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RANDOMIZED ALGORITHMS FOR STABILIZING SWITCHING SIGNALS

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Abstract. Qualitative behaviour of switched systems has attracted considerable research attention in the recent past. In this article we study 'how likely' is it for a family of systems to admit stabilizing switching signals. A weighted digraph is associated to a switched system in a natural fashion, and the switching signal is expressed as an infinite walk on this digraph. We provide a linear time probabilistic algorithm to find cycles on this digraph that have a desirable property (we call it "contractivity"), and under mild statistical hypotheses on the connectivity and weights of the digraph, demonstrate that there exist uncountably many stabilizing switching signals derived from such cycles. Our algorithm does not require the vertex and edge weights to be stored in memory prior to its application, has a learning/exploratory character, and naturally fits very large scale systems.

1. **Introduction.** Switched systems are typically employed to model dynamical systems that are prone to known or unknown abrupt parameter variations [4, p.3]. These systems naturally arise in a multitude of contexts such as networked systems, quantization, variable structure systems, etc.; see e.g., [15, 13, 14] and the references therein for extensive lists of application areas. It is well-known that the qualitative behaviour of a switched system depends not only on those of the individual subsystems, but also crucially on the properties of the switching signal. For instance, divergent trajectories may be generated by switching appropriately among stable subsystems, while a suitably constrained switching signal may ensure stability of the switched system even if all the subsystems are unstable (see, e.g.,

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[13, p.19] for examples with two subsystems). This interesting feature motivates the identification of classes of switching signals that ensure stability of given switched systems.

Recently in [9, 10] the authors proposed a class of time-dependent [13] switching signals for global asymptotic stability of discrete-time switched linear systems. This class of switching signals is characterized solely in terms of certain asymptotic properties of the switching signals, that neither involve nor imply point-wise bounds on the number of switches — unlike in the case of the classical average dwell time switching [5, 21]. The aim of the present article is to study the existence of the above class of stabilizing switching signals. To address this question, we take recourse to probability theory: in broad strokes, considering the ensemble of switched systems as the sample space, we ask how likely a switched system sampled from this ensemble is to admit a stabilizing switching signal. Of course, while the existence of stabilizing switching signals is an important issue in its own right, algorithmic synthesis of such switching signals is important for obvious reasons. Studies addressing the latter aspects have appeared before in the switched systems literature: see, e.g., [6], [17]. In this article we address both the issues at once.

Our approach begins with associating a weighted digraph to a family of systems and the admissible transitions among them in a natural way, and express by switching signal as an infinite walk on the above digraph.[1] In particular, here we are interested in the class of infinite walks corresponding to the class of stabilizing switching signals proposed in [9, Theorem 1]. (We shall henceforth freely switch between system-theoretic and the corresponding graph-theoretic terminology in the above sense.) Towards answering the question of existence of an infinite walk that corresponds to a stabilizing switching signal, we turn to probability theory and provide probabilistic guarantees for this existence problem. Under mild conditions, we provide a randomized algorithmic mechanism to identify a class of switched systems that satisfy the conditions proposed in [9, Theorem 1], thereby addressing the matter of synthesis of stabilizing switching signals.

Our solution comprises of the following stages: Firstly, we propose a probabilistic algorithm to synthesize cycles on “typical” digraphs. Secondly, conditions on the connectivity and weights of the underlying digraph of a switched system are identified that ensure that the cycles synthesized as above satisfy a certain good property with high probability. We call this property “contractivity”. We demonstrate that switching signals corresponding to infinite walks constructed out of contractive cycles are stabilizing. Finally, under mild hypotheses, we demonstrate that there exist uncountably many stabilizing infinite walks — these correspond to stabilizing switching signals for the switched system, see Remark 5 for a detailed discussion.

More specifically, our contributions in this article may be viewed from the following perspectives:

- Standard deterministic algorithms that are employed to synthesize cycles on a weighted digraph [2, p. 646], [19], [12, 20] may not be applicable to switched systems whose underlying digraphs are large (e.g., in variable structure systems with a large number of substructures), especially if their sizes are so large that not all the weights can be kept in memory at once. For such large digraphs our algorithm provides probabilistic guarantees in the spirit of randomized algorithms for

[1] Digraphs have appeared before in switched systems literature, see e.g., [16], [7].

synthesis and design of contractive cycles. In today's era of large-scale networked systems, this is an extremely important and positive feature of our results.

