The List Distinguishing Number of Kneser Graphs

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Abstract

A graph G is said to be k-distinguishable if every vertex of the graph can be colored from a set of k colors such that no non-trivial automorphism fixes every color class. The distinguishing number $D(G)$ is the least integer k for which G is k-distinguishable. If for each $v \in V(G)$ we have a list $L(v)$ of colors, and we stipulate that the color assigned to vertex v comes from its list $L(v)$ then G is said to be $\mathcal{L}$-distinguishable where $\mathcal{L} = \{L(v)\}_{v\in V(G)}$. The list distinguishing number of a graph, denoted $D_l(G)$, is the minimum integer k such that every collection of lists $\mathcal{L}$ with $|L(v)| = k$ admits an $\mathcal{L}$-distinguishing coloring. In this paper, we prove that $D_l(G) = D(G)$ when G is a Kneser graph.

Keywords: List Distinguishing Number, Distinguishing Number, Kneser Graphs

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1 Introduction

Let G be a graph and let $Aut(G)$ denote the full automorphism group of G. By an r-vertex coloring of G, we shall mean a map $f : V(G) \to \{1, 2, \dots, r\}$, and the sets $f^{-1}(i)$ for $i \in \{1, 2 \dots, r\}$ shall be referred to as the color classes of f. An automorphism $\sigma \in Aut(G)$ is said to fix a color class C of f if $\sigma(C) = C$, where $\sigma(C) = \{\sigma(v) : v \in C\}$. A vertex coloring of the graph G with the property that no non-trivial automorphism of G fixes all the color classes is called a distinguishing coloring of the graph G.

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Albertson and Collins [1] defined the distinguishing number of graph G, denoted $D(G)$, as the minimum r such that G admits a distinguishing r-vertex coloring.

An interesting variant of the distinguishing number of a graph, due to Ferrara, Flesch, and Gethner [7] goes as follows. Given an assignment $\mathcal{L} = \{L(v)\}_{v\in V(G)}$ of lists of available colors to vertices of G, we say that G is $\mathcal{L}$-distinguishable if there is a distinguishing coloring f of G such that $f(v) \in L(v)$ for all v. The list distinguishing number of G, denoted $D_l(G)$, is the minimum integer k such that G is $\mathcal{L}$-distinguishable for any list assignment $\mathcal{L}$ with $|L(v)| = k$ for all v. The list distinguishing number has generated a bit of interest recently (see [7, 8, 9] for some relevant results) primarily due to the following question that appears in [7]:

$$\text{Is } D_l(G) = D(G) \text{ for all graphs } G?$$

As they state themselves, one of the authors of [7] believes this to be the case, while another author was more circumspect about the same. The authors of [7], prove the same for cycles of size at least 6, cartesian products of cycles, and for graphs whose automorphism group is a dihedral group D_{2n}. The paper [8] settles this question in the affirmative for trees, and [9] establishes it for interval graphs.

Let $r \geq 2$, and $n \geq 2r + 1$. The Kneser graph $K(n, r)$ is defined as follows: The vertex set of $K(n, r)$ consists of all r-element subsets of $[n]$; vertices u, v in $K(n, r)$ are adjacent if and only if $u \cap v = \emptyset$. The distinguishing number of the Kneser graphs is well known (see [2]): $D(K(n, r)) = 2$ when $n \neq 2r + 1$ and $r \geq 3$; for $r = 2$, $D(K(5, 2)) = 3$, and $D(K(n, 2)) = 2$ for all $n \geq 6$ (see [1]).

Our main result in this paper settles the aforementioned question in the affirmative for the family of Kneser graphs.

Theorem 1. *$D_l(K(n, r)) = D(K(n, r))$ for all $r \geq 2, n \geq 2r + 1$.*

Before we proceed to the proof of the theorem, we describe the main idea of the proof. We choose randomly (uniformly) and independently for each vertex v, a color from its list $L(v)$, and we calculate/bound the expected number of non-trivial automorphisms that fix every color class for this random set of choices. This line of argument features in some other related contexts, for e.g., [3, 4, 6, 11] most notably under the umbrella of what is called the 'Motion Lemma', and some of its variants. For $r = 2$, the cases $8 \leq n \leq 22$ include some explicit computation using a SAGE code. These methods however do not work in the case $r = 2$ and $n = 6$ or $n = 7$, so we need different arguments to settle this case. As it turns out, the case with $r \geq 3$ is simpler than the case $r = 2$.

The rest of the paper is organized as follows. In the next couple of sections, we detail the proof for $r = 2$. The case $r \geq 3$ is considered in Section 4. We conclude with a few remarks and a conjecture in the final section. We also include an Appendix that provides the details of the SAGE code and related calculations that settle the proof for $r = 2, 8 \leq n \leq 22$.

2 List Distinguishing Number of $K(n,2)$ when $n \geq 8$.

As mentioned in the Introduction, the distinguishing number of Kneser graphs is known ([2]):

Theorem 2. *$D(K(n,2)) = 2$ for $n \geq 6$, and $D(K(5,2)) = 3$.*

Let S_n denote the symmetric group on n symbols. Observe that every permutation $\sigma \in S_n$ induces an automorphism of $K(n,r)$ as follows: If $v = \{i_1, i_2, \ldots, i_r\}$, then $\sigma(v) := \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_r)\}$. Hence S_n is contained in the full automorphism group of $K(n,r)$. If $n \geq 2r+1$, it is a well known consequence (see [5], Lemma 7.8.2, pg. 147 for instance) of the Erdős-Ko-Rado theorem that S_n is in fact the full automorphism group of $K(n,r)$.

Note that the Kneser graph $K(n,2)$ is the complement of the line graph of K_n, so a list distinguishing coloring of the vertices of $K(n,2)$ is easier to understand as a coloring of the edges of K_n. It is also quite straightforward to see that $D(K(n,2)) = 2$ for $n \geq 6$. Indeed, for each $n \geq 6$, there exists a graph on n vertices with a trivial automorphism group. Fix such a graph G, color the edges of G red (say), and color the remaining edges of K_n blue (say). If $\sigma \in S_n$ is an automorphism of $K(n,2)$ that fixes both these color classes, then in particular, σ also acts as an automorphism of G as well as its complement $\overline{G}$. But this implies that σ is the identity map. The same argument also extends to the Kneser graph $K(n,3)$ for $r \geq 3$. However, this argument fails when the color of each vertex of $K(n,r)$ has to be an element of the list of colors assigned to v.

