Persistence based convergence rate analysis of consensus protocols for dynamic graph networks[☆]

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Abstract

This article deals with the consensus problem of agents communicating via time-varying communication links in undirected graph networks. The highlight of the current work is to provide practically computable rates of convergence to consensus that hold for a large class of time-varying edge weights. A novel analysis technique based on classical notions of persistence of excitation and uniform complete observability is proposed. The new analysis technique for consensus laws under time-varying graphs provides explicit bounds on rate of convergence to consensus for single integrator dynamics. In the case of double integrators a minor modification to the standard relative state feedback law is shown to guarantee exponential convergence to consensus under similar assumptions as the single integrator case. The consensus problem is re-formulated in the edge agreement framework to which persistence of excitation based results apply. A novel application of results from Ramsey theory allows for proof of consensus and convergence rate estimation under switching graph topology. The current work connects classical results from nonlinear adaptive control and combina-

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torics to the modern theory of consensus over multi-agent networks.

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1. Introduction

Convergence analysis of consensus algorithms is an important research problem and has received attention in recent times. Classically, proof of consensus in directed and undirected switching graph networks has been accomplished by utilizing notions of *Stochastic Indecomposable Aperiodic* (SIA) matrices (see for example Ren and Beard [1]). The aforementioned approach however, did not yield an estimate of the convergence rate in switching graphs. For static graphs, the convergence rate is well established and known to depend on the second smallest eigenvalue of the graph Laplacian [2, Chapter 2]. For discrete dynamics with time-varying graphs Cao et al. [3] were able to utilize notions of SIA matrices to propose results on the convergence rate to consensus. They however discuss graph composition rather than the more practical graph union to arrive at the convergence result. Further, an assumption on existence of self-arc at each vertex is made implying that each agent knows its own state.

More recently Martin and Girard [4] employed a novel *persistent connectivity* and *cut-balance interaction* assumption to prove consensus and evaluate the convergence rate for single integrator dynamics. However, computation of a bound on convergence rate based on the aforementioned result requires explicit function form of the edge weights ($a_{ij}(t)$) or their value at each time instant. Further, Shi and Johansson [5] have proposed a continuous and a discrete time update law to achieve consensus in single integrator agent dynamics communicating via directed persistent graphs while also estimating convergence rates to consensus. Their result however, assumes a stringent *arc-balance* [5, Sec. III] assumption and the rate depends on *cycle edges* i.e., arcs which are not persistent. Both [4] and [5] utilize the notion of persistent graphs to arrive at their results. In the context of double integrator dynamics, Zhu et al. [6] illustrate a generalized feedback law based on the local position and velocity along with distributed relative information to achieve consensus in linear, periodic and positive exponential second order agent dynamics. Additionally, they infer that in the periodic consensus problem for undirected graphs, convergence rate depends on the largest and the second smallest eigenvalues of the graph laplacian. The feedback law is however, not practicable in cases where adequate localization sensors are not available.

The current article presents results on convergence rates in consensus by considering multi-agent systems to be copies of linear systems with time-varying control gain of the form [7],

$$\dot{x} = g(t)u \tag{1}$$

The time-varying scalar gain $g(t)$ is assumed to satisfy the *persistence of excitation* condition [8, p. 72] which implies that although the signal may pass through singular phases over several instants of time, there exists a window of time T_{per}, over which, the signal is active. Stability of the aforementioned dynamics has been well documented in [9], [10]. Morgan and Narendra [9] establish a novel adaptive update law $u = -g(t)x$ and also propose that, persistence of excitation of $g(t)$ is both necessary and sufficient to ensure exponential stability for the above dynamics. Notable contributions in stabilizing control design for above dynamics with drift, i.e. $\dot{x} = Ax + g(t)u$ can be attributed to authors of [10] and [11]. However, the aforementioned references do not address the stability in the context of multi-agent systems. In a multi-agent framework, the time-varying scaling $g(t)$ represents the on-off nature of inter-agent communication (edge weights). The central objective of this article is to provide persistence based analysis of the consensus protocols for single and double integrator agent dynamics and establish a bound on convergence rates for the same. Diverse inter-agent communication topologies are considered using $g_i(t)$ as the time-varying weight associated with edge e_i.

A preliminary investigation of the persistence based consensus analysis technique has been carried out by the authors in [12] for single integrator agent dynamics. The analysis is based on the edge agreement framework for undirected graphs [13] and utilizes notions of *persistence of excitation* (PE) and *Uniform Complete Observability* (UCO) [8], to obtain an explicit convergence rate estimate for consensus. The results in [12] are however restrictive in their practical applicability due to the assumption of repetition of the same spanning tree in the union graph. The current work significantly extends [12] in two ways, (a) the convergence rate bound is estimated for the more practical case of the union graph containing different spanning trees by employing results from *Combinatorics*, specifically *Ramsey theory* [14], (b) a convergence rate bound is also computed for the double integrator consensus dynamics that represent the structure of most mechanical systems.

The consensus law used in the analysis of double integrator dynamics is inspired by [15] where the authors suggest a local velocity feedback in addition to relative information for achieving consensus in switching digraphs.

The article unfolds as follows. In section 2 an overview of graph theory and Ramsey theory with some fundamental ideas regarding Persistence of Excitation (PE) and stability theory are covered. The main results are proposed in section 3 with supporting proofs. Section 5 presents the simulation results for two different test problems. The conclusions of this work are summarized in section 6.

1.1. Nomenclature

Throughout the article $|\cdot|$ operator denotes the absolute value of a scalar argument; $\|\cdot\|$ denotes the Euclidean norm for vectors and matrices; $\mathbb{Z}^+$ denotes the set of positive integers excluding zero; for a symmetric matrix, $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimum and the maximum eigenvalue of M; $M \geq N$ (respectively $M > N$) for symmetric matrices M and N implies that $M-N$ is positive semi-definite (respectively positive definite); $tr(M)$ denotes the trace of M; I_n is the identity matrix of dimension n; $B_h(x)$ denotes a closed ball of radius $h > 0$ centered at $x \in \mathbb{R}^n$ and defined as, $B_h(x) = \{y \in \mathbb{R}^n : \|y - x\| \leq h\}$. An arithmetic progression is defined as a sequence of $n \in \mathbb{N}$ numbers, $a_k =$

$\{a_0 + kd\}_{k=0}^{n-1}$ such that the differences between successive terms is a constant $d \in \mathbb{N}$. Duty cycle ($D = \frac{T_{on}}{T_{tot}}$) of a periodic signal is specified as the percentage of one time period over which a signal is active, where, T_{on} is the time duration of signal activity and T_{tot} is the total time period. The notation square(T) refers to a square wave of time-period T with an associated duty cycle.

2. Mathematical Preliminaries

2.1. Algebraic graph theory fundamentals

In this section, some preliminary ideas of algebric graph theory are introduced. An elaborate discussion of the same is available in [16]. An undirected graph ($\mathcal{G}$) is a pair (V, E) where, $V = \{v_1, v_2, \cdots, v_n\}$ is a finite non empty node set and $E = \{e_1, e_2, \ldots, e_m\}$ is an edge set. $\{v_i, v_j\} \in E$ is an undirected edge if agents v_i and v_j exchange information with each other. If there exists an edge between vertices v_i and v_j then, we call them adjacent and denote it by $v_i \sim v_j$. In this case, edge $\{v_i, v_j\}$ is called incident with vertices v_i and v_j. A path of length p in graph $\mathcal{G}$ is given by a sequence of distinct vertices $v_{i_0}, v_{i_1}, \cdots, v_{i_p}$ such that for $k = \{0, 1, \cdots (p-1)\}$ the vertices v_{i_k} and $v_{i_{k+1}}$ are adjacent. In this case, v_{i_o} and v_{i_p} are called end vertices of the path and $v_{i_1}, \cdots, v_{i_{p-1}}$ are called the inner vertices. When the vertices of the path are distinct except for its end vertices, the path is called a simple cycle. A graph is connected, if for every pair of vertices in $V(\mathcal{G})$, there is a path that has them as its end vertices. Any graph $\tilde{\mathcal{G}} = \left(\tilde{V}, \tilde{E}\right)$ is a subgraph of $\mathcal{G} = (V, E)$ if $\tilde{V} \subseteq V$ and $\tilde{E} \subseteq E$. A tree is defined as a connected graph without any cycle. If for a subgraph $V = \tilde{V}$, then it is referred to as a spanning subgraph. A spanning tree for a graph $\mathcal{G}$ is thus a spanning subgraph of $\mathcal{G}$ that is also a tree. The incidence matrix $D(\mathcal{G}^{\mathcal{O}}) \in \mathbb{R}^{n \times m}$ of an undirected graph $\mathcal{G}$ with arbitrary orientation $\mathcal{O}$ is defined as,

$$D(\mathcal{G}^{\mathcal{O}}) = [d_{ij}]$$

where,

$[d_{ij}] = -1$ if, v_i is the tail of e_j

$[d_{ij}] = 1$ if, v_i is the head of e_j

$[d_{ij}] = 0$ otherwise

The graph laplacian matrix $L(\mathcal{G}^{\mathcal{O}}) \in \mathbb{R}^{n \times n}$ of an arbitrarily oriented graph $\mathcal{G}^{\mathcal{O}}$ is defined as,

$$L(\mathcal{G}^{\mathcal{O}}) = D(\mathcal{G}^{\mathcal{O}})D(\mathcal{G}^{\mathcal{O}})^T \tag{2}$$

For a weighted graph the graph laplacian matrix is redefined as,

$$L(\mathcal{G}^{\mathcal{O}}) = D(\mathcal{G}^{\mathcal{O}})WD(\mathcal{G}^{\mathcal{O}})^T \tag{3}$$

where, $W \in \mathbb{R}^{m \times m}$ is the diagonal matrix with the weights $w(e_i),\ i = \{1, 2, ..., m\}$ on the diagonal entry. The symmetric graph laplacian matrix is invariant with respect to choice of orientation. The edge laplacian $L_e(\mathcal{G}^{\mathcal{O}}) \in \mathbb{R}^{m \times m}$ matrix is defined as,

$$L_e(\mathcal{G}^{\mathcal{O}}) = D(\mathcal{G}^{\mathcal{O}})^T D(\mathcal{G}^{\mathcal{O}}) \tag{4}$$

For a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ the weighted edge laplacian matrix is defined as [17], $L_e(\mathcal{G}^{\mathcal{O}}) := W^{\frac{1}{2}} D^{\intercal}(\mathcal{G}^{\mathcal{O}})D(\mathcal{G}^{\mathcal{O}})W^{\frac{1}{2}}$ and is also orientation invariant. Since, the current article deals only with undirected graphs and all choices of orientation lead to the same Laplacian, the more cumbersome $D(\mathcal{G}^{\mathcal{O}})$ notation will be exchanged for $D(\mathcal{G})$. The graph laplacian matrix is positive semi-definite with eigenvalues ordered as,

$$0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_n(\mathcal{G})$$

The graph $\mathcal{G}$ is connected, if and only if $\lambda_2(\mathcal{G}) > 0$. For an undirected graph $\lambda_2(\mathcal{G})$ is defined as the algebraic connectivity and determines the convergence rate of the time invariant consensus algorithm.