- Our algorithm exhibits an "online learning" character in the following sense: starting with a rough probabilistic description of the underlying weighted digraph, (i.e., without knowledge of the precise values of the weights,) we explore the digraph and synthesize a cycle *during* this exploration that is contractive with high probability. On the other hand, the traditional algorithms for synthesizing cycles require complete knowledge of the digraph and the vertex and edge weights a priori.
- If the constituent subsystems of a switched system are prone to evolve/drift over time in a manner that is not precisely known but certain statistical estimates of the nature of evolution are available, our algorithm can be applied, and it will construct a contractive cycle with uniform probabilistic guarantees over all such evolutions.

The remainder of this article exposes as follows: In §2 we briefly recall the class of stabilizing switching signals proposed in [9] and formulate the problem under consideration. The association of a weighted digraph with a switching system is described in §3. Our main results appear in §4, and numerical experiments illustrating our results are provided in §5.

Some notations used in this article: $\mathbb{N} = \{1, 2, \cdots\}$ is the set of natural numbers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\mathbb{R}$ is the set of real numbers. For a finite set M, $|M|$ denotes its cardinality and $M \sqcup N$ denotes the disjoint union of M with another finite set N. For a digraph $G(V, E)$, $d^+(v)$ denotes the outdegree of a vertex $v \in V$. For $V' \subset V$ we let $N^+_{V'}(v) := \{u \in V' \mid (v, u) \in E\}$ denote the set of outneighbours of a vertex v in V', and let $d^+_{V'}(v) := \left|N^+_{V'}(v)\right|$ denote the outdegree of v in V'. For a walk W on $G(V, E)$, $|W|$ denotes its length.

2. Problem statement.

We consider a family of discrete-time linear systems (written as initial-value recursions)

$$x(t+1) = A_i x(t), \quad x(0) \text{ given}, \quad i \in \mathcal{P}, \quad t \in \mathbb{N}_0, \tag{1}$$

where $x(t) \in \mathbb{R}^d$ is the vector of states at time t, $\mathcal{P} = \{1, 2, \cdots, N\}$, and $A_i \in \mathbb{R}^{d\times d}$, $i \in \mathcal{P}$, are fixed full-rank matrices. Let $\sigma : \mathbb{N}_0 \to \mathcal{P}$ be a *switching signal* that specifies at every time t, the index of the active system from family (1). The discrete-time *switched linear system* generated by the family of systems (1) and the switching signal σ is given by

$$x(t+1) = A_{\sigma(t)} x(t), \quad x(0) \text{ given}, \quad t \in \mathbb{N}_0. \tag{2}$$

We are interested in switching signals that lead to the property of global asymptotic stability of (2). In this connection, recall:

Definition 2.1. The switched system (2) is *globally asymptotically stable* (GAS) for a given switching signal σ if (2) is

- Lyapunov stable, and
- globally asymptotically convergent: for all $x(0)$, $\lim_{t\to+\infty} x(t) = 0$.

Let $\mathcal{P}_S$ be the set of indices of the asymptotically stable systems in the family (1), and let $\mathcal{P}_U := \mathcal{P} \setminus \mathcal{P}_S$ denote the rest of the systems in (1).[2] Let $E(\mathcal{P})$ denote

[2]The set $\mathcal{P}_U$ includes (by definition) all the unstable systems and the marginally stable ones.

the set of ordered pairs (i,j) such that switches from system i to system j are admissible, $i,j \in \mathcal{P}$. We recall two facts off the shelf:

Fact 1 ([9, Fact 1])**.** For each $i \in \mathcal{P}$ there exists a pair (P_i, λ_i), where $P_i \in \mathbb{R}^{d\times d}$ is a symmetric and positive definite matrix, and

- if $i \in \mathcal{P}_S$, then $0 < \lambda_i < 1$;
- if $i \in \mathcal{P}_U$, then $\lambda_i \geqslant 1$;

such that, with $\mathbb{R}^d \ni \xi \longmapsto V_i(\xi) \coloneqq \xi^\top P_i \xi \geqslant 0$, we have

$$V_i(\gamma_i(t+1)) \leqslant \lambda_i V_i(\gamma_i(t)), \quad t \in \mathbb{N}_0, \tag{3}$$

where $\gamma_i(\cdot)$ solves the $i-$th recursion in (1), $i \in \mathcal{P}$. The members of the family $\{V_i\}_{i\in\mathcal{P}}$ are called Lyapunov-like functions.