Suppose $n \geq 6$ and suppose $\{L(e)\}_{e \in E(K_n)}$ is a collection of lists of colors of size 2 for the edges of K_n. For each edge of K_n we choose a color uniformly and independently at random from its given list of colors. We shall refer to this as the random coloring of $K(n,r)$ in the rest of the paper. As mentioned in the introduction, we seek to compute the expected number of non-trivial automorphisms that fix all the colors class of this random coloring.

First, we set up some notations.

a. If the disjoint cycle decomposition of a permutation $\sigma \in S_n$ consists of l_i cycles of length λ_i, for $i = 1, 2, \ldots, t$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_t$, then we say σ is of type Λ where $\Lambda := (\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots, \lambda_t^{l_t})$. Note that $\sum_i l_i \lambda_i = n$.

b. $CT^{(n)}$ shall denote the set of all permutation types in S_n, i.e.,

$$CT^{(n)} := \{(\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots, \lambda_t^{l_t}) \text{ with } \sum_i l_i \lambda_i = n \text{ and } \lambda_1 < \lambda_2 < \cdots < \lambda_t\}.$$

c. $CT^{(n)}_{\geq r}$, $CT^{(n)}_r$ shall denote the sets of all permutation types with minimum cycle length at least r, and with minimum cycle length exactly r, respectively.

d. For positive integers a, b, we shall denote by (a,b), the g.c.d. of a and b.

e. $g(x) := \left\lfloor \frac{(x-1)^2}{2} \right\rfloor$ and $g(x,y) := xy - (x,y)$. Here, the functions $g(x)$ and $g(x,y)$ are defined for non-negative integers x, y.

First, observe that if a non-trivial automorphism σ fixes each of the color classes (as sets) in the random coloring of $E(K_n)$, then every edge in the orbit of an edge $e \in E(K_n)$ under the action of σ must be assigned the same color. In particular, one can compute an upper bound for the probability that σ preserves every color class as a function of the permutation type of σ.

Our current goal is the following: For a non-trivial $\sigma \in S_n$, we seek an upper bound $P(\sigma)$ on the probability that σ fixes all the color classes (as sets) in the random coloring. We then set $P(\Lambda) := \sum_{\sigma \text{ of type } \Lambda} P(\sigma)$.

Lemma 3. *Let $\sigma \in S_n$ be a non-trivial permutation of type $\Lambda = (\lambda_1^{l_1}, \lambda_2^{l_2}, \ldots, \lambda_t^{l_t})$. Furthermore, for $i \leq j$ let $l_j^*(i) := l_i(l_i - 1)/2$ when $i = j$ and $l_j^*(i) = l_i l_j$ for all $j > i$. Then the probability that σ fixes every color class in a random coloring of $K(n, 2)$ is at most*

$$P(\sigma) := \frac{1}{2^\mu},$$

where

$$\mu = \sum_{i=1}^{t} \left(g(\lambda_i) l_i + \sum_{j \geq i}^{t} g(\lambda_i, \lambda_j) l_j^*(i) \right),$$

where the functions $g(\cdot)$ and $g(\cdot, \cdot)$ are as defined in our notations. Consequently, for $\Lambda \in CT^{(n)}$,

$$P(\Lambda) = n! 2^{-\mu} \prod_{i=1}^{t} \frac{\lambda_i^{-l_i}}{(l_i)!}.$$

Proof. Let $\sigma \in S_n$. First, we consider the case where $\sigma = (12 \cdots r)$. As was observed earlier, if σ fixes every color class, then for each edge e, every edge in the set $\{e, \sigma(e), \sigma^2(e), \ldots, \sigma^k(e)\}$ has the same color. Here, the integer $k \geq 1$ is the smallest integer satisfying $\sigma^{k+1}(e) = e$. In particular, for each $1 \leq i \leq \lfloor r/2 \rfloor$, the set of edges $\mathcal{E}_i := \{(1, i+1), (2, i+2), \ldots, (r, i+r)\}$ is monochromatic, where the addition is performed modulo r. Note that the sets $\mathcal{E}_i$ are pairwise disjoint, and furthermore, if r is odd then $|\mathcal{E}_i| = r$ for each i, whereas for r even, $|\mathcal{E}_i| = r$ for $1 \leq i < \frac{r}{2} - 1$, while $|\mathcal{E}_{r/2}| = r/2$. Hence the probability that each of the $\mathcal{E}_i$ is monochromatic under the random coloring is at most $2^{-g(r)}$ where $g(r) = \left\lfloor \frac{(r-1)^2}{2} \right\rfloor$.

Now, suppose $\sigma = (12 \cdots r)(r+1 \;\; r+2 \;\; \cdots \;\; r+s) = C_1 C_2$, say. Then besides the sets $\mathcal{E}_i \subset \{1, 2 \ldots, r\}$ and $\mathcal{F}_j \subset \{r+1, \ldots, r+s\}$ as described above, the set of edges between C_1 and C_2 is partitioned into monochromatic sets each of size, the least common multiple (lcm) of r, s.

Hence, the probability that σ preserves all the color classes under the random coloring is at most $2^{-\mu}$ where

$$\mu = \lfloor \frac{(r-1)^2}{2} \rfloor + \lfloor \frac{(s-1)^2}{2} \rfloor + \frac{rs}{lcm(r,s)}(lcm(r,s) - 1) = g(r) + g(s) + g(r,s)$$

as in the statement of the lemma.

Now, finally, if $\sigma = C_1 C_2 \cdots C_u$ for disjoint cycles C_i, and if σ preserves all the color classes of the random coloring, then the sets of edges between each C_i and C_j are partitioned into sets of size

$lcm(|C_i|, |C_j|)$, each of which must be monochromatic by the same argument as above. It is then easy to see that the probability that σ preserves all the color classes under the random coloring is at most $2^{-\mu}$ where μ is as in the statement of the lemma. This establishes the first part of the lemma.