Remark 1. *In the text that follows, it will invariably be assumed without loss of generality, that in representing the laplacian corresponding to a* time-varying *graph* ($L(\tilde{\mathcal{G}}(t)) = D(\mathcal{G})W(t)D(\mathcal{G})^T$ *), the incidence matrix* $D(\mathcal{G})$ *in (3) is constant while the edge weight matrix* $W(t)$ *captures the time-varying nature. The* *graph ($\mathcal{G}$) corresponding to the laplacian $D(\mathcal{G})D(\mathcal{G})^T$ (i.e. $W(t) = I_m$) will be referred to frequently as the* underlying *graph. $W(t)$ is assumed to be at least piecewise continuous. Further, in section 3.3 a* switching *graph topology is simulated by designing $D(\mathcal{G})$ to represent the incidence matrix of a complete underlying graph corresponding and $W(t)$ are the respective edge weights that serve to turn the edge on or off at time instant t. Therefore, starting from a complete underlying graph any general communication graph can be obtained by appropriate assignment of edge weights, $W(t)$. However, theorems 5 and 6 hold for all connected underlying graphs, $\mathcal{G}$ including a complete graph.*

2.2. Fundamental results

In this section, we introduce fundamental notions and results from system theory and combinatorics to be used later. The following is an important result from *Ramsey theory* [14, p. 29] namely *Van der Waerden's theorem.*

Theorem 1. *[14, p. 29] For any two given positive integers r and k there is some natural number N such that, if the integers $[1, N] = \{1, 2, \cdots, N\}$ are colored, each with one of r different colors, then there are at least k integers in arithmetic progression of all the same color.*

The current best known upper bound on least possible N, given (r, k) (called Van Der Waerden's number $S(r, k)$) is [18],

$$S(r, k) \leq 2^{2^{r^{2^{2^{k+9}}}}}$$

The following two definitions and theorem introduce persistence of excitation of signals and uniform complete observability under output injection.

Definition 1. *[8, p. 72] The signal $g(\cdot) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{n \times m}$ is* Persistently Exciting *(PE) if there exist finite positive constants μ_1, μ_2, T_{per} such that,*

$$\mu_2 I_n \geq \int_t^{t+T_{per}} g(\tau)g(\tau)^T d\tau \geq \mu_1 I_n \qquad \forall t \geq t_0 \tag{5}$$

The notion of persistence of excitation is common in systems identification. Although the mathematical formulation is identical, in the current context it will be used to determine union graph connectivity. We also define a notion called *finite-time* persistence of excitation. *Finite-time* persistence of excitation requires the existence of $t_f > t_0 + T_{per} > 0$ such that condition (5) holds for all $t \in [t_0, t_f - T_{per}]$.

Definition 2. *[8, p. 35] Consider the linear time-varying system* $[C(t), A(t)]$ *defined by,*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \qquad x(t_0) = x_0 \\ y(t) &= C(t)x(t) \end{aligned} \tag{6}$$

where, $x(t) \in \mathbb{R}^n$*,* $y(t) \in \mathbb{R}^m$ *and* $A(t) \in \mathbb{R}^{n \times n}$*,* $C(t) \in \mathbb{R}^{m \times n}$ *are piecewise continuous functions of time.The system defined in equation (6) is called* uniformly completely observable *(UCO) if there exist finite and strictly positive scalar constants* β_1, β_2, δ *such that, for all* $t_0 \geq 0$

$$\beta_2 I_n \geq \int_{t_0}^{t_0+\delta} \Phi_A^T(\tau, t_0) C^T(\tau) C(\tau) \Phi_A(\tau, t_0) d\tau \geq \beta_1 I_n$$

Here, $\Phi_A(\tau, t_0) \in \mathbb{R}^{n \times n}$ is the state transition matrix corresponding to $A(t)$ starting at t_0.

Theorem 2. *[8, pp. 73-74] Consider the linear time-varying system* $[C(t), A(t)]$ *as follows,*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \qquad x(0) = x_0 \\ y(t) &= C(t)x(t) \end{aligned}$$

where, $x(t), y(t), A(t)$ *and* $C(t)$ *are as per Definition 2 and* $[C(t), A(t)+K(t)C(t)]$ *is the system with output feedback given by,*

$$\begin{aligned} \dot{x}(t) &= (A(t) + K(t)C(t))\, x(t) \\ y(t) &= C(t)x(t) \end{aligned}$$

In the above, $K(t) \in \mathbb{R}^{n \times m}$ *is the time-varying output feedback gain. Now, assume that, for all* $\delta > 0$*, there exists* $K_\delta \geq 0$ *such that for all,* $t_0 \geq 0$*,*

$$\int_{t_0}^{t_0+\delta} \parallel K(\tau) \parallel^2 d\tau \leq K_\delta$$

Then, the system $[C(t), A(t)]$ is uniformly completely observable if and only if $[C(t), A(t) + K(t)C(t)]$ is uniformly completely observable. Moreover, if the observability gramian of the system $[C(t), A(t)]$ satisfies,

$$\beta_2 I_n \geq \int_{t_0}^{t_0+\delta} \Phi_A^T(\tau, t_0) C^T(\tau) C(\tau) \Phi_A(\tau, t_0) d\tau \geq \beta_1 I_n$$

for all $t_0 \geq 0$ then the observability grammian of the system $[C(t), A(t) + K(t)C(t)]$ also satisfies the above mentioned inequalities with identical choice of δ and,

$$\tilde{\beta}_1 = \frac{\beta_1}{\left(1 + \sqrt{K_\delta \beta_2}\right)^2}$$

$$\tilde{\beta}_2 = \beta_2 e^{(K_\delta \beta_2)}$$

The following results state the standard exponential stability theorem in the sense of Lyapunov and its converse for linear systems.

Theorem 3. *[8, pp. 31-32] Let, $B_h(0)$ be a closed ball of radius h centered at $0 \in \mathbb{R}^n$. If there exists a function $V(t, x) : \mathbb{R}^{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ and strictly positive scalar constants $\alpha_1, \alpha_2, \alpha_3, \delta$ such that for all $x \in B_h(0)$, $t \geq 0$*

$$\alpha_1 \|x(t)\|^2 \leq V(t, x) \leq \alpha_2 \|x(t)\|^2$$

$$\frac{dV(t, x(t))}{dt} \leq 0$$

$$\int_t^{t+\delta} \frac{dV(\tau, x(\tau))}{d\tau} d\tau \leq -\alpha_3 \|x(t)\|^2$$

then, $x(t) \in \mathbb{R}^n$ converges exponentially to $0 \in \mathbb{R}^n$. Further, $V(t, x)$ evolves according to,

$$V(t, x(t)) \leq m_v e^{-\alpha_v (t - t_0)} V(t_0, (x(t_0))) \qquad t \geq t_0 \geq 0$$

where,

$$m_v = \frac{1}{\left(1 - \frac{\alpha_3}{\alpha_2}\right)}$$

$$\alpha_v = \frac{1}{\delta} \ln \frac{1}{\left(1 - \frac{\alpha_3}{\alpha_2}\right)}$$

Thus, $0 \in \mathbb{R}^n$ is uniformly exponentially stable.

Theorem 4. *[19, p. 120] Consider the linear time-varying system defined by,*

$$\dot{x}(t) = A(t)x(t) \qquad x(t_0) = x_0 \tag{7}$$

where, $x(t) \in \mathbb{R}^n$ and $A(t) \in \mathbb{R}^{n\times n}$. Suppose that, the linear state equation (7) is uniformly exponentially stable*, and there exists a finite and strictly positive scalar constant β such that, $\|A(t)\| \le \beta$ for all $t \ge t_0$. Then, there exists a matrix function $P(t) \in \mathbb{R}^{n\times n}$ that for all t is symmetric, continuously differentiable and,*

$$P(t) = \int_t^{\infty} \Phi_A^T(\tau, t)\, \Phi_A(\tau, t)\, d\tau \tag{8}$$

Further, the $P(t) \in \mathbb{R}^{n\times n}$ as stated satisfies,

$$\begin{aligned} \eta I_n \le P(t) &\le \rho I_n \\ A^T(t)P(t) + P(t)A(t) + \dot{P}(t) &\le -I_n \end{aligned}$$

In the above theorem, $\Phi_A(\tau, t) \in \mathbb{R}^{n\times n}$ is the state transition matrix for $A(t)$, I_n is the identity matrix of dimension n and η, ρ are finite and strictly positive scalar constants.