Fact 2 ([9, Fact 2, Proposition 1])**.** There exist numbers $\mu_{ij} > 0$ such that

$$V_j(\xi) \leqslant \mu_{ij} V_i(\xi) \text{ for all } \xi \in \mathbb{R}^d, \ (i,j) \in E(\mathcal{P}). \tag{4}$$

In particular, the smallest such constants μ_{ij} are given by $\mu_{ij} = \lambda_{\max}(P_j P_i^{-1}), i,j \in \mathcal{P}$, where for a matrix $M \in \mathbb{R}^{n\times n}$ having real spectrum, $\lambda_{\max}(M)$ denotes its maximal eigenvalue.

In [9, Theorem 1] we identified a large class of switching signals σ under which the resulting switched system (2) is GAS; we recall the key result here:

Theorem 2.2 ([9, Theorem 1])**.** *Consider the switched system (2). For $t \in \mathbb{N}$ let N_t^σ be the number of switches before (and including) t. Then the switched system (2) is GAS under all switching signals σ that satisfy*

$$\varliminf_{t\to+\infty} \frac{N_t^\sigma}{t} > 0, \tag{5}$$

and

$$\varlimsup_{t\to+\infty} \frac{\displaystyle\sum_{(k,\ell)\in E(\mathcal{P})} (\ln \mu_{k\ell}) \sharp\{k \to \ell\}_t + \sum_{j\in\mathcal{P}_U} \lvert \ln \lambda_j \rvert \, \sharp\{j\}_t}{\displaystyle\sum_{j\in\mathcal{P}_S} \lvert \ln \lambda_j \rvert \, \sharp\{j\}_t} < 1, \tag{6}$$

where $\{\lambda_j | j \in \mathcal{P}\}$ and $\{\mu_{ij} | (i,j) \in \mathcal{P}\}$ are extracted from the family (1) using Fact 1 and Fact 2, respectively, $\sharp\{k \to \ell\}_t$ denotes the number of times a switch from system k to system ℓ is made by σ till time t, and $\sharp\{j\}_t$ denotes the number of times that the system j is activated by σ till time t.

The condition (5) is sufficient to ensure that the switched system (2) does not eventually adhere to an unstable system in (1); it may be dropped from the theorem if $\mathcal{P}_U = \emptyset$. The condition (6) says that the limit superior of the ratio

$$\frac{\displaystyle\sum_{(k,\ell)\in E(\mathcal{P})} (\ln \mu_{k\ell}) \sharp\{k \to \ell\}_t + \sum_{j\in\mathcal{P}_U} \lvert \ln \lambda_j \rvert \, \sharp\{j\}_t}{\displaystyle\sum_{j\in\mathcal{P}_S} \lvert \ln \lambda_j \rvert \, \sharp\{j\}_t},$$

which is a function of time t, is strictly less than 1 as $t \to +\infty$. The term $\sum_{(k,\ell)\in E(\mathcal{P})} (\ln \mu_{k\ell}) \cdot \sharp\{k \to \ell\}_t$ in the numerator of the above ratio captures the number of times each admissible transition $(k,\ell) \in E(\mathcal{P})$ occurs in σ till time t, weighted by $\ln \mu_{k\ell}$'s, where $\mu_{k\ell}$ is as in Fact 2. The terms $\sum_{j\in\mathcal{P}_S} \lvert \ln \lambda_j \rvert \, \sharp\{j\}_t$ and

$\sum_{j\in\mathcal{P}_U} |\ln\lambda_j| \sharp\{j\}_t$ capture the number of times a system $j \in \mathcal{P}_S$ (resp. $\mathcal{P}_U$) is activated till time t by σ, weighted by the numbers λ_j's obeying Fact 1.