For the second part, observe that if $\Lambda = (\lambda^{l_1}, \lambda^{l_2}, \dots, \lambda_t^{l_t})$ is a permutation type, then there are exactly $n! \prod_{i=i}^{t} \frac{\lambda_i^{-l_i}}{(l_i)!}$ permutations of type Λ. This is easily illustrated by the following example. If $n = 8, \Lambda = (1, 2, 2, 3)$, then for any permutation $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8)$, we associate a permutation $\pi(\sigma) := (\sigma_1)(\sigma_2\ \sigma_3)(\sigma_4\ \sigma_5)(\sigma_6\ \sigma_7\ \sigma_8)$. But then any other permutation σ' obtained by permuting the sets of elements corresponding to cycles of the same type, or by a cyclic permutation within the elements of a cycle in the association above, yield the same π. For instance, for the permutations $\sigma' = (\sigma_1, \sigma_4, \sigma_5, \sigma_2, \sigma_3, \sigma_6, \sigma_7, \sigma_8)$ (with the 2-cycles $(\sigma_2\ \sigma_3)$ and $(\sigma_4\ \sigma_5)$ swapped) or $\sigma'' = (\sigma_1, \sigma_4, \sigma_5, \sigma_2, \sigma_3, \sigma_7, \sigma_8, \sigma_6)$ (including a cyclic permutation of $(\sigma_6, \sigma_7, \sigma_8)$) we have $\pi(\sigma) = \pi(\sigma') = \pi(\sigma'')$. Hence the number of permutations of permutation type Λ is precisely $\frac{8!}{(1 \cdot 2 \cdot 2 \cdot 3) \cdot 2!}$. The general argument is similar, so we skip the details. $\square$

For each $\Lambda \in CT^{(n)}$ we associate a permutation type $\Gamma_\Lambda \in CT_{\geq \lambda_1}^{(n-\lambda_1)}$ as follows: Suppose $\Lambda = (\lambda_1^{l_1}, \lambda_2^{l_2}, \dots, \lambda_t^{l_t})$ we define

$$\begin{aligned} \Gamma_\Lambda &:= (\lambda_1^{l_1-1}, \lambda_2^{l_2}, \dots, \lambda_t^{l_t}), && \text{if } l_1 > 1 \\ &:= (\lambda_2^{l_2}, \dots, \lambda_t^{l_t}), && \text{if } l_1 = 1. \end{aligned}$$

In this case we say that Λ extends Γ. When the context is clear, we shall write Γ to mean Γ_Λ.

Suppose Λ extends Γ. Following Lemma 3, let us write

$$P(\Lambda) = R_{\lambda_1}(\Lambda) P(\Gamma)$$

where

$$R_{\lambda_1}(\Lambda) := \frac{n(n-1)(n-2)\dots(n-\lambda_1+1)}{\lambda_1 l_1} 2^{-g(\lambda_1) - g(\lambda_1,\lambda_1)(l_1-1) - \sum_{j=2}^{t} g(\lambda_1,\lambda_j) l_j}.$$

Lemma 4. *Suppose $\Lambda = (\lambda_1^{l_1}, \lambda_2^{l_2}, \dots, \lambda_t^{l_t})$, $\Lambda \neq (1^n)$ is a permutation type in $CT^{(n)}$. Then*

$$\begin{aligned} R_{\lambda_1}(\Lambda) &< 2^{-n\lambda_1/19} \ \textit{if } \lambda_1 \geq 2 \textit{ and } n \geq 23. \\ R_1(\Lambda) &\leq \frac{n}{2(n-2)} \ \textit{and equality is achieved precisely if } \Lambda = (1^{n-2}, 2), \\ R_1(\Lambda) &\leq \frac{n}{4(n-4)} \ \textit{if } \Lambda \neq (1^{n-2}, 2) \end{aligned} \tag{1}$$

Proof. Let $2 \leq \lambda_1 \leq \lfloor \frac{n}{2} \rfloor$, and set $s = \frac{\lambda_1 n}{19}$; observe that

$$\log n < \frac{15n}{76} \leq \frac{n}{2} - \frac{s}{\lambda_1} - \frac{\lambda_1}{2}$$

holds for $n \geq 23$. Here by log we shall mean the logarithm to the base 2. As $\lambda_1 \geq 2$ and $n = \sum_{i=1}^{t} \lambda_i l_i$, we have $\sum_{j=1}^{t} l_j \leq n/2$, so

$$\frac{n}{2} - \frac{s}{\lambda_1} - \frac{\lambda_1}{2} < n - \sum_{j=1}^{t} l_j - \frac{s}{\lambda_1} - \frac{\lambda_1}{2} + \frac{1}{\lambda_1}\left(\log(l_1) + \log(\lambda_1)\right).$$

Since $n = \sum_{j=1}^{t} \lambda_j l_j$ we have (by rearranging the terms)

$$n - \sum_{j=1}^{t} l_j - \frac{s}{\lambda_1} - \frac{\lambda_1}{2} + \frac{1}{\lambda_1}\left(\log(l_1) + \log(\lambda_1)\right) \tag{2}$$

$$= \sum_{j=1}^{t} \lambda_j l_j - \left(l_1 + \frac{1}{\lambda_1}\sum_{j=2}^{t} \lambda_1 l_j\right) - \frac{s}{\lambda_1} - \frac{\lambda_1}{2} + \frac{1}{\lambda_1}\left(\log(l_1) + \log(\lambda_1)\right) \tag{3}$$

$$= \frac{-s}{\lambda_1} + \frac{\lambda_1}{2} + l_1\lambda_1 - \lambda_1 - l_1 + \sum_{j=2}^{t} \lambda_j l_j + \frac{1}{\lambda_1}\left(\log(l_1) + \log(\lambda_1) - \sum_{j=2}^{t} \lambda_1 l_j\right) \tag{4}$$

$$= \frac{-s}{\lambda_1} + \frac{\lambda_1}{2} + l_1\lambda_1 - \lambda_1 - l_1 + \frac{1}{\lambda_1}\left(\sum_{j=2}^{t}(\lambda_1\lambda_j - \lambda_1) l_j + \log(l_1) + \log(\lambda_1)\right). \tag{5}$$

To elaborate, we rewrite $n = \sum_j \lambda_j l_j$ in (2) and write $\sum_j l_j$ as $l_1 + \frac{1}{\lambda_1}\sum_{j=2} \lambda_1 l_j$ to get (3); (4) results from (3) by writing $-\frac{\lambda_1}{2}$ as $\frac{\lambda_1}{2} - \lambda_1$, isolating the term $\lambda_1 l_1$ from $\sum_j \lambda_j l_j$, and rearranging terms, and finally (5) is another suitable rearrangement of (4).