3. Analysis of time-varying Consensus Protocols

3.1. Consensus control for single integrator dynamics

Consider n identical single integrator agent dynamics as [2, p. 25],

$$\dot{x}_i = u_i \quad i = 1, 2, \cdots, n \tag{9}$$

with a feedback law of the following form,

$$u_i = -k \sum_{j=1}^{n} a_{ij}(t)\,(x_i - x_j) \qquad a_{ij}(t) \ge 0 \tag{10}$$

Combining the control laws, $u(t) = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^T \in \mathbb{R}^n$ defined in (10) for individual agents, the closed loop agent dynamics $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^T \in \mathbb{R}^n$ comes out as,

$$\begin{aligned} \dot{x}(t) &= -kL(\tilde{\mathcal{G}}(t))x \\ &= -kD(\mathcal{G})W(t)D(\mathcal{G})^T x \end{aligned} \tag{11}$$

where, $k > 0$, $D(\mathcal{G}) \in \mathbb{R}^{n\times m}$, is the time-invariant incidence matrix for the underlying graph $\mathcal{G}$ and $W(t) = \text{diag}\,[g_1^2\, g_2^2\, \dots\, g_m^2]^T \in \mathbb{R}^{m\times m}$ represents the edge weights. $W(t)$ as defined codes the notion of time-varying and diverse inter-agent communication topology. It is assumed that the underlying graph $\mathcal{G}$ (i.e. for $W(t) = I_m$) is *connected*. $\mathcal{G}$ therefore contains a spanning tree and it is possible to partition $W(t) \in \mathbb{R}^{m\times m}$ as two block diagonal matrices, $W(t) = \text{diag}\begin{bmatrix} W_\tau(t) & W_c(t) \end{bmatrix}$ where, $W_\tau(t) \in \mathbb{R}^{p\times p}$ represents the weighting functions corresponding to the spanning tree edges whereas, $W_c(t) \in \mathbb{R}^{(m-p)\times(m-p)}$ represents the weights corresponding to the cycle edges. Here, p ($= n-1$ for undirected graphs) is the number of spanning tree edges. With this background we are now ready to state the first result in this paper.

Theorem 5. *Consider the closed-loop consensus dynamics (11). Assume that, the underlying graph $\mathcal{G}$ ($W(t) = I_m$) is connected. The states of the closed-loop dynamics $x(t)$ with time-varying communication topology characterized by $W(t)$, achieve consensus exponentially, if there exists a spanning tree with corresponding edge-weight matrix $W_\tau(t)$ persistently exciting. Further the convergence rate α_v to consensus is bounded below by,*

$$\boxed{\alpha_v \geq \frac{1}{2T_{per}} \ln \frac{1}{\left[1 - \frac{2k\lambda_{min}(\Lambda)\mu_1}{\left(1+k\sqrt{p}\|\Lambda\|\mu_2\right)^2}\right]}}$$

where, T_{per}, μ_1 and μ_2 are the constants appearing in Definition 1 and Λ is a diagonal matrix containing the eigenvalues of the spanning tree edge-laplacian matrix.

We begin with a transformation to the edge laplacian and the edge agreement protocol as defined in [13], [16] to prove the aforementioned theorem. Consider the following transformation to the edge states,

$$x_e = D(\mathcal{G})^T x \tag{12}$$

Differentiating (12) leads to the resulting expression,

$$
\begin{aligned}
\dot{x}_e &= D(\mathcal{G})^T \dot{x} \\
&= -kD(\mathcal{G})^T D(\mathcal{G}) W(t) D(\mathcal{G})^T x \\
&= -k\tilde{L}_e(\mathcal{G}) W(t) x_e
\end{aligned} \tag{13}
$$

In the above expression, $\tilde{L}_e(\mathcal{G}) \in \mathbb{R}^{m\times m}$ is used to represent the edge laplacian matrix of the underlying graph $\mathcal{G}$. The edge agreement problem pertains to stabilization instead of the consensus problem in node states. Since $\mathcal{G}$ is a *connected graph* it can be described as the union of two sub-graphs as $\mathcal{G}_\tau \cup \mathcal{G}_c$, where $\mathcal{G}_\tau$ represents the spanning tree and $\mathcal{G}_c$, the cycle edges. Using an appropriate permutation of the edge indices we can partition the incidence matrix of $\mathcal{G}$ as,

$$
D(\mathcal{G}) = \begin{bmatrix} D(\mathcal{G}_\tau) & D(\mathcal{G}_c) \end{bmatrix} \tag{14}
$$

It is useful to represent the edge-laplacian matrix, defined in equation (13), in terms of this new permutation,

$$
\begin{aligned}
\tilde{L}_e(\mathcal{G}) &= \begin{bmatrix} D(\mathcal{G}_\tau) & D(\mathcal{G}_c) \end{bmatrix}^T \begin{bmatrix} D(\mathcal{G}_\tau) & D(\mathcal{G}_c) \end{bmatrix} \\
&= \begin{bmatrix} D(\mathcal{G}_\tau)^T D(\mathcal{G}_\tau) & D(\mathcal{G}_\tau)^T D(\mathcal{G}_c) \\ D(\mathcal{G}_c)^T D(\mathcal{G}_\tau) & D(\mathcal{G}_c)^T D(\mathcal{G}_c) \end{bmatrix} \\
&= \begin{bmatrix} \tilde{L}_e(\mathcal{G}_\tau) & D(\mathcal{G}_\tau)^T D(\mathcal{G}_c) \\ D(\mathcal{G}_c)^T D(\mathcal{G}_\tau) & \tilde{L}_e(\mathcal{G}_c) \end{bmatrix}
\end{aligned} \tag{15}
$$

As mentioned earlier, $W(t) \in \mathbb{R}^{m\times m}$ can also partitioned into two block diagonal matrices,

$$
W(t) = \begin{bmatrix} W_\tau(t) & 0 \\ 0 & W_c(t) \end{bmatrix} \tag{16}
$$

The edge state vector can be identically partitioned as,

$$
x_e = \begin{bmatrix} x_\tau \\ x_c \end{bmatrix} \tag{17}
$$

The columns of the cycle edges $D(\mathcal{G}_c) \in \mathbb{R}^{n\times(m-p)}$ are linearly dependent on the columns of $D(\mathcal{G}_\tau) \in \mathbb{R}^{n\times p}$. This relationship can be expressed as follows,

$$D(\mathcal{G}_\tau)Z = D(\mathcal{G}_c) \tag{18}$$

where, the matrix $Z \in \mathbb{R}^{p\times(m-p)}$ is defined as,

$$Z = \left(D(\mathcal{G}_\tau)^T D(\mathcal{G}_\tau)\right)^{-1} D(\mathcal{G}_\tau)^T D(\mathcal{G}_c) \tag{19}$$

Substituting (15), (16) and (17) in (13) we have,

$$\dot{x}_e = -k\begin{bmatrix} \tilde{L}_e(\mathcal{G}_\tau) & D(\mathcal{G}_\tau)^T D(\mathcal{G}_c) \\ D(\mathcal{G}_c)^T D(\mathcal{G}_\tau) & \tilde{L}_e(\mathcal{G}_c) \end{bmatrix}\begin{bmatrix} W_\tau(t) & 0 \\ 0 & W_c(t) \end{bmatrix} x_e \tag{20}$$

Therefore, the states corresponding to the spanning tree edges ($x_\tau \in \mathbb{R}^p$) and the cycle edges ($x_c \in \mathbb{R}^{(m-p)}$) evolve according to,

$$\dot{x}_\tau = -k\tilde{L}_e(\mathcal{G}_\tau)W_\tau(t)x_\tau - kD(\mathcal{G}_\tau)^T D(\mathcal{G}_c)W_c(t)x_c \tag{21}$$

$$\dot{x}_c = -kD(\mathcal{G}_c)^T D(\mathcal{G}_\tau)W_\tau(t)x_\tau - k\tilde{L}_e(\mathcal{G}_c)W_c(t)x_c \tag{22}$$

The aforementioned transformation to edge dynamics precisely mimics references [13], [16] and presented here for reference. Our interest is in the behavior of the edges corresponding to the spanning tree since they represent the minimal edge subset that must go to zero for consensus to be achieved. The cycle edges can be reconstructed from the spanning tree edges as follows,

$$x_c(t) = Z^T x_\tau(t) \tag{23}$$

With the aforementioned transformation, (21) reduces to,

$$\begin{aligned}
\dot{x}_\tau &= -k\tilde{L}_e(\mathcal{G}_\tau)W_\tau(t)x_\tau - kD(\mathcal{G}_\tau)^T D(\mathcal{G}_c)W_c(t)x_c \\
&= -k\tilde{L}_e(\mathcal{G}_\tau)\left[W_\tau(t) + ZW_c(t)Z^T\right]x_\tau \\
&= -k\tilde{L}_e(\mathcal{G}_\tau)\begin{bmatrix} I_p & Z \end{bmatrix}\begin{bmatrix} W_\tau(t) & 0 \\ 0 & W_c(t) \end{bmatrix}\begin{bmatrix} I_p \\ Z^T \end{bmatrix} x_\tau \\
&= -k\tilde{L}_e(\mathcal{G}_\tau)RW(t)R^T x_\tau
\end{aligned} \tag{24}$$

where, $R = \begin{bmatrix} I_p & Z \end{bmatrix} \in \mathbb{R}^{p\times m}$. It is evident that $\tilde{L}_e(\mathcal{G}_\tau) \in \mathbb{R}^{p\times p}$ is symmetric and positive definite. Therefore, it can be decomposed as $\tilde{L}_e(\mathcal{G}_\tau) = \Gamma\Lambda\Gamma^T$, where $\Gamma \in \mathbb{R}^{p\times p}$ is an orthogonal matrix and $\Lambda \in \mathbb{R}^{p\times p}$ is a diagonal matrix with positive real entries. Substituting into (24) yields,

$$\dot{x}_\tau = -k\Gamma\Lambda\Gamma^T RW(t)R^T x_\tau \tag{25}$$

We can now introduce a set of modified states (say, $y \in \mathbb{R}^p$) defined by the similarity transformation, $y = \Gamma^T x_\tau$, with dynamics,

$$\dot{y} = \Gamma^T \dot{x}_\tau = -k\Lambda\Gamma^T RW(t)R^T\Gamma y \tag{26}$$

