1}{n}(s - (r-s) - (r-s) + (n-2r+s)) \\
&= \frac{n-4r+4s}{n}
\end{aligned}$$

For graph to be vector 3-colorable

$$\begin{aligned}
\langle v_A, v_B \rangle &\leq -\frac{1}{3-1} \\
\frac{n-4r+4s}{n} &\leq -\frac{1}{2} \\
2n - 8r + 8s &\leq -n \\
3n &\leq 8r - 8s
\end{aligned}$$

We put n to be 8s and r to be 3s which satisfies the above inequality making our graph vector 3-colorable

Theorem 75. *If $\mathcal{F} \subseteq \binom{[n]}{r}$ such that for any $A \neq B \in \mathcal{F}$, $|A \cap B| > s$ then*

$$|\mathcal{F}| \leq \sum_{i=0}^{r-s-1} \binom{n}{i}$$

If we start with a graph G defined as $V(G) = \binom{[8s]}{4s}$ and A,B adjacent when $|A \cap B| \leq s$. Then G is also vector 3-colorable but by the above theorem on the family of maximal independent set $\mathcal{I}$ we get

$$\begin{aligned}\alpha(G) &\leq \sum_{i=0}^{4s-s-1} \binom{8s}{i} \\ &\leq 2^{H(3/8)8s}\end{aligned}$$

$$\begin{aligned}\chi(G) &\geq \frac{\binom{8s}{4s}}{\alpha(G)} \\ &\geq \Omega\left(\frac{2^{8s}/\sqrt{s}}{2^{H(3/8)8s}}\right) \\ &\geq \Omega\left(\frac{2^{s(8+3\log_2(\frac{3}{8})+5\log_2(\frac{5}{8}))}}{\sqrt{s}}\right) \\ &\geq \Omega(n^{\delta})\end{aligned}$$

where $\delta \leq 1 + \frac{3}{8}\log_2(\frac{3}{8}) + \frac{5}{8}\log_2(\frac{5}{8}) = 0.0455$ and $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$

Chromatic Number of Kneser graphs

Assuming $n \geq 2r+1$ One natural way for coloring is as follows consider
$\mathcal{F}_1 =$ all subsets containing 1
$\mathcal{F}_2 =$ all subsets containing 2 but not 1
$\mathcal{F}_3 =$ all subsets containing 3 but not 1 or 2

We keep going till we are left with only $\{n, n-1, .., n-2r-2\}$. We have used $n-2r-1$ colors and If we take any 2 of remaining set they intersect thus forming an independent set so 1 color suffices. So we get a coloring of n-2r+2 colors. Next conjecture states that we can't do better.

Conjecture 76. $\chi(K(n,r)) = n - 2r + 2$.

Theorem 77. *(Borsuk-Ulam) If $f : S^d \to \mathbb{R}^d$ is continuous then there exists $x_0 \in s^d$ such that*

$$f(x_0) = f(-x_0)$$

Alternate formulation of the above theorem is if $S^d = U_1 \cup U_2 \cup ... \cup U_{d+1}$ where each U_i is either closed or open for all $1 \leq i \leq d$ then there exist some $1 \leq i \leq d+1$ such that U_i contains antipodal pair.

Proof of Conjecture

Take $d = n - 2r$, pick n points on S^{d+1} in GENERAL position i.e. the equator has atmost $d+1$ points.These represent the set $\{1, 2, ..., n\}$. Assume $\chi(K(n,r)) \leq d+1$ Then

$$\binom{[n]}{r} = V_1 \bigsqcup V_2 \bigsqcup ... \bigsqcup V_{d+1}$$

Define sets $U_i \subseteq S^{d+1}$ as

$$U_i = \{x \in S^{d+1} | \text{The open hemisphere centered at x contains some r-subset of } V_i\}$$