Since $\lambda_1 \geq (\lambda_1, \lambda_j)$, we have for $n \geq 23$,

$$\log n < \frac{-s}{\lambda_1} + \frac{\lambda_1 - 2}{2} + (\lambda_1 - 1)(l_1 - 1) + \frac{1}{\lambda_1}\left(\sum_{j=2}^{t} g(\lambda_1, \lambda_j) l_j + \log(l_1) + \log(\lambda_1)\right)$$

Since the function $g(x) = \lfloor (x-1)^2/2 \rfloor$ satisfies $g(x) \geq (x^2 - 2x)/2$, we have

$$\lambda_1 \log n < -s + g(\lambda_1) + g(\lambda_1, \lambda_1)(l_1 - 1) + \sum_{j=2}^{t} g(\lambda_1, \lambda_j) l_j + \log(l_1) + \log(\lambda_1)$$

and thus,

$$n^{\lambda_1} < 2^{-s} l_1 \lambda_1 2^{g(\lambda_1) + g(\lambda_1,\lambda_1)(l_1-1) + \sum_{j=2}^{t} g(\lambda_1,\lambda_j) l_j}$$

and the first part of the lemma follows immediately.

When $\lambda_1 = 1$, then note that

$$R_1(\Lambda) = \frac{n}{l_1} 2^{\left(\sum_{j=2}^{t}(1-\lambda_j) l_j\right)} = \frac{n}{l_1} 2^{L-n},$$

where $L = \sum_{j=1}^{t} l_j$. Since $\lambda_2 \geq 2$, it follows that $n - L \geq (n - l_1)/2$, so we have

$$R_1(\Lambda) \leq \frac{n}{l_1 2^{(n-l_1)/2}}.$$

It follows by elementary calculus (for instance) that the function $h(x) = x2^{(n-x)/2}$ defined on $[1, n-2]$ achieves its minimum value of $2(n-2)$ at $x = n-2$, so in particular,

$$R_1(\Lambda) \leq \frac{n}{2(n-2)}$$

as required.
If Λ corresponds to a permutation type of a non-trivial permutation and $\Lambda \neq (1^{n-2}, 2)$ then arguing as before, the function $h(x)$ defined on $[1, n-4]$ achieves its minimum value of $4(n-4)$ at $x = n-4$ and $R_1(1^{n-3}, 3) = \frac{n}{4(n-3)} < \frac{n}{4(n-4)}$. Therefore we have $R_1(\Lambda) \leq \frac{n}{4(n-4)}$. This completes the proof. □

Let $S_n^* := S_n \setminus \{I\}$, where I denotes the identity map, and set

$$f(n) := \sum_{\sigma \in S_n^*} P(\sigma) = \sum_{\Lambda \in CT^{(n)} \setminus \{(1^n)\}} P(\Lambda).$$

Let $f_{\geq i}(n)$ denote the corresponding sum over all those permutation types $\Lambda \in CT_{\geq i}^{(n)}$ with every cycle of size at least i (and not including $\Lambda = (1^n)$). Also let $P(n) := P(\Lambda)$ for the permutation type $\Lambda = (n)$. Observe that $f(n)$ gives an upper bound for the probability that there exists a non-trivial permutation in S_n, which preserves all the color classes in the random coloring.

Lemma 5. *For any $n \geq 23$,*

$$f(n) < \frac{n}{2(n-2)} f(n-1) + \sum_{i=2}^{\lfloor n/2 \rfloor} 2^{-ni/19} f_{\geq i}(n-i) + P(n).$$

Proof. Observe that any permutation type in $CT_i^{(n)}$ is an extension of a unique permutation type in $CT_{\geq i}^{(n-i)}$, and conversely, every permutation type in $CT_{\geq i}^{(n-i)}$ gives rise to a unique permutation type in $CT_i^{(n)}$. Therefore,

$$\begin{aligned} f(n) &= \sum_{\Lambda \in CT^{(n)} \setminus \{(1^n)\}} P(\Lambda) = \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{\Lambda \in CT_i^{(n)} \setminus \{(1^n)\}} P(\Lambda) + P(n) \\ &= \sum_{\Lambda \in CT_1^{(n)} \setminus \{(1^n)\}} R_1(\Lambda) P(\Gamma_\Lambda) + \sum_{i=2}^{\lfloor n/2 \rfloor} \sum_{\Lambda \in CT_i^{(n)}} R_i(\Lambda) P(\Gamma_\Lambda) + P(n) \end{aligned}$$

By the bounds for $R_{\lambda_1}(\Lambda)$ from Lemma 4, and observing that $\sum_{\Lambda \in CT_i^{(n)}} P(\Gamma_\Lambda) \leq \sum_{\Gamma \in CT_{\geq i}^{(n-i)}} P(\Gamma)$ we have

$$\begin{aligned} f(n) &< \sum_{\Gamma \in CT^{(n-1)} \setminus \{(1^{n-1})\}} \frac{n}{2(n-2)} P(\Gamma) + \sum_{i=2}^{\lfloor n/2 \rfloor} \sum_{\Gamma \in CT_{\geq i}^{(n-i)}} 2^{-ni/19} P(\Gamma) + P(n) \\ &< \frac{n}{2(n-2)} f(n-1) + \sum_{i=2}^{\lfloor n/2 \rfloor} 2^{-ni/19} f_{\geq i}(n-i) + P(n). \end{aligned}$$

□

Theorem 6. *$f(n) < 1$ for all $n \geq 8$. In fact, for all $n \geq 8$,*

$$f(n) \leq \frac{Cn^2}{2^n}$$

for some absolute constant C. Consequently, $D_l(K(n,2)) = 2$ for $n \geq 8$.

Proof. We induct on n. We first verify the theorem for all $8 \leq n \leq 22$ by an explicit computation using a SAGE code. For further details, see the Appendix.

Assume $f(k) < 1$ for $22 \leq k \leq n-1$. By Lemma 5 we have

$$f(n) \leq \frac{n}{2(n-2)} f(n-1) + \sum_{r=2}^{\lfloor n/2 \rfloor} 2^{\frac{-nr}{19}} f_{\geq r}(n-r) + P(n)$$

so

$$f(n) \leq \frac{n}{2(n-2)} + \sum_{r=2}^{\lfloor n/2 \rfloor} 2^{\frac{-nr}{19}} + 0.0000093.$$

Since $\sum_{r=2}^{\lfloor n/2 \rfloor} 2^{\frac{-nr}{19}} < (2^{n/19}(2^{n/19}-1))^{-1} < 0.33$ for $n \geq 23$, we have $f(n) < 1$ when $n \geq 23$.

For the exponential-decay upper bound, we again proceed inductively. The only difference is that this time, we are slightly more careful with our bounds, though we do not attempt to optimize for the constant C. We shall show that $f(n) \leq 10^7 n^2/2^n$ for all $n \geq 9$.