In order to prove of exponential convergence, we define a Lyapunov-like function $V(\cdot) : \mathbb{R}^p \to \mathbb{R}$,

$$V(y) = y^T\Lambda^{-1}y \tag{27}$$

The time derivative of $V(y)$ along closed-loop dynamics (26) can be written as,

$$\begin{aligned}
\dot{V}(y) &= y^T\Lambda^{-1}\dot{y} + \dot{y}^T\Lambda^{-1}y \\
&= y^T\Lambda^{-1}\left(-k\Lambda\Gamma^T RW(t)R^T\Gamma\right)y + y^T\left(-k\Gamma^T RW(t)R^T\Gamma\Lambda^T\right)\Lambda^{-1}y \\
&= -2ky^T\left(\Gamma^T RW(t)R^T\Gamma\right)y
\end{aligned} \tag{28}$$

Integrating both sides we get,

$$\begin{aligned}
\int_t^{t+T_{per}} \dot{V}(\sigma, y)\, d\sigma &= -2k\int_t^{t+T_{per}} y^T(\sigma)\Gamma^T RW(\sigma)R^T\Gamma y(\sigma)d\sigma \\
&= -2k\int_t^{t+T_{per}} y^T(\sigma)\left(\Gamma^T RW(\sigma)^{\frac{1}{2}}\right)\left(W(\sigma)^{\frac{1}{2}}R^T\Gamma\right)y(\sigma)d\sigma
\end{aligned} \tag{29}$$

Lemma 1. *Let $\Gamma \in \mathbb{R}^{p\times p}$ be an orthogonal matrix and $R = \begin{bmatrix} I_p & Z \end{bmatrix} \in \mathbb{R}^{p\times m}$. Then, $W_\tau(t)^{1/2}$ is Persistently exciting (PE) implies that $RW(t)^{1/2}$ is Persistently exciting (PE) which in turn implies $\Gamma^T RW(t)^{1/2}$ is Persistently exciting (PE).*

Proof. We assume that the persistent signal $W_\tau(t)$ has its smallest eigenvalue bounded below by $\mu_1 > 0$, and the persistence window is $T_{per} \in [0, \infty)$, $\forall t \geq t_0$ as per Definition 1. In order to prove the above, we begin by establishing persistence of $RW(t)^{1/2} \in \mathbb{R}^{m \times p}$.

$$\int_t^{t+T_{per}} RW(\sigma)R^T d\sigma = \int_t^{t+T_{per}} \left[W_\tau(\sigma) + ZW_c(\sigma)Z^T\right] d\sigma$$

Consider, any vector α such that $\|\alpha\| = 1$,

$$\begin{aligned}
&\alpha^T \left\{ \int_t^{t+T_{per}} \left[W_\tau(\sigma) + ZW_c(\sigma)Z^T\right] d\sigma \right\} \alpha \\
&= \alpha^T \text{diag}\left[\int_t^{t+T_{per}} g_1^2(\sigma) d\sigma, \ldots, \int_t^{t+T_{per}} g_p^2(\sigma) d\sigma \right] \alpha \\
&+ \left(Z^T\alpha\right)^T \text{diag}\left[\int_t^{t+T_{per}} g_{p+1}^2(\sigma) d\sigma, \ldots, \int_t^{t+T_{per}} g_m^2(\sigma) d\sigma \right] \left(Z^T\alpha\right) \\
&\geq \mu_1 \alpha^T \alpha = \mu_1
\end{aligned}$$

Similar arguments can be employed to prove that a similarity transform does not impact the persistence of excitation of a signal and therefore $\Gamma^T RW(t)^{1/2} \in \mathbb{R}^{p \times m}$ is also persistently exciting with identical T_{per} and $\mu_1 > 0$.

□

From the PE and UCO Definitions 1 and 2, it is evident that $\left[W(t)^{1/2}R^T\Gamma, 0\right]$ is UCO. Now define, $K(t) = -k\Lambda\Gamma^T RW(t)^{1/2}$. The integral of $K(t) \in \mathbb{R}^{p \times m}$ over a window of time, T_{per} can be evaluated as,

$$\begin{aligned}
\int_t^{t+T_{per}} \parallel K(\sigma) \parallel^2 d\sigma &= \int_t^{t+T_{per}} k^2 \parallel \Lambda\Gamma^T RW(\sigma)^{1/2} d\sigma \parallel^2 \\
&\leq k^2 \parallel \Lambda \parallel^2 \left[tr \int_t^{t+T_{per}} \Gamma^T RW(\sigma)R^T\Gamma d\sigma \right] \\
&\leq k^2 \parallel \Lambda \parallel^2 p\mu_2
\end{aligned}$$

Now, as per Theorem 2 the system $\left[W(t)^{1/2}R^T\Gamma, 0\right]$ is UCO if and only if $\left[W(t)^{1/2}R^T\Gamma, -k\Lambda(\Gamma^T RW(t)^{1/2})(W(t)^{1/2}R^T\Gamma)\right]$ is UCO. The observability grammian for this modified system with, $A(t) = -k\Lambda\Gamma^T RW(t)R^T\Gamma$ and $C(t) = W(t)^{1/2}R^T\Gamma$ is as follows for all, $t \geq 0$,

$$\tilde{\mu}_2 I_n \geq \int_t^{t+T_{per}} \Phi_A^T(\tau, t_0) C^T(\tau) C(\tau) \Phi_A(\tau, t_0) d\tau \geq \tilde{\mu}_1 I_n \tag{30}$$

where,

$$\tilde{\mu}_1 = \frac{\mu_1}{\left(1 + k\sqrt{p} \parallel \Lambda \parallel \mu_2\right)^2}$$
$$\tilde{\mu}_2 = \mu_2 e^{k^2 p \|\Lambda\|^2 \mu_2^2}$$

Therefore, for all $t \geq t_0$, the integral defined in equation (29) evaluates to,

$$\int_t^{t+T_{per}} \dot{V}(\sigma) d\sigma = -2ky^T(t) \left[\int_t^{t+T_{per}} \Phi_A^T(\sigma, t) C^T(\sigma) C(\sigma) \Phi_A(\sigma, t) d\sigma \right] y(t) \tag{31}$$

which by equation (30) evaluates to,

$$\int_t^{t+T_{per}} \dot{V}(\sigma) d\sigma \leq -\frac{2k\mu_1}{\left(1 + k\sqrt{p} \parallel \Lambda \parallel \mu_2\right)^2} \|y\|^2 \tag{32}$$

Now, comparing the result in (32) with the exponential stability Theorem 3, we have,

$$\frac{\alpha_3}{\alpha_2} = \frac{2k\lambda_{\min}(\Lambda)\mu_1}{\left(1 + k\sqrt{p} \parallel \Lambda \parallel \mu_2\right)^2}$$
$$m_v = \frac{1}{\left(1 - \frac{\alpha_3}{\alpha_2}\right)} = \frac{1}{\left[1 - \frac{2k\lambda_{\min}(\Lambda)\mu_1}{\left(1+k\sqrt{p}\|\Lambda\|\mu_2\right)^2}\right]}$$
$$2\alpha_v = \frac{1}{\delta} \ln \frac{1}{\left(1 - \frac{\alpha_3}{\alpha_2}\right)} = \frac{1}{T_{per}} \ln \frac{1}{\left[1 - \frac{2k\lambda_{\min}(\Lambda)\mu_1}{\left(1+k\sqrt{p}\|\Lambda\|\mu_2\right)^2}\right]}$$

and the explicit solution can be computed as $\forall t \geq t_0$,

$$V(y(t)) \leq m_v e^{-2\alpha_v(t-t_0)} V(y_0)$$
$$\Rightarrow \parallel y(t) \parallel \leq \left\{ \sqrt{\frac{\lambda_{\max}(\Lambda) m_v}{\lambda_{\min}(\Lambda)}} \right\} e^{-\alpha_v(t-t_0)} \parallel y(t_0) \parallel \tag{33}$$

The bound on convergence rate follows,

$$\boxed{\alpha_v \geq \frac{1}{2T_{per}} \ln \frac{1}{\left[1 - \frac{2k\lambda_{\min}(\Lambda)\mu_1}{\left(1+k\sqrt{p}\|\Lambda\|\mu_2\right)^2}\right]}} \tag{34}$$

$y(t)$ and $x_\tau(t)$ are related to each other via the similarity transformation, and therefore the spanning tree edges converge at an identical rate.

Note 1. *The decay rate expression (33) will continue to hold for an interval* $[t_0, t_f)$ *if* $W_\tau(t)$ *satisfies only a* finite-time persistence of excitation *condition. This makes it possible to compute the bound on convergence rate if all parameters in Eq. (34) are known only over a finite future time-horizon.*

Remark 2. *The connected* underlying *graph,* $\mathcal{G}$ *required for proof of Theorem 5 may have more than one spanning tree that is persistently excited. In such an event the convergence rate can be computed based on the 'fastest' spanning tree to get a tighter lower bound.*

A pertinent query regarding (34) is the possibility of influencing the bound on convergence rate solely by changing the value of gain $k \in \mathbb{R}^{>0}$. For a time invariant graph network, arbitrarily pushing up the value of k improves the rate of convergence. However, this does not hold true in the time-varying scenario where increasing k, increases the convergence rate to an extent, but the increments saturate for larger k values. The effect of the scalar gain is evident from (34) and will be illustrated through simulations later.