U_i are open so

$$S^{d+1} = U_1 \cup U_2 \cup ... \cup U_{d+1} \cup C$$

By theorem 5, one them contains an antipodal pair, But U_i can't contain an antipodal pair. Therefore C contains an antipodal pair say $\{x_0, -x_0\}$. But a hemisphere with x_0 contains $\leq r-1$ points of r-sets, similarly hemisphere with $-x_0$ contains $\leq r-1$ points and equator contains $\leq d+1$ points since GENERAL position. This implies

$$n \leq (r-1) + (r-1) + (d+1) = d - 2r - 1 = n - 1$$

Contradiction! Therefore $\chi(K(n,r)) = n - 2r + 2$

Proof of Theorems used

Proof of Theorem 5

If $\mathcal{F} \subseteq \binom{[n]}{r}$ such that for any $A \neq B \in \mathcal{F}$, $|A \cap B| > s$ then

$$|\mathcal{F}| \leq \sum_{i=0}^{r-s-1} \binom{n}{i}$$

We will try to define a vector space V over some field $\mathcal{F}$ and associate the members of $\mathcal{F}$ as vectors in V such that those memebers in $\mathcal{F}$ are linearly independent. Take V to be multilinear polynomial in $[x_1, ..., x_n]$ with degree $\leq d = r - s - 1$ which implies $dim(V) = \sum_{i=0}^{d} \binom{n}{i}$. We will associate a polynomial f_A for each set $A \in \mathcal{F}$ If $A \neq B$, $|A \cap B| \in \{s+1, ..., r-1\}$ which is a size d set

$$f_A(x_1, ..., x_n) = \prod_{j=s+1}^{r-1} (\langle \tilde{x}, v_A \rangle - j)$$

where v_A is 0-1 characteristic vector of A and $\tilde{x} = (x_1, ..., x_n)$

$f_A(v_B) = 0$ if $B \neq A$ and $f_A(v_A \neq 0$ write $f_A = \sum_{0 \leq \alpha_i \leq d-1} C_{\tilde{\alpha}} x_1^{\alpha_1} ... x_n^{\alpha_n}$. Replace each $x_i^{\alpha_i}$ for $\alpha_i \geq 1$ with x_i .

Thus we get a multilinear map g_A such that for $\varepsilon_i \in \{0,1\}$

$$\begin{aligned}
&g_A(\varepsilon_1, ..., \varepsilon_n) = f_A(\varepsilon_1, ..., \varepsilon_n)\\
\Longrightarrow &g_A(v_B) = f_A(v_B) = 0 \text{ and } g_A(v_B) = f_A(v_B) \neq 0\\
\Longrightarrow &\{g_A\}_{A\in\mathcal{F}} \text{ are linearly independent}\\
\Longrightarrow &|\mathcal{F}| \leq \sum_{i=0}^{r-s-1} \binom{n}{i}
\end{aligned}$$

Proof of Alternate reformulation of Theorem 7

Lusternik-Schnirelmann Theorem version of Boruk-Ulam states that $S^d = U_1 \cup U_2 \cup ... \cup U_{d+1}$ where each U_i is either closed or open for all $1 \leq i \leq d$ then there exist some $1 \leq i \leq d+1$ such that U_i contains antipodal pair.

L-S version is simple application of B-U Theorem. Suppose no antipodal pair exists in any U_i $f : S^n \to \mathbb{R}^n$ given by

$$f(x) = (d(x, U_1), d(x, U_2)..., d(x, U_n))$$

where $d(x, A) = inf_{y\in A} d(x, y)$, f is continuous as distance to any set is continuous. By B-U Theorem $\exists x_0 \in S^n$ such that $f(x_0) = f(-x_0)$. By assumption $x_0, -x_0$ not in U_{n+1}. Either x_0 or $-x_0$ is in U_i for some $i \leq n$. WLOG $x_0 \in U_i$ We have $f(x_0)_i = 0 \Longrightarrow f(-x_0)_i = 0$. If U_i is closed. $-x_0 \in U_i$. Contradiction! Therefore U_i is open and $-x_0 \in \bar{U_i}$.

$U_i \cap (-U_i) = \varphi$ by assumption, therefore $U_i \subseteq S^n \setminus (-U_i)$ which is closed

$$\begin{aligned}
&\Longrightarrow \bar{U_i} \subseteq S^n \setminus (-U_i)\\
&\Longrightarrow -x_0 \in \bar{U_i} \subseteq S^n \setminus (-U_i)\\
&\Longrightarrow -x_0 \notin -U_i.
\end{aligned}$$

Contradiction ! Therefore $\exists$ U_i which contains an antipodal pair.