It is easy to see that this statement holds for $n \leq 33$, since $10^7 n^2/2^n$ is greater than 1 for all these values of n. We proceed as in the proof of Lemma 5 but isolate the terms arising from permutations of type $(1^{n-2}, 2)$ and note that their contribution to the sum $f(n)$ is precisely $n(n-1)/2^{n-1}$; for the other Λ with $\lambda_1 = 1$, by inequality 1 in Lemma 4, we have $R_1(\Lambda) \leq \frac{n}{4(n-4)}$. Combining these observations, and by induction as well as the trivial inequalities $f_{\geq r}(n-r) \leq f(n-r)$, we have

$$f(n) < 2\frac{n^2}{2^n} + \frac{n}{2(n-4)} \cdot \frac{10^7 n^2}{2^n} + \frac{10^7 n^2}{2^n} \left(2^{-n/19}(2^{n/19}-2)^{-1}\right) < 10^7 \frac{n^2}{2^n}$$

for $n \geq 34$, and the induction is complete. □

Remark: As observed in the proof, $f(n) \geq \frac{\binom{n}{2}}{2^{n-2}}$, so we actually have $f(n) = \Theta(n^2/2^n)$.

3 List Distinguishing Number of $K(n, 2)$ for $n = 6, 7$.

Consider a graph G with a collection of color lists $\mathcal{L} = \{L(e)|e \in E(G)\}$ for its edges. By the *color palette of a vertex v of G*, we mean the multi-set of colors assigned to the edges incident at the vertex v in a list coloring of the edges of G. A *maximum potentially monochromatic path (MPM path)* P shall refer to a maximum sized path in G such that $\bigcap_{e \in E(P)} L(e) \neq \emptyset$. We use $l(P)$ to denote the length of P and $|P|$ to denote the number of vertices in P.

Lemma 7. *Let $n \geq 6$, suppose we have a collection of lists $\mathcal{L} = \{L(e)|e \in E(K_n)\}$ of size 2. If there is no potentially monochromatic path in K_n of length two then there is a distinguishing list coloring of the edges of K_n from the lists in $\mathcal{L}$.*

Proof. As before, consider a random coloring of the edge set. We claim that the probability that there exist distinct vertices u, v with the same color palette is at most $\binom{n}{2}/2^{n-2}$. Indeed, if we fix a pair of distinct vertices u, v, then for any color incident at vertex u, and not on the edge uv, there is at most one edge incident with v that can have that color in its list, by the hypothesis. Hence the probability that the palette of v is the same as the palette of u is at most $1/2^{n-2}$. The claim now follows by the union bound. Since $\binom{n}{2}/2^{n-2} < 1$ for $n \geq 6$, we are through. □

In what follows, we restrict our attention to $K(6, 2)$ and $K(7, 2)$. As part of the setup, we shall assume that we have a collection of lists of colors of size 2 for the edges of K_7 and K_8.

We also introduce some further terminology. Let P be an MPM path in K_n. By G', we shall denote the complete subgraph on $[n] \setminus V(P)$. The edges between G' and P will be referred to as *crossing edges*, e_{ij} shall denote the edge between vertex i and j, and c_{ij} shall denote the color assigned to the edge e_{ij}. The available common color on the edges of P is denoted c_1. Without loss of generality we assume $V(P) = \{1, 2, \ldots, |P|\}$.

Theorem 8. $D_l(K(6, 2)) = 2$.

Proof. Let P be an MPM path. By virtue of Lemma 7, we may assume that $|P| \geq 3$. Our coloring scheme is as follows. We color each edge of P using the color c_1. To describe the coloring on the other edges, we consider the following cases:

1. $|P| = 6$**:** For $e \notin E(P)$, $e \neq e_{24}, e_{35}$, color e using a color from $L(e) \setminus \{c_1\}$. Now color e_{24} and e_{35} using different colors, i.e., ensure that $c_{24} \neq c_{35}$. This coloring is distinguishing since the color class c_1 is fixed (as a set) only by two maps - the identity and the permutation $\sigma = (16)(25)(34)$. But since $\sigma(e_{24}) = e_{35}$, and they are colored differently, σ does not fix every color class.

2. $|P| = 5$**:** For each of the other edges, color the edge by a color different from c_1 from its list. Again, ensure that $c_{16} \neq c_{56}$; G' consists of the lone vertex 6 and $|P| = 5$, so c_1 does not appear on the lists of both e_{16} and e_{56}, so this arrangement is possible. By our choices, no crossing edge is colored c_1, so the monochromatic set of edges colored c_1 is again precisely P. This coloring is distinguishing for very similar reasons as above.

3. $|P| = 4$: For each $e \notin E(P)$ pick a color $c(e) \in L(e) \setminus \{c_1\}$ and ensure that $c_{45} \notin \{c_{15}, c_{16}, c_{46}\}$; again, these arrangements are possible by the maximality of P as none of the crossing edges from the end vertices of P contain c_1 in the given lists. It is now easy to check that this coloring is distinguishing.

4. $|P| = 3$: Color the edges e_{16} and e_{46} arbitrarily from their lists, and for the remaining edges, impose a restriction on the color that has to be assigned to it as in Table 1 below. Again, note that the maximality of P ensures that all these restrictions can be respected.

Edges	Restriction on the color choice
e_{12}, e_{23}	Assign c_1
$e_{24}, e_{25}, e_{26}, e_{13}, e_{45}$	Avoid c_1
$e_{34}, e_{35}, e_{36}, e_{14}, e_{15}$	Avoid c_{16}
e_{56}	Avoid c_{46}

Table 1: Coloring Scheme when $n = 6$

To see why this is distinguishing, suppose σ is an automorphism that fixes each of these color classes. By the restrictions of our choices, the only path of length 2 comprising of edges all of which are colored c_1, is the path P. Our choices also ensure that the palettes of vertices 1 and 3 are different, so it follows that σ fixes $1, 2, 3$. Since $c_{46} \neq c_{56}$, we must have $\sigma \neq (45), (456), (465)$ and since $c_{14}, c_{15} \neq c_{16}$, we must have $\sigma \neq (46), (56)$; this implies that σ is the identity map on $[6]$. $\square$

Theorem 9. $D_l(K(7, 2)) = 2$.