3.2. Convergence analysis for the second-order consensus protocol

We now consider the class of multi-agent systems with double integrator dynamics as follows [2, p. 78],

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= u_i \qquad i = 1, 2, \cdots, n \end{aligned} \tag{35}$$

and feedback,

$$u_i = u_{i_1} + u_{i_2} \tag{36}$$

where,

$$u_{i_1} = -\kappa y_i(t) \tag{37}$$

is the local velocity feedback, with feedback gain $\kappa > 0$ and,

$$u_{i_2} = -\alpha \sum_{j=1}^{n} a_{ij}(t) \left(x_i(t) - x_j(t)\right) - \gamma \sum_{j=1}^{n} a_{ij}(t) \left(y_i(t) - y_j(t)\right) \tag{38}$$

where, $a_{ij}(t) \geq 0$ are the elements of the adjacency matrix. By combining the control laws, $u(t) = \begin{bmatrix} u_{1_1} & u_{2_1} & \cdots & u_{n_1} \end{bmatrix}^T + \begin{bmatrix} u_{1_2} & u_{2_2} & \cdots & u_{n_2} \end{bmatrix}^T \in \mathbb{R}^n$ defined in (37) and (38) for individual agents, the overall closed loop dynamics can be expressed similar to (11) as,

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -\kappa y - \alpha L(\tilde{\mathcal{G}}(t))x - \gamma L(\tilde{\mathcal{G}}(t))y \\
&= -\kappa y - \alpha D(\mathcal{G})W(t)D(\mathcal{G})^T x - \gamma D(\mathcal{G})W(t)D(\mathcal{G})^T y
\end{aligned} \tag{39}
$$

Here, $\alpha > 0$, $\gamma > 0$ are constant scalar gains. $D(\mathcal{G}) \in \mathbb{R}^{n\times m}$ and $W(t) \in \mathbb{R}^{m\times m}$ are as defined in the single integrator case. The following theorem outlines the extension of the single integrator convergence result to double integrator agent dynamics.

Theorem 6. *Consider the closed-loop consensus dynamics (39). Assume that, the underlying graph $\mathcal{G}$ ($W(t) = I_m$) is connected and edge weights are uniformly bounded. The states of the double integrator dynamics, $x(t)$ and $y(t)$ with a time-varying communication topology qualified by $W(t)$, achieve consensus exponentially, if the spanning tree weight matrix $W_\tau(t)$ is persistently exciting. Moreover, the convergence rate is lower bounded by,*

$$
\boxed{\xi \geq \min\left\{\kappa, \frac{1}{2\lambda_{\max}(P(t))}\right\} \quad \forall t \geq t_0}
$$

where κ and $P(t)$ are defined in Eq. (37) and (56) respectively.

Proof. We consider as in the single integrator case, a transformation to the edge state vectors as follows,

$$x_{e_1} = D^T(\mathcal{G})x \tag{40}$$

$$x_{e_2} = D^T(\mathcal{G})y \tag{41}$$

Differentiating (40), (41) results in the following edge dynamics,

$$\dot{x}_{e_1} = D^T(\mathcal{G})\dot{x} = x_{e_2} \tag{42}$$

$$\begin{aligned} \dot{x}_{e_2} &= D^T(\mathcal{G})\dot{y} \\ &= -\kappa D^T(\mathcal{G})y - \alpha D^T(\mathcal{G})D(\mathcal{G})W(t)D(\mathcal{G})^T x - \gamma D^T(\mathcal{G})D(\mathcal{G})W(t)D(\mathcal{G})^T y \\ &= -\kappa x_{e_2} - \alpha \tilde{L}_e(\mathcal{G})W(t)x_{e_1} - \gamma \tilde{L}_e(\mathcal{G})W(t)x_{e_2} \end{aligned} \tag{43}$$

where, $\tilde{L}_e(\mathcal{G}) \in \mathbb{R}^{m\times m}$ is the time-invariant edge laplacian matrix corresponding to the underlying graph $\mathcal{G}$. The consensus problem is reduced to the classical stabilization problem in edge states as before. The underlying graph $\mathcal{G}$, has been assumed to be *connected*. Therefore, it can be represented as the union of two sub-graphs $\mathcal{G}_\tau \cup \mathcal{G}_c$. The edge state vectors can be partitioned as,

$$x_{e_1} = \begin{bmatrix} x_{\tau_1} \\ x_{c_1} \end{bmatrix} \qquad x_{e_2} = \begin{bmatrix} x_{\tau_2} \\ x_{c_2} \end{bmatrix} \tag{44}$$

where, the notations $x_{\tau_1} \in \mathbb{R}^p$, $x_{\tau_2} \in \mathbb{R}^p$ are used to represent the edge states of the spanning tree corresponding to the position and velocity interaction topology associated with x_i and y_i. Similarly, $x_{c_1} \in \mathbb{R}^{(m-p)}, x_{c_2} \in \mathbb{R}^{(m-p)}$ denote the states symbolizing the cycle edges for the same. The columns of the cycle edges $D(\mathcal{G}_c) \in \mathbb{R}^{n\times(m-p)}$ are linearly dependent on the columns of $D(\mathcal{G}_\tau) \in \mathbb{R}^{n\times p}$ and the corresponding relationship has been previously noted in equations (18) and (19).

Expanding out (42) and (43) we have,

$$\dot{x}_{e_1} = x_{e_2} \tag{45}$$

$$\begin{aligned} \dot{x}_{e_2} &= -\kappa x_{e_2} \\ &\quad - \alpha \begin{bmatrix} \tilde{L}_e(\mathcal{G}_\tau) & D(\mathcal{G}_\tau)^T D(\mathcal{G}_c) \\ D(\mathcal{G}_c)^T D(\mathcal{G}_\tau) & \tilde{L}_e(\mathcal{G}_c) \end{bmatrix} W(t)x_{e_1} \\ &\quad - \gamma \begin{bmatrix} \tilde{L}_e(\mathcal{G}_\tau) & D(\mathcal{G}_\tau)^T D(\mathcal{G}_c) \\ D(\mathcal{G}_c)^T D(\mathcal{G}_\tau) & \tilde{L}_e(\mathcal{G}_c) \end{bmatrix} W(t)x_{e_2} \end{aligned} \tag{46}$$

As before we are interested only in the behavior of the spanning tree edges, since they represent the minimal edge subset that must go to zero for consensus to be achieved. Employing the following relation between spanning tree and cycle edge states,

$$\begin{aligned} x_{c_1}(t) &= Z^T x_{\tau_1}(t) \\ x_{c_2}(t) &= Z^T x_{\tau_2}(t) \end{aligned} \tag{47}$$

the dynamics of x_{τ_1} and x_{τ_2} can be written as,

$$\dot{x}_{\tau_1} = x_{\tau_2} \tag{48}$$

$$\dot{x}_{\tau_2} = -\kappa x_{\tau_2} - \alpha \tilde{L}_e(\mathcal{G}_\tau) R W(t) R^T x_{\tau_1} - \gamma \tilde{L}_e(\mathcal{G}_\tau) R W(t) R^T x_{\tau_2} \tag{49}$$

where, $R \in \mathbb{R}^{p\times(m-p)}$ is as defined for the single integrator dynamics. For a connected graph, $\tilde{L}_e(\mathcal{G}_\tau) \in \mathbb{R}^{p\times p}$ is symmetric and positive definite. Therefore, with the eigenvalue decomposition $\tilde{L}_e(\mathcal{G}_\tau) = \Gamma\Lambda\Gamma^T$ and modified states defined by the following similarity transformation,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \Gamma^T & 0 \\ 0 & \Gamma^T \end{bmatrix} \begin{bmatrix} x_{\tau_1} \\ x_{\tau_2} \end{bmatrix} \tag{50}$$

we obtain,

$$\dot{z}_1 = z_2 \tag{51}$$

$$\dot{z}_2 = -\kappa z_2 - \alpha \Lambda \Gamma^T R W(t) R^T \Gamma z_1 - \gamma \Lambda \Gamma^T R W(t) R^T \Gamma z_2 \tag{52}$$

We are now left to prove the exponential convergence of (51)-(52) instead of (48)-(49). For this purpose, we define a time-varying potential function $V(\cdot,\cdot) : \mathbb{R}^p \times \mathbb{R}^{\geq 0} \to \mathbb{R}$,

$$V(z,t) = (z_2 + \kappa z_1)^T P(t) (z_2 + \kappa z_1) \tag{53}$$

where, $P(t) = P^T(t) > 0$ will be defined later. The time derivative of $V(z,t)$ along closed-loop dynamics (51)-(52), can be written as,

$$\begin{aligned} \dot{V}(z,t) = &(z_2 + \kappa z_1)^T P(t) \left\{-\Lambda\Gamma^T R W(t) R^T \Gamma (\gamma z_2 + \alpha z_1) - \kappa z_2 + \kappa z_2\right\} \\ &+ \left\{-\Lambda\Gamma^T R W(t) R^T \Gamma (\gamma z_2 + \alpha z_1) - \kappa z_2 + \kappa z_2\right\}^T P(t) (z_2 + \kappa z_1) \\ &+ (z_2 + \kappa z_1)^T \dot{P}(t) (z_2 + \kappa z_1) \end{aligned} \tag{54}$$

In the aforementioned equation, the design parameters are chosen as $\alpha = \gamma\kappa$. Equation (54) is reduced to,

$$\dot{V}(z,t) = \zeta^T(t)\left\{P(t)M(t) + M^T(t)P(t) + \dot{P}(t)\right\}\zeta(t) \tag{55}$$

with, $\zeta(t) \triangleq (z_2(t) + \kappa z_1(t)) \in \mathbb{R}^p$ and $M(t) = -\gamma\Lambda\Gamma^T RW(t)R^T\Gamma \in \mathbb{R}^{p\times p}$ is an exponentially stabilizing matrix under the assumption that $W_\tau(t) \in \mathbb{R}^{p\times p}$ is persistently exciting and underlying graph $\mathcal{G}$ is connected (from Theorem 5). Furthermore, $\|M(t)\|$ bounded as long as we assume a bounded weight matrix $W(t)$. From converse Lyapunov theorem 4, we have existence of a continuously differentiable and symmetric matrix $P(t) \in \mathbb{R}^{p\times p}$ such that,

$$\left\{P(t)M(t) + M^T(t)P(t) + \dot{P}(t)\right\} \leq -I_p \tag{56}$$

Therefore, the solution of (55) evolves according to,

$$\begin{aligned} \dot{V}(z,t) &\leq -\zeta^T(t)\zeta(t) \\ &\leq -\left\{\frac{1}{\lambda_{\max}(P(t))}\right\}V(z,t) \qquad \forall t \geq t_0 \end{aligned} \tag{57}$$

which implies,

$$\begin{aligned} V(t) &\leq e^{-\frac{1}{\lambda_{\max}(P(t))}(t-t_0)}V(t_0) \\ \Rightarrow \|\zeta(t)\| &\leq \left\{\sqrt{\frac{\lambda_{\max}(P(t_0))}{\lambda_{\min}(P(t))}}\right\} e^{-\frac{1}{2\lambda_{\max}(P(t))}(t-t_0)} \parallel \zeta(t_0) \parallel \end{aligned} \tag{58}$$