Graph Theory	2018
Vector colourings and the KMS algorithm	
Lecturer: Niranjan Balachandran	Scribe: Madhur Agrawal

In this part, we define the notion of vector k-colourings and describe an algorithm for finding a vector 3-colouring in 3 colourable graphs.

Definition 78. *Given $k \in \mathbb{N}, k > 1$ a graph $G = ([n], E)$ is said to be vector k-colourable if there exist unit vectors $v_1, v_2, ..., v_k \in \mathbb{R}^n$ such that $\langle v_i, v_j \rangle \leq \frac{-1}{k-1}$ if $i, j \in E$.*

We have a *strong* vector k-colouring if $\langle v_i, v_j \rangle = \frac{-1}{k-1}$

Observation 79. *If G is k-colourable, then it is (strong) vector k-colourable.*

Consider the vectors given by $v_i = \frac{e_i - \frac{1}{k}\sum_{k=1}^{n} e_k}{|e_i - \frac{1}{k}\sum_{k=1}^{n} e_k|}$ and assign vertex i the colour v_i

Finding the minimum k for which a graph is vector k-colourable is rewritable as an SDP (Semi definite programming) problem.

Prototype SDP

Given symmetric matrices $A_1, A_2, ..., A_m, C$, we seek to maximize

$$\begin{gathered} \text{MAX } C \odot X, \text{ subject to} \\ A_i \odot X = b_i \\ X \succeq 0 \end{gathered}$$

where $A_{nxn} \odot B_{nxn}$ is the inner product in $\mathbb{R}^{2n}$

Fact 80. *Under very minimal assumptions, SDP admits a polynomial time algorithm giving a solution upto any given accuracy $\epsilon > 0$*

Strong Vector Colouring as SDP

$$\begin{gathered} \text{Min t subject to} \\ Y_{ij} = -\frac{1}{t-1} \text{ if } i, j \in E \\ Y_{i,i} = 1 \\ Y = [Y_{i,j}] \succeq 0 \end{gathered}$$

Remark 81. *As seen in the next scribe, there exist graphs which are vector 3-colourable but with chromatic number $\geq n^{\delta}$ for some $\delta \geq 0$*

Fact 82. *Determining if $\chi(G) \leq K$ is NP-Complete if $K \geq 3$*

Fact 83. *Unless P=NP, it is NP-HARD to approximate $\chi(G)$ to within a factor of $n^{1-\epsilon}$*

Given a 3-colourable graph, what is the least k such that we have a polynomial time algorithm to find a k-colouring?
We present an idea of Wigderson (early 90's) that uses $O(\sqrt{n})$ colours

Idea: If G is 3-colourable, N(v) is bipartite for any v $\in$ V. Bipartite graphs are 2-colourable in polytime. We consider a new parameter Δ. As long as we have a vertex of degree $> \Delta$, we colour N(v) using 2 new, distinct colours and delete N(v). When the max degree drops to or below Δ, we colour greedily using $\Delta + 1$ colours.
We therefore use $\leq 2 * \frac{n}{\Delta} + \Delta + 1$ colours. When taking $\Delta = \sqrt{2n}$, we get that we use $\leq 2\sqrt{2n} + 1$ colours.

KARGER-MOTWANI-SUDAN(1998-99) gave us a randomized version of this algorithm to improve complexity to $\widetilde{O}(n^{\frac{1}{4}})$, where $\widetilde{O}(f(n)) \leq kf(n)(\log n)^{O(1)}$
Idea 1: Since G is 3-colourable, it is vector 3-colourable in polynomial time.
Idea 2: Pick a "zone" on the unit sphere, and let $I_0 = \{i | v_i \in zone\}$. Let I be a subset of I_0 consisting of isolated vertices in G(I_0). Clearly, I is independent in G.
Idea 3: We pick a random gaussian vector $X \sim N(0, I_n)$, and set $I_0 = \{i : \langle v_i, X\rangle \geq t\}$ for some parameter t which we fix later.