Proof. We proceed as in Theorem 8. Let P be an MPM path. In all the cases, our coloring scheme assigns the color c_1 for each $e \in E(P)$. We shall impose certain restrictions on how the colors are assigned for $e \notin E(P)$ as well as certain other 'special' edges. We have the following cases.

1. $|P| \geq 5$: For each $e \notin (E(P) \cup \{e_{24}, e_{35}, e_{46}, e_{56}\})$, assign $c(e) \in L(e) \setminus \{c_1\}$. For the edges $e_{24}, e_{35}, e_{46}, e_{56}$ the restriction(s) imposed are as follows:

   (a) if $|P| = 7$: Ensure that $c_{24} \neq c_{46}$.

   (b) if $|P| = 6$: Ensure that $c_{24} \neq c_{35}$.

   (c) if $|P| = 5$: Ensure that $c_{56} \notin \{c_{16}, c_{17}, c_{57}\}$.

   It is straightforward to check that the restrictions imposed are all feasible by the hypothesis. The proof that these colorings are all distinguishing is similar to the argument in Theorem 8, so we omit the details.

2. $|P| = 4$: Ensure that $c_{56} \notin \{c_1, c_{67}, c_{57}\}$, $c_{17} \neq c_{47}$ and $c_{16} \neq c_{15}$. Finally, for any crossing edge e, choose $c(e) \in L(e) \setminus \{c_1\}$.
Our choice of coloring guarantees that any automorphism σ that fixes every color class necessarily maps the sets $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$ into themselves respectively. Since $c_{57}, c_{67} \neq c_{56}$, $\sigma(7) = 7$ and since $c_{17} \neq c_{47}$ it follows that σ fixes each of $1, 2, 3, 4$. Finally, since $c_{16} \neq c_{15}$, σ fixes $5, 6$ as well, so σ is the identity.

3. $|P| = 3$: If the given color lists of edges incident on the vertex 2 have the common color c_1 then assign c_1 to all the edges incident on vertex 2. Note that since $|P| = 3$, no list on an edge not containing 2 can contain the color c_1. Color the remaining edges according to Theorem 8. If there is some edge (say e_{27}) which does not contain the color c_1 in its list, then assign the color c_1 on the edges of the MPM path P and color the edges e_{16} and e_{46} arbitrarily from their lists. For the remaining edges not in $E(P)$ we color the edge using any color from its list other than the one forbidden for it as listed in Table 2.

   Note that since P is by assumption an MPM path, all these restrictions mentioned in Table 2 are feasible.

Sub case. $c_1 \notin L(e_{27})$.

Edges	Restriction on the color choice
e_{12}, e_{23}	Assign c_1
$e_{24}, e_{25}, e_{26}, e_{13}, e_{45}, e_{47}, e_{57}, e_{67}$	Avoid c_1
$e_{34}, e_{35}, e_{36}, e_{14}, e_{15}, e_{37}$	Avoid c_{16}
e_{56}	Avoid c_{46}
e_{17}	Avoid c_{15}
e_{27}	Avoid c_{24}
e_{37}	Avoid c_{36}

Table 2: Coloring scheme when $n = 7$ and $c_1 \notin L(e_{27})$

We show this is a distinguishing coloring. Suppose $\sigma \in S_7$ fixes every color class for this coloring. If all the edges incident on the vertex 2 have the color c_1, then as note above, no other edge of K_7 can have c_1 in the given list of colors by the assumption on the length of MPM path. Therefore $\sigma(2) = 2$. By Theorem 8, it follows that σ is the identity map.
Suppose $c_1 \notin e_{27}$. A consequence of the restrictions we have imposed (as in Table 2) implies that P is the unique path of length 2 which consists of edges colored c_1. Since the palettes of 1 and 3 are different, we must have $\sigma(i) = i$ for $i = 1, 2, 3$. We claim that $\sigma(7) = 7$. Indeed, the color restriction imposed here ensures that neither $\sigma(4) = 7$ nor $\sigma(7) = 4$ is possible since $c_{24} \neq c_{27}$. Furthermore, if $\sigma(6) = 7$, then $\sigma(e_{36}) = e_{37}$ but by our color assignments, these edges are colored differently. Similarly, $\sigma(5) \neq 7$ since $c_{15} \neq c_{17}$. This establishes the claim. Finally, by similar arguments as in Theorem 8, it follows that σ fixes $4, 5, 6$ as well, so σ is the identity map. $\square$

The last result in this section deals with $K(5, 2)$. This case is slightly different since $D(K(5, 2)) = 3$, but the argument we shall use is identical to the one for $K(n, 2)$ for $n \geq 8$.

Theorem 10. $D_l(K(5, 2)) = 3$.

Proof. Again, we assign a color uniformly at random and independently across the edges of K_5 from their respective lists. There are precisely 6 different permutation types in S_5 (see Table 4). This time, the expected number of non-trivial permutations that fix every color class is given by

$$\frac{10}{3^3} + \frac{15}{3^4} + \frac{20}{3^6} + \frac{20}{3^7} + \frac{30}{3^7} + \frac{24}{3^8} < 1,$$

so there exists a color assignment for the edges from their respective lists which is distinguishing. □

4 List Distinguishing Number of $K(n, r)$ when $r \geq 3$

In this section we show that $D_l(K(n, r)) = 2$ for $r \geq 3, n \geq 2r + 1$ holds. Recall that the vertices of $K(n, r)$ correspond to r-subsets of $[n] := \{1, 2 \ldots, n\}$ and vertices $u, v \in V(K(n, r))$ are adjacent if and only if $u \cap v = \emptyset$.

Again, given a collection of lists $\{L(v)\}$ of size 2 for each vertex v of $K(n, r)$, consider a random coloring of the vertices of $K(n, r)$. As in the case of $K(n, 2)$ we show that with positive probability, the random coloring is distinguishing. Recall again every automorphism of $K(n, r)$ is induced by a permutation in S_n, where the action of the permutation on the vertices of K_n induced the action on the vertices of $K(n, r)$.