It is possible to re-write $\zeta(t) \in \mathbb{R}^p$ using (51) as follows,

$$\zeta(t) = \dot{z}_1(t) + \kappa z_1(t) \tag{59}$$

It is evident that $\zeta(t)$ is bounded for all time due to boundedness of $V(t)$. Furthermore, it has been shown that $\zeta(t)$ exponentially converges to zero. This implies, from the definition of $\zeta(t)$, that $z_1(t) \in \mathbb{R}^p$ remains bounded for all time and converges to the origin exponentially. This also implies exponential convergence of $z_2(t) \in \mathbb{R}^p$ to the origin. From equation (59), the solution of $z_1(t)$ evolves according to,

$$z_1(t) = e^{-\kappa(t-t_0)}z_1(t_0) + \int_{t_0}^{t}\zeta(\delta)e^{-\kappa(t-\delta)}d\delta \tag{60}$$

Keeping in mind the exponential decay of $\zeta(t)$, it can be concluded that the convergence rate for z_1 and z_2 is,

$$\boxed{\xi \geq \min\left\{\kappa, \frac{1}{2\lambda_{\max}(P(t))}\right\} \quad \forall t \geq t_0} \tag{61}$$

□

Note 2. *The convergence rate bound (61) depends on* $\lambda_{\max}(P(t))$. *Here,* $P(t) \in \mathbb{R}^{p\times p}$ *is defined as the symmetric positive definite solution of a Lyapunov equation. The following straightforward computation helps bound* $\lambda_{\max}(P(t))$. *The single integrator spanning edge convergence expression (33) states that,*

$$\begin{aligned} &\| y(t) \| \leq \left\{\sqrt{\frac{\lambda_{\max}(\Lambda)m_v}{\lambda_{\min}(\Lambda)}}\right\} e^{-\frac{\alpha_v}{2}(t-t_0)} \| y(t_0) \| \\ \Rightarrow &\|\Phi_M(\tau, t)\| \leq \Omega e^{-\frac{\alpha_v}{2}(\tau-t)} \end{aligned}$$

where $\Omega \triangleq \sqrt{\lambda_{\max}(\Lambda)m_v/\lambda_{\min}(\Lambda)}$. *Since,* $P(t)$ *is symmetric, we have from the choice of* $P(t)$ *in (8),*

$$\lambda_{\max}(P(t)) = \|P(t)\| \leq \int_t^\infty \|\Phi_M(\tau, t)\|^2 d\tau \leq \frac{\Omega^2}{\alpha_v}$$

As is evident from above, the convergence rate of the double integrator algorithm is closely related to the single integrator case. Therefore, the convergence rate saturation pointed out for the single integrator case still exists and arbitrarily increasing γ in (38) does not result in faster convergence to consensus beyond a point.

3.3. Consensus in networks with switching topology

The previous sections outline the proof of consensus of the classical algorithm [2] along with determination of an explicit bound on rate of convergence. However, the relationship between the sufficient condition (PE of the spanning tree weight matrix $W_\tau(t)$) to an actual multi-agent system is not directly evident. In this section, we present corollaries to Theorem 5 and 6 that establish the aforementioned relation. The following special case has been proved in our earlier work [12, Corollary. III.1] and presented here for reference.

Corollary 1. *[12, Corollary. III.1] Let,* $t_1, t_2, \cdots$ *be the infinite time sequence of graph switching instants with,* $\tau_i = t_{i+1} - t_i$*,* $i = 0, 1, \cdots$*. Let* $\mathcal{G}_n(t_i)$ *be the undirected graph at time* $t = t_i$ *with non-negative edge weights where* $\mathcal{G}_n(t)$ *is assumed piecewise continuous and bounded. Considering the agent dynamics (9), the continuous time update laws (10) achieve consensus asymptotically, if there exists an infinite sequence of contiguous, nonempty and uniformly bounded time intervals* $\left[t_{i_j}, t_{i_{j+1}}\right)$*;* $j = 1, 2, \cdots$ *starting at* $t_{i_1} = t_0$*, with the property that the union of the undirected graph across each such interval has the same spanning tree.*

The aforementioned corollary is a special case of the more general result in [2, pp. 45–46]. Our result however allows for estimation of bounds on convergence rate to consensus.

Sketch of Proof: We give a gist of the proof here. For details, the reader is referred to [12]. Let us assume that the underlying graph $\mathcal{G}$ complete (i.e. all nodes are in the neighborhood of the other). It is possible to write the laplacian corresponding to any graph $\mathcal{G}_n(t)$ as, $L(\mathcal{G}_n(t)) = D(\mathcal{G})W(t)D(\mathcal{G})^T$ by appropriate choice of edge weights $W(t)$ which cycle between zero and positive values. The only variable quantities in the aforementioned description are the diagonal entries of $W(t)$. Now, we have to prove that the conditions in the corollary imply that $W_\tau(t) \in \mathbb{R}^{p \times p}$ is *persistently exciting*. We have already assumed that all contiguous intervals with union graph containing the spanning tree are uniformly bounded (say by t_{max}). Moreover, union of graphs in each contiguous interval contain the *same* spanning tree, which implies that the spanning tree edge states remain unaltered between intervals (i.e. x_τ and x_c signify the same edge states in (17) over all intervals).

If we now carefully choose, $T_{per} > 2t_{\max}$ in Definition 1 and integrate $W_\tau(t)$ over any window of time T_{per} (integration of $W_\tau(t)$ over a time window is uniquely identifiable with taking the union of graphs over the same period) we can show that it satisfies a positive definite lower bound as required by (5). This proves PE of $W_\tau(t)$ and Theorem 5 can be directly applied to prove convergence.

Moreover, It was also established that the spanning edge dynamics converge as in equation (33) which is repeated here for convenience,

$$\parallel y(t) \parallel \leq \left\{ \sqrt{\frac{\lambda_{\max}(\Lambda) m_v}{\lambda_{\min}(\Lambda)}} \right\} e^{-\frac{\alpha_v}{2}(t-t_0)} \parallel y(t_0) \parallel \tag{62}$$

The above-mentioned corollary is restrictive as it assumes the repetition of the *same* spanning tree in successive time intervals. The following corollary extends above to the general case where different spanning trees out of a finite set potentially appear on successive time intervals. Similar theorems for directed and undirected graphs have been proven in the past (for example [2, pp. 45-46], [20]). The distinction here being the ability to compute a practically verifiable bound on convergence rate to consensus. Before stating the corollary we define the notion of *distinct* graphs. Two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ will henceforth be called *distinct* if there exists an edge weight $g_i(t)$ that is strictly positive for one graph and exactly equal to zero for the other.

Corollary 2. *Let $t_1, t_2, \cdots$ be the infinite time sequence of graph switching instants with, $t_{i+1} - t_i \geq t_L$ for some positive t_L, and $i = 0, 1, \cdots$. Let $\mathcal{G}_n(t_i)$ be the undirected graph at time $t = t_i$ with non-negative edge weights which are piecewise continuous and bounded in nature. Considering the agent dynamics (9) and (35), the continuous time update laws (10) and (36) achieve consensus asymptotically, if there exists an infinite sequence of contiguous, nonempty and uniformly bounded time intervals $\left[t_{i_j}, t_{i_{j+1}}\right)$; $j = 1, 2, \cdots$ starting at $t_{i1} = t_0$, with the property that the union of the undirected graph across each such interval has a spanning tree. Furthermore, the bounds on convergence rate are governed by (34) and (61) respectively, computed corresponding to the slowest spanning tree.*

Note 3. *In the context of the above theorem, the* slowest *spanning tree is the one that results in smallest possible lower bounds in (34) and (61) assuming all other quantities (k, γ, μ_1, μ_2, κ, p) remain constant.*

Proof. Since the multi-agent system has finite number of nodes the possible set of distinct graphs is finite. Therefore let, $\bar{\mathcal{G}}_j = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_j\}$ denote the set of all possible distinct undirected graphs by which the agents communicate with each other. Likewise, consider, $\bar{\mathcal{G}}_s$ to be the set of all possible distinct spanning trees. Since, $\bar{\mathcal{G}}_s \subset \bar{\mathcal{G}}_j$, the set of distinct spanning trees is also finite. We assume, r to be the cardinality of $\bar{\mathcal{G}}_s$.

The fact that union of graphs over each interval $[t_{i_j}, t_{i_{j+1}})$ contains a spanning tree $\mathcal{G}_s \in \bar{\mathcal{G}}_s$ implies in terms of our mathematical setup that, the edge weight matrix $W_\tau(t) \in \mathbb{R}^{p\times p}$ corresponding to $\mathcal{G}_s$ satisfies, $\int_{t_{i_j}}^{t_{i_{j+1}}} W_\tau^2(t)dt \geq \mu_1 I_p$ for some $\mu_1 > 0$. Therefore, the decay rate expression (62) will continue to hold for each interval $[t_{i_j}, t_{i_{j+1}})$, if $W_\tau(t)$ satisfies only a *finite-time persistence of excitation* condition (Definition 1). The notion of the existence of a spanning tree in the union graph over contiguous intervals can be identified with coloring the natural numbers with a finite set of colors. Therefore by the *Van der Waerden's theorem* (Theorem 1) given $r \in \mathbb{Z}^+$ as the number of spanning trees and $k \in \mathbb{Z}^+$ being a positive integer of our choice, there exists an interval $[t_0, t) = [t_{i_1}, t_{i_s})$ and a specific spanning tree $\mathcal{G}_s \in \bar{\mathcal{G}}_s$ that appears in an arithmetic progression of length k within $[t_0, t)$. In case of a switching graph scenario, an arithmetic progression of length k is defined as a set of intervals of the form, $A = \left\{[t_{i_a}, t_{i_{a+1}}), [t_{i_{a+d}}, t_{i_{a+d+1}}), \cdots, [t_{i_{a+(k-1)d}}, t_{i_{a+(k-1)d+1}})\right\}$ where $[t_{i_a}, t_{i_{a+1}})$ denotes the initial block of time when the union graph contains the spanning tree $\mathcal{G}_s$ and d signifies the distance between two successive appearances of $\mathcal{G}_s$ in the union graph.