$$\mathbb{E}\,|I| = \mathbb{E}\,|I_0| - \mathbb{E}\,|I_0 \setminus I|$$

$$\mathbb{E}\,|I_0| = \sum_{i=1}^{n} \mathbb{P}(v_i \in I_0)$$

Note $v_i \in I_0$ iff $\langle v_i, X\rangle \geq t$. But, by the radial symmetry of X, $\langle v_i, X\rangle$ is also gaussian. In fact, it is $N(0, |v_i|^2) = N(0,1)$. So, if we write the tail probability formula as

$$\Phi(t) = \int_t^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}\, dx$$

then

$$\mathbb{P}(v_i \in I_0) = \Phi(t)$$

hence

$$\mathbb{E}\,|I_0| = n\Phi(t)$$

$$\mathbb{P}(v_i \in I_0 \setminus I) = \mathbb{P}(v_i \in I, and \exists v_j \,|\, \{i,j\} \in E, v_j \in I_0)$$

Suppose G has max degree Δ. Then, we get the above probability

$$\mathbb{P}(v_i \in I_0 \setminus I) \leq \Delta \cdot \mathbb{P}(v_i, v_j \in I_0)$$

for some fixed arbitrary pair i,j.
Note, the product $\langle v_i, X\rangle \geq t$ creates a half plane with distance t away from the origin. We also note, as $\langle v_i, v_j\rangle \leq -\frac{1}{2}$, the angle between v_i and v_j is greater than or equal to 120 degrees. Hence, the acceptable range for X so that both v_i and v_j are in I can be contained within a half plane along their angle bisector with distance 2t from the origin. Hence, again by radial symmetry,

$$\mathbb{P}(v_i, v_j \in I_0) \leq \Phi(2t)$$

$$\mathbb{P}(v_i \in I_0 \setminus I) \leq \Delta \cdot \Phi(2t)$$

$$\mathbb{E}\,|I_0 \setminus I| \leq n\Delta \cdot \Phi(2t)$$

$$\mathbb{E}\,|I| \geq n(\Phi(t) - \Delta\Phi(2t))$$

We recall the standard facts

$$\Phi(t) \leq \frac{e^{\frac{-t^2}{2}}}{t\sqrt{2\pi}}$$

$$\Phi(t) \geq (\frac{1}{t} - \frac{1}{t^3})\frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}}$$

$$\implies \mathbb{E}\,|I| \geq \frac{n}{\sqrt{2\pi}}[(\frac{1}{t} - \frac{1}{t^3})e^{\frac{-t^2}{2}} - \frac{\Delta}{2t}e^{-2t^2}]$$

Pick $t = \sqrt{\frac{2}{3}\log(2\Delta)}$

$$\implies \frac{1}{2t}e^{\frac{-t^2}{2}} = \frac{\Delta}{t}e^{-2t^2}$$

As long as $\Delta \geq \frac{e^6}{2}$, we have $t \geq 2$ and $(\frac{1}{t} - \frac{1}{t^3}) \geq \frac{1}{2t}$

$$\implies \mathbb{E}\,|I| \geq \frac{n}{\sqrt{2\pi}}\frac{\Delta}{2t}e^{-4\log(2\Delta)/3}$$
$$= \widetilde{\Omega}(\frac{n}{\Delta^{1/3}\sqrt{\log(\Delta)}})$$

Hence, there exists a randomized algorithm to get an independent set of size $\geq \widetilde{\Omega}(\frac{n}{\Delta^{1/3}\sqrt{\log(\Delta)}})$ with high probability. Set $\Delta = n^{3/4}$. If $\Delta(G) \leq \Delta$, we perform this process to get an independent set of size $\widetilde{\Omega}(n^{\frac{3}{4}})$

The complete idea: Start with Wigderson's trick. 2-colour neighbourhoods of v and delete them until the max degree drops below $\Delta = n^{\frac{3}{4}}$. This uses $O(\frac{n}{\Delta}) = O(n^{\frac{1}{4}})$ colours. After that, we find independent sets and colour them with 1 colour each. This takes $O(\frac{n}{n/\Delta^{1/3}\sqrt{\log\Delta}}) = O(\Delta^{1/3}\sqrt{\log\Delta}) = O(n^{\frac{1}{4}}\log(n))$ colours.

Hence, we can colour the graph in $O(n^{\frac{1}{4}}\log(n))$ colours.

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