Lemma 11. *Let $\sigma \in S_n$ be a non-trivial permutation. Then the probability that the automorphism of $K(n, r)$ induced by σ fixes every color class in a random coloring is at most $\frac{1}{2^m}$ where $m = \binom{n-2}{r-1}$.*

Proof. Without loss of generality suppose σ has the cycle $(1, 2, \ldots, t)$ for some $2 \leq t \leq n$. Let v be a vertex corresponding to a set containing the element 1, but not the element 2 in $[n]$. Then since $2 \in \sigma(v)$ it follows that $v \neq \sigma(v)$. Therefore, if σ fixes every color class, then the sets $(v, \sigma(v))$ form a monochromatic pair of vertices on $K(n, r)$. The probability that for every such v, the pair $(v, \sigma(v))$ is a monochromatic pair is at most 2^{-m} as stated in the lemma. □

Theorem 12. *If $r \geq 3$ and $n \geq 2r + 1$, then $D_l(K(n, r)) = 2$.*

Proof. Set $m = \binom{n-2}{r-1}$. Consider the random coloring of $K(n, r)$ as described earlier. By Lemma 11, the probability that there exists a non-trivial automorphism that fixes every color class under this random coloring is at most

$$\frac{|Aut(K(n, r))|}{2^m} = \frac{n!}{2^{\binom{n-2}{r-1}}} \leq \frac{n!}{2^{\binom{n-2}{2}}}$$

since $r \geq 3$ and $n \geq 2r$. It is straightforward to check that the last expression is less than 1 for $n \geq 9$.

This leaves us with the lone remaining case(s): $r = 3, n = 7, 8$. In these cases we look at the corresponding expressions a little closer. We bifurcate the set of non-trivial automorphisms into two categories: We say a permutation $\sigma \in S_n$ is of Category I if all the cycles in the cycle decomposition of σ have size at most 2, otherwise we say σ is a Category II permutation.

For $n = 7$ there are $\frac{7!}{2.5!} + \frac{7!}{2^2.2!.3!} + \frac{7!}{2^3.3!} = 231$ non-trivial permutations of Category I and 4808 permutations of Category II, while for $n = 8$ there are 973 non-trivial permutations of Category I and 39346 permutations of Category II. Let E_I and E_{II} denote the events that there exists a non-trivial automorphism of Category I, Category II respectively, that fixes every color class. We shall now describe a set $\mathcal{I} := \mathcal{I}_\sigma \subset V(K(n,3))$ $(n = 7, 8)$ depending only on σ that satisfies the following:

- Every $v \in \mathcal{I}_\sigma$ satisfies $\sigma(v) \neq v$.
- For $n = 7$, $|\mathcal{I}_\sigma| = 10$ if σ is a Category I permutation, and $|\mathcal{I}_\sigma| = 16$ if σ is a Category II permutation. For $n = 8$, $|\mathcal{I}_\sigma| = 15$ if σ is a Category I permutation, and $|\mathcal{I}_\sigma| = 25$ if σ is a Category II permutation.
- If $\mathcal{E}_v$ denotes the event that $v, \sigma(v)$ have the same color in a random coloring then the events $\{\mathcal{E}_v | v \in \mathcal{I}\}$ are independent.

Suppose σ is of Category I, and let σ contain the 2-cycle (12); then define $\mathcal{I}_\sigma := \{v \subset [n] \mid 1 \in v, 2 \notin v\}$. Suppose σ is a Category II permutation and contains the cycle $(123\cdots)$ (it might well be just a 3-cycle); then set $\mathcal{I}_\sigma := \mathcal{I}^1_\sigma \cup \mathcal{I}^2_\sigma := \{v \subset [n] \mid 1 \in v, 2 \notin v\} \cup \{v \subset [n] \mid 1, 2 \notin v, 3 \in v\}$. We shall henceforth omit the subscript σ for convenience.

The first and second conditions listed above are straightforward to verify in all cases. To see the third, note that if σ is a Category I permutation, then $\mathcal{I} \cap \sigma(\mathcal{I}) = \emptyset$, so the events $\mathcal{E}_v$ are all independent. If σ is of Category II, then note that in addition to the above observation, we also have $\sigma(\mathcal{I}^1) \cap \mathcal{I}^2 = \emptyset$, so again the events $\{\mathcal{E}_v | v \in \mathcal{I}\}$ are independent. Hence it follows that

$$P(E) = P(E_I) + P(E_{II}) < \frac{231}{2^{10}} + \frac{4808}{2^{16}} < 1.$$

Similarly when $n = 8$ we have

$$P(E) = P(E_I) + P(E_{II}) < \frac{973}{2^{15}} + \frac{39346}{2^{25}} < 1$$

and this completes the proof. $\square$

Remark: The above proof in particular gives an alternate proof of the fact that $D(K(n,r)) = 2$ which is shorter than the one that appears in [2].

5 Concluding Remarks

- While we have only stressed on the fact that with positive probability, a random list-coloring of the vertices of $K(n,r)$ (for $r \geq 2$) is distinguishing, the proofs in fact demonstrate that these are asymptotically almost sure events. In particular, these give very efficient randomized algorithms to obtain distinguishing list colorings for the Kneser graphs.

- Our ideas and techniques allow us to give simple(r) proofs of (some of) the results of [7]. For instance, one can prove that $D_l(C_n) = D(C_n)$ for all cycles C_n in a more-or-less straightforward manner by these methods. We believe that these methods may possibly also extend to yield similar results for several other families of graphs. An instructive instance would be to consider an r-fold cartesian product of complete graphs; the distinguishing number of cartesian products of complete graphs was shown to be 2 in [10] though it is not yet known if the list distinguishing number also equals 2, and we believe that the same ideas may turn out to be useful there (though the computations get more complicated).

- Observe that for a non-trivial $\sigma \in S_n$, $P(\sigma)$ is a *strict* upper bound for the probability that σ fixes all the color classes, unless the color lists of all the vertices (of $K(n,r)$) in every orbit of σ are identical. Thus, our expression for the expected number of non-trivial automorphisms that fix every color class is a strict upper bound unless all the lists are identical. This leads us to propose the following conjecture:

  Conjecture 13. *For a connected vertex-transitive graph G, with a collection of equal sized (size k, say) lists $\mathcal{L} = \{L_v | v \in V\}$, if $p(\mathcal{L})$ denotes the probability that a random coloring (obtained by choosing for each vertex, a color from its list uniformly and independently) admits a non-trivial automorphism which preserves all the color classes, then $p(\mathcal{L})$ is maximized when the lists are identical.*

  Our results, while not quite proving this stronger statement exactly (since computing these probabilities exactly would be cumbersome) in fact prove that the expected number of non-trivial automorphisms that fix all the color classes is actually maximized when the lists are identical, for the Kneser graphs.