We denote the uniform upper and lower bounds on the contiguous time intervals by $t_{\max}$ and $t_{\min}$ respectively, with the assumption that, $t_{\max} > t_{\min} > 0$. Therefore if we re-define the contiguous intervals of time as $[t_{i_a}, t_{i_{a+d}}), [t_{i_{a+d}}, t_{i_{a+2d}})$ and so on we have the appearance of the same spanning tree $\mathcal{G}_s$ in the union graph over all contiguous intervals from $[t_0, t)$ thus satisfying the assumptions of Corollary 1. Further, the persistence window (Definition 1) can be chosen as, $T_{per} = (d+1)t_{\max} + \sigma$ for any $\sigma > 0$, to ensure $W_\tau(t)$ is persistently exciting over interval $[t_0, t)$ as required by Theorem 5. Therefore, a norm decay rate expression as given by Equation (34) holds for $\|y(t)\|$ (and hence by linearity for $\|x_\tau(t)\|$). However, the rate expression holds true only for finite final time

$t_f = t_{i_s}$.

Based on the preceding arguments, it is evident that we only need to resolve the exponential decay of the norm to zero asymptotically. Let us assume to the contrary that $\|y(t)\|$ as in Equation (33) does not attain a value below some $\epsilon > 0$. By continuity of solutions we can assume that there exists some time, t^* such that, $\|y(t^*)\| = \epsilon + \delta$ for any small positive δ. The interval $[t^*, \infty)$ thus contains contiguous, non-empty, uniformly bounded intervals containing a spanning tree in the union graph as per the assumptions of the corollary. Since the number of spanning trees is still finite, Van der Waerdens theorem can still be applied and for a sufficiently large k^* of our choice an interval obtained (say $[t^*, t')$) containing k^* contiguous intervals with an identical spanning tree as before. Thus a decay rate as follows can be obtained,

$$\| y(t) \| \leq \beta e^{-\alpha_v (t - t^*)} \| y(t^*) \| \qquad \forall t \in [t^*, t') \tag{63}$$

Since $\beta > 0$ as defined in Equation (33) is independent of initial time, it is evident from above that we can choose k^* large enough to ensure $\|y(t')\| < \epsilon$ thus contradicting our assumption on a positive lower bound for $\|y(t)\|$. This implies that we will have exponential convergence of $\|y(t)\|$ and by virtue of a linear transformation that of $\|x_\tau(t)\|$ to the origin. Further the rate of convergence denoted by α_v as in Equation (34) will be at least as fast as that corresponding to the slowest spanning tree. The extension from the finite time decay to the asymptotic convergence can also be accomplished directly by employing the *Rado selection principle* [21, p. 77].

The proof for the double integrator dynamics (35) with control law (36) follows immediately as evident from the proof of theorem 6. $\square$

The convergence rate is bounded as in expressions (34) and (61) corresponding to the slowest spanning tree (i.e., the spanning tree resulting in the smallest convergence rate). In each of these computations the persistence window is, $T_{per} = (d+1)t_{\max} + \sigma$ for any $\sigma > 0$ based on the proof to the corollary. Here $d = (S(r,k) - 1)/(k-1)$. The dependence on the Van Der Waerden number,

$S(r,k)$ makes the bounds conservative. However, the computation of $S(r,k)$ is independent of sequence of graphs appearing at each time instant as required by [4]. The computation can therefore be carried out offline and only requires knowledge of number of spanning trees expected. $S(r,k)$ has indeed been computed exactly for several special cases.

4. Discussion

A comparison with some prevalent assumptions in proof of consensus for continuous time systems is made in this section.

Assumption 1 (Martin and Girard [4]) (Persistent Connectivity) The graph $(\mathcal{N},\mathcal{E}^p)$ (called Persistent Graph) is strongly connected where

$$\mathcal{E}^p = \left\{(j,i) \in \mathcal{N}\times\mathcal{N} \,\middle|\, \int_0^\infty a_{ij}(s)\mathrm{d}s = +\infty\right\} \tag{64}$$

with $a_{ij}(t)$ being the elements of the edge weight matrix $W(t)$ in our terminology.

In the current work we make the persistence of excitation assumption (Definition 1) on the spanning tree edge weight matrix (Theorems 5–6 and Corollaries 1–2). The assumption in this article is identical in spirit to the one made by Moreau [22]. The notion of persistent graphs is a weaker assumption compared to persistence of excitation of spanning tree edge weights. Authors in [4] have cited an example of a two agent system with edge weight $a_{ij}(t) = 1/t$ to illustrate a case which does not satisfy the persistence of excitation condition in Eq. (5). The above choice of $a_{ij}(t)$ however does form a persistent graph as per (64) and the agents achieve consensus. On the other hand, our assumption on persistence of excitation of $W_\tau(t)$ implies that the spanning tree graph satisfies the persistent connectivity assumption. However, the primary highlight of current work is to provide a bound on the convergence rate. Practical computation of convergence rate using techniques in [4], requires a time-rescaling to be evaluated that relies on exact information of edge weights $a_{ij}(t)$ for all future times ([4, Eq. (6)]). On the other hand, the convergence rate bound presented in Eq. (34)

relies only on global properties of edge weights like T_{per}, μ_1 and μ_2 rather than the actual value of $a_{ij}(t)$ for all time. Therefore convergence rates to consensus for a class of signals with identical T_{per}, μ_1 and μ_2 are equal. For the switching graph case, there is additional dependence on the Van Der Waerden number, $S(r,k)$ which depends only on the number of agents in the system.

Assumption 2 (Hendrickx and Tsitsiklis [23]) (Cut-Balance) There exists a constant $K \geq 1$ such that for all t, and any nonempty proper subset S of $\{1, 2, \ldots, n\}$, we have

$$K^{-1} \sum_{i \in S, j \notin S} a_{ji}(t) \leq \sum_{i \in S, j \notin S} a_{ij}(t) \leq K \sum_{i \in S, j \notin S} a_{ji}(t) \tag{65}$$

Hendrickx and Tsitsiklis [23] utilize the cut-balance assumption along with weak connectivity (equivalent to connectivity for undirected graph networks) to prove consensus. However, their work does not provide a rate of convergence. In the context of the current work since the communication topology is assumed to be undirected, the cut-balance assumption is automatically satisfied.

Assumption 3 (Shi and Johansson [5]) (Arc Balance) There exists a constant $P > 1$ such that for any two edges (j,i) and (m,k) in the persistent graph $\mathcal{E}^p$ and $t \geq 0$, we have

$$P^{-1} a_{ij}(t) \leq a_{km}(t) \leq P a_{ij}(t) \tag{66}$$

The above assumption along with continuity of edge weights and a rooted directed spanning tree in $\mathcal{E}^p$ is shown to be sufficient to achieve consensus by Shi and Johansson [5]. In the current work too a rooted spanning tree is required in the persistent graph for consensus. However, the persistence of excitation condition does not require the edge weights to satisfy an arc balance condition. This can be illustrated with a 3 agent network example. Let us consider the following edge weights,

$$a_{12}(t) = 1, \quad a_{23}(t) = \begin{cases} 0 & [2k, 2k+1),\ k \in \mathbb{N} \\ 1 & [2k+1, 2k+2) \end{cases} \quad a_{31}(t) = 0$$

It is evident from above that the edges $(1,2)$ and $(2,3)$ form a rooted spanning tree and the corresponding edge weights satisfy the persistence of excitation condition (5). Consensus is therefore achieved as per Theorem 5. However, over intervals of time, $[2k, 2k+1)$ the arc balance condition (66) cannot be satisfied between $a_{12}(t)$ and $a_{23}(t)$ for any $P > 1$ and therefore results from [5] cannot be applied. The continuity of edge weights is also not required in current work.

5. Simulation Results

In this section, we consider illustrative examples to validate Theorem 6, Corollary 1 and 2. Simulations corresponding to Theorem 5 are available in [12]. First, let us look at a multi-agent system with four agents and dynamics as follows,

$$\dot{x}_i = y_i$$

$$\dot{y}_i = u_i$$

where, $x_i, y_i \in \mathbb{R}$. denote the position and velocity state respectively, of the i^{th} agent. The underlying graph $\mathcal{G}$ of the communication topology (with arbitrary orientation) for the above mentioned multi-agent system is shown in Fig 1 with $g_i^2(t)$ representing the weights corresponding to edge e_i. The incidence matrix is computed as follows,

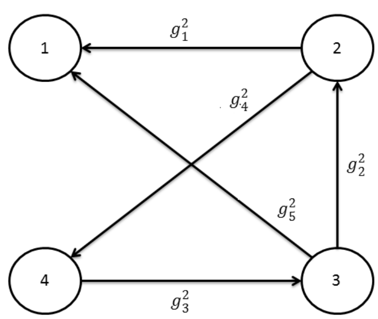

Figure 1: Information-exchange topologies between four agents (underlying graph $\mathcal{G}$)

$$D(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

The weight matrix is chosen as, $W(t) = \text{diag}[g_1^2(t), g_2^2(t), g_3^2(t), g_4^2(t), g_5^2(t)]$ where $g_i^2(t) = \{\text{square}(8(t - d_i)) + 1\} \sin^2(it)$ for $i = \{1,2,3,4,5\}$ with a duty cycle of 0.2 (defined in Section 1.1) and time shift $d_i = 0, 0.157, 0.316, 0.4724, 0.62$ seconds respectively. The design parameters are chosen as, $\kappa = \alpha = \gamma = 1$. The initial conditions for the position and velocity coordinates are selected as, $x_i = y_i = [0.1, 0.2, 0.4, 0.7]^T$. The control law defined in (36) directs the agents to move from their respective initial positions and velocities, smoothly to the consensus value as shown in Fig. 2 and Fig. 3 as expected. Fig. 4 plots the convergence of the spanning edge states along with the exponential envelope. The spanning edge state trajectories always lie within the estimated exponential convergence rate envelope as illustrated by our theoretical analysis.