  Vertex-transitivity and connectivity may be necessary conditions as can be seen from the following simple example. Let $r < s < n/2$. Take vertex disjoint copies of $K(n,r)$ and $K(n,s)$ and let G be the graph obtained from adding a vertex ω adjacent to all the vertices of $K(n,r)$ and $K(n,s)$. First, note that the full automorphism group of G is S_n where the action of S_n keeps each connected component mapped into itself, and maps ω to itself. One can also check that the expected number of automorphisms that fix each color class in a random 2-list coloring of G is the product of expectations of the corresponding expressions arising from the random 2-list colorings of $K(n,r)$ and $K(n,s)$ respectively. However, the 2-lists for $K(n,r)$ and the 2-lists for $K(n,s)$ need not be the same.

  It would be interesting to check if this strengthened conjecture also holds for the other classes of graphs [7, 8, 9] for which the list distinguishing conjecture has been proven.

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Appendix: Proving $f(n) < 1$ for $8 \le n \le 22$

It is straightforward, though a little tedious, to check that $f(8) \approx 0.874 < 1$ by calculating $\sum_{\Lambda \in CT_8} P(\Lambda)$ directly (see Table 4 for a listing of all permutation types of $n \le 8$); we also check that $f_{\ge 4}(5)$, $f_{\ge 3}(6)$ and $f_{\ge 2}(7)$ are strictly less than one:

$$\begin{aligned}
f_{\ge 4}(5) =& \frac{5!}{5} 2^{-8} \approx 0.0937. \\
f_{\ge 3}(6) =& \frac{6!}{3^2 2!} 2^{-2-2-6} + \frac{6!}{6} 2^{-12} \approx 0.0683. \\
f_{\ge 2}(7) =& \frac{7!}{2^2 2! 3} 2^{-12} + \frac{7!}{12} 2^{-2-4-11} + \frac{7!}{10} 2^{-8-9} + \frac{7!}{7} 2^{-18} \approx 0.061.
\end{aligned}$$

Also, $P(n) = \frac{n!}{n} 2^{-\lfloor \frac{(n-1)^2}{2} \rfloor}$. Since $P(n)$ is monotonically strictly decreasing for $n \ge 3$, we may bound $P(n) < P(9) = 8! 2^{-32} \approx 0.0000093$. For $n \le 22$, we compute bounds for $f(n)$ using a computer SAGE code; the details of the code and the SAGE programming syntax appear below. In all these cases, we have $f(n) < 1$ for $n \le 22$ (See Table 3).

n	Upper bound of $f(n)$
9	0.566
10	0.422
11	0.268
12	0.171
13	0.103
14	0.061
15	0.036
16	0.021
17	0.012
18	0.007
19	0.004
20	0.003
21	0.002
22	0.001

Table 3: $f(n)$, $n < 23$

SAGE Code to Calculate $f(n), 9 \le n \le 22$.

A SAGE programming code is given below, which calculate an upper bound of $f(n)$ defined in Theorem 6. The idea is to use the recurrence relation

$$f(n) \le \frac{n}{2(n-2)} f(n-1) + \sum_{i=2}^{\lfloor n/2 \rfloor} \sum_{\Lambda \in CT_{\ge i}^{(n-i)}} R_i(\Lambda) P(\Gamma_\Lambda) + P(n)$$

with $P(n) \leq 0.0000093$ and

$$\sum_{i=2}^{\lfloor n/2 \rfloor} \sum_{\Lambda \in CT_{\geq i}^{(n-i)}} R_i(\Lambda) P(\Gamma_\Lambda) \leq \sum_{i=2}^{\lfloor n/2 \rfloor} f(n-i) \sum_{\Lambda \in CT_{\geq i}^{(n-i)}} R_i(\Lambda).$$

We use the bounds of $f_{\geq 4}(5), f_{\geq 3}(6), f_{\geq 2}(7)$ and $f(8)$ given in Theorem 6 as initial values in the calculation. The SAGE code syntax and output of the program in SageMath Version 7.3 is given below.

Syntax

```
import sys
from sage.all import *
def w(a,b):
    s=a*b-gcd(a,b)
    return s
def f(a):
    s=(a-1)**2/2.0
    s=s.floor()
    return s
def R(g,n,p):
    M=set(g)
    N=list(M)
    lam=p
    l1=list(g).count(p)
    k= len(N)
    L=[]
    sum=0
    for j in range (k):
        d=N[j]
        l = list(g).count(d)
        t=w(lam,d)*l
        sum=sum+t
        L.append(l)
    mu=f(lam) + sum
    E = 1/float(2^m u)
    pro=1
    for r in range(lam):
        pro=(n-r)*pro
    F=pro/float((l1+1)*lam)
    Rlam=F*E
    return Rlam
def SR(n,p):
    Z=0
    for g in Partitions(n-p, min_part=p):
```

```
        Z=R(g,n,p)+Z
    return Z
A=[0,0,0,0,0,0.0937,0.0683,0.061,0.874]
for n in range(9,23):
    t=floor(n/2)
    X=0
    for p in range (2, t+1):
        X=A[n-p]*SR(n,p)+ X
    FX= (n*A[n-1])/float(2*(n-2)) + 0.0000093+ X
    print "f(",n,")is",FX
    A.append(FX)
```

Output

```
f( 9 )is 0.565502029643413
f( 10 )is 0.421002282515914
f( 11 )is 0.267868741717338
f( 12 )is 0.170601031045550
f( 13 )is 0.102346113079601
f( 14 )is 0.0609053489833814
f( 15 )is 0.0353110371363751
f( 16 )is 0.0203048671548423
f( 17 )is 0.0115310549618470
f( 18 )is 0.00650639470158091
f( 19 )is 0.00364658006324804
f( 20 )is 0.00203611883735761
f( 21 )is 0.00113463671742483
f( 22 )is 0.000633429447998104
```

Partition Table of $5 \leq n \leq 8$.

The following is a table containing all possible disjoint cycle decompositions of elements of S_n when $5 \leq n \leq 8$. We denote number of integer partition of n by $I(n)$.

n	$I(n)$	Partitions
8	22	$11111111, 1111112, 111122, 11222, 2222,$ $111113, 11123, 1223, 1133, 233,$ $11114, 1124, 224, 134, 44,$ $1115, 125, 35, 116, 26, 17, 8$
7	15	$1111111, 111112, 11122, 1222, 11113, 1123, 223, 133,$ $1114, 124, 34, 115, 25, 16, 7$
6	11	$111111, 11112, 1122, 222, 1113, 123, 33,$ $114, 24, 15, 6$
5	7	$11111, 1112, 122, 113, 23, 14, 5$

Table 4: Partition table