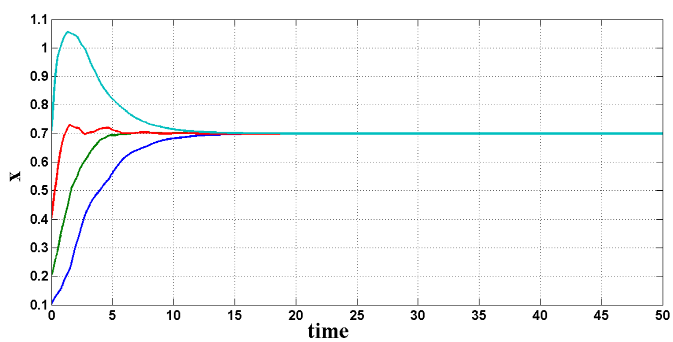

Figure 2: Position trajectories of the four agents

For the double integrator dynamics above, consider the controller defined in (36) with incremental values of gain γ to verify convergence rate. Fig. 5 shows plots of the theoretical convergence bound and actual rate versus γ. The theoretical bounds are computed using (61) while the actual rates are obtained via curve fitting. The plots show that the theoretical bounds reach a peak and start to decay. The actual bound on the other hand exhibits saturation on

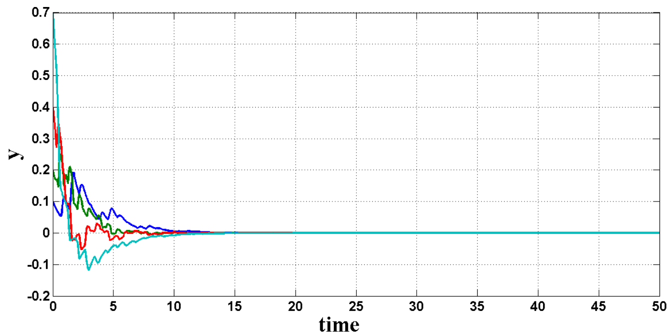

Figure 3: Velocity trajectories of the four agents

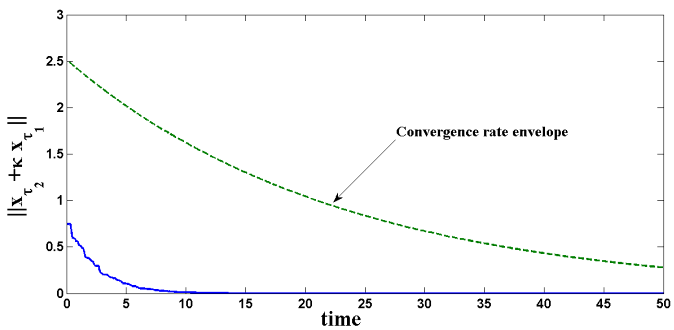

Figure 4: Double integrator convergence envelope

increasing γ.

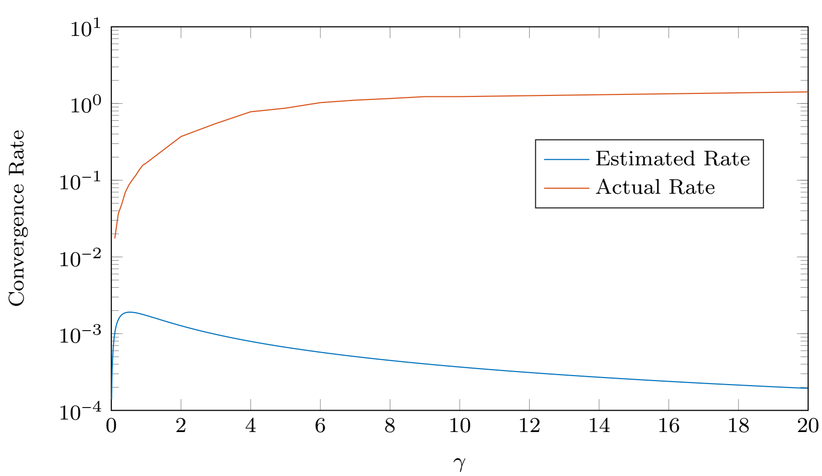

Figure 5: Convergence rates vs control gain, γ

The next set of simulations aim to illustrate Corollary 1 and Corollary 2. A system of four agents with single integrator dynamics is used for these simulations.

$$\dot{x}_i = u_i \tag{67}$$

where $x_i \in \mathbb{R}$ denotes the position coordinates of the i^{th} agent. The underlying graph $\mathcal{G}$ (with arbitrary orientation) is shown in Fig. 6 with $g_i^2(t)$ representing the weights corresponding to edge e_i. The underlying graph is complete as indicated in Section 3.3. The spanning trees appearing in the union graph are

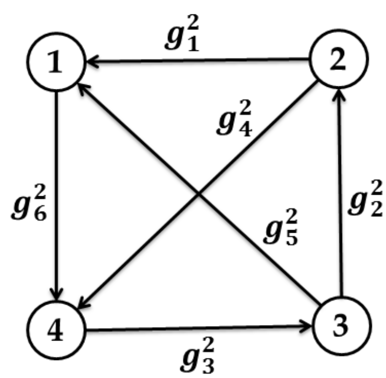

Figure 6: Interaction topology (with arbitrary orientation) between the four agents

depicted in Fig. 7. Fig. 8 shows the simulation results when the *same* spanning

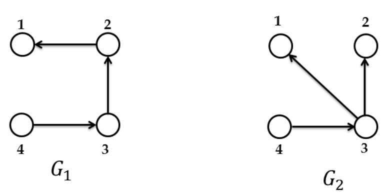

Figure 7: Evaluation of different spanning tree

tree $\mathcal{G}_2$ repeats in each consecutive time interval. The initial conditions are chosen as, $x_\tau(0) = [0.1, 0.2, 0.3]^T$. The plots indicate convergence to consensus as expected and an exponential rate envelope estimated using properties of $\mathcal{G}_2$ in (34).

In the second scenario, the case of spanning trees $\mathcal{G}_1$ and $\mathcal{G}_2$ appearing randomly in contiguous intervals is considered. Fig. 9 depicts the results for the

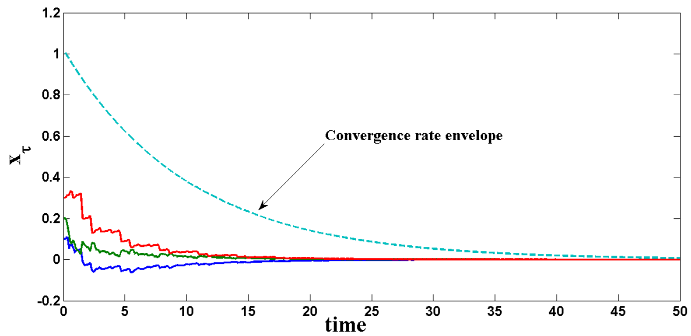

Figure 8: Resultant agent trajectories with *same* spanning tree repeated over contiguous intervals

general switching graph case with initial conditions on edge states, $x_e(0) = [0, 0.1, 0.77, 0.8]^T$ corresponding to edges $\{2,1\}, \{3,2\}, \{4,3\}, \{3,1\}$ respectively. The control law defined in (10) with control gain, $k = 1$, directs the four agents to move from their initial locations to the consensus value as evident from the plot. The plot also shows the calculated convergence rate envelope using the 'slowest' spanning tree and a Van Der Waerden number, $W(2,3) = 9$ utilized to calculate T_{per} as illustrated in proof of Corollary 2.

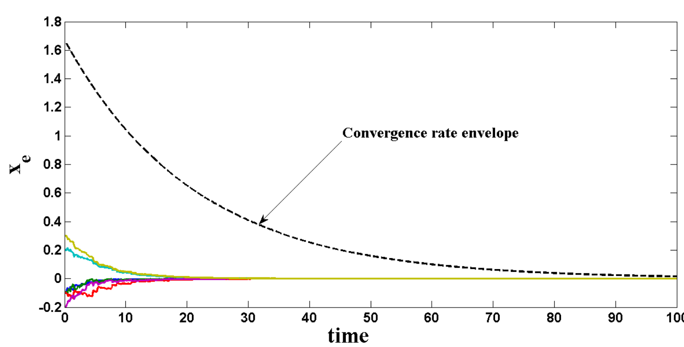

Figure 9: Resultant agent trajectories with *different* spanning trees in contiguous intervals

6. Concluding Remarks

Consensus algorithms for two separate classes of multi-agent systems communicating through diverse inter-agent communication topologies are studied in this work. Time-varying weights are assigned to each edge that potentially pass through singular phases representing communication dropouts. However, these time-varying weights are assumed to satisfy a persistence of excitation condition. The consensus control laws are analyzed by transforming the node agreement problem to an edge agreement one by a suitable coordinate transformation. This allows treatment of a stabilization problem, which is conducive to utilization of classical results of adaptive and nonlinear control to prove consensus. The time-dependent control schemes have been shown to exponentially stabilize the edge set vector for the multi-agent system with dynamic communication topology. Results from Ramsey theory were used to further prove exponential convergence to consensus under a switching communication graph topology. The novel analysis technique for consensus algorithms in undirected graphs has great utility since it helps compute an explicit bound on the rate of convergence for agents communicating via time varying networks. It has also been deduced that arbitrary increasing scalar gains in the consensus control law may not lead to faster convergence. Therefore, future research will target application of the novel analysis technique to different coordinated control problems and improving rate of convergence to consensus.

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