It should be noted that sufficient conditions in Theorem 2.2 rely on the Lyapunov-like functions $\{V_i\}_{i\in\mathcal{P}}$ via Fact 1 and Fact 2. Since there may be many viable choices of $\{V_i\}_{i\in\mathcal{P}}$, it is not immediately clear that there exists a switching signal σ for which (6) can be verified for a particular choice of the family $\{V_i\}_{i\in\mathcal{P}}$. Indeed, as demonstrated in [9, Example 2], given a family of systems (1) and the set of admissible transitions among the systems in the family, there may not exist a switching signal σ that satisfies condition (6). This motivates the question:

Problem 1. *What class of switched systems admits the class of switching signals that satisfies the conditions in Theorem 2.2?*

In this article we provide a description of a class of switched systems that admit switching signals satisfying (6).

3. Preliminaries.

Associating a weighted digraph with (2). We associate a weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ with the switched system (2) in the following natural fashion:

- the set of vertices of G is the index set $\mathcal{P}$;
- the set of edges $E(\mathcal{P})$ consists of:
  - a directed edge from vertex i to vertex j whenever it is admissible to switch from system i to system j, and
  - a self-loop at vertex j, $j \in \mathcal{P}$, whenever it is admissible to dwell on system j for at least two consecutive time steps;
- edge and vertex weights $w(i,j) := \ln\mu_{ij}$, $(i,j) \in E(\mathcal{P})$ (Fact 2) and $w(j) := |\ln\lambda_j|$, $j \in \mathcal{P}$ (Fact 1) denote the edge weights and vertex weights of $G(\mathcal{P}, E(\mathcal{P}))$, respectively. Clearly, $w(i,j) = 0$ for a self-loop.

Various definitions. We abbreviate $G(\mathcal{P}, E(\mathcal{P}))$ by G whenever there is no risk of confusion. Recall that the *size* of G is the number of its edges, and its *order* is the number of its vertices. A *walk* W on a digraph $G(V,E)$ [1, p. 4] is an alternating sequence of vertices and (directed) edges $W = v_0, e_1, v_1, e_2, v_2, \cdots, v_{\ell-1}, e_\ell, v_\ell$, where $v_i \in V$, $e_i = (v_{i-1}, v_i) \in E$, $0 < i \leqslant \ell$. The *initial vertex* of W is v_0 and the *final vertex* of W is v_ℓ. If $v_0 = v_\ell$, we say that the walk is closed. We follow the convention: A closed walk [1, p. 4] W on a directed graph $G(V,E)$ is a *circuit* if all its edges are distinct; W is a *cycle* if the vertices $\{v_i\}_{i=1}^{\ell-1}$ are distinct from each other and v_0. By the term *infinite walk* we mean a walk of infinite length [1, p. 5], i.e., it has infinitely many edges. An *initial subwalk* W' of a walk W is an initial segment of W, which we write as $W' \leqslant W$.

Fact 3 ([9, Fact 3])**.** The set of switching signals $\sigma : \mathbb{N}_0 \to \mathcal{P}$ and the set of infinite walks on $G(\mathcal{P}, E(\mathcal{P}))$ (defined as above) are in bijective correspondence.

For a walk W on G, we let N_W denote the number of edges excluding self-loops that appear in W, and let

$$\nu(W) := \frac{N_W}{|W|}, \quad |W| > 0, \tag{7}$$

be the *transition frequency* of W. On the family of all finite walks W on G we define the function

$$\Xi(W) \coloneqq \frac{\displaystyle\sum_{(k,\ell)\in E(\mathcal{P})} w(k,\ell)\sharp\{k \to \ell\}_W + \sum_{j\in\mathcal{P}_U} w(j)\sharp\{j\}_W}{\displaystyle\sum_{j\in\mathcal{P}_S} w(j)\sharp\{j\}_W}, \tag{8}$$

where $\sharp\{k \to \ell\}_W$ and $\sharp\{j\}_W$ denote the number of times the edge (k, ℓ) and the vertex j appear in W, respectively, $\mathcal{P}_S$ and $\mathcal{P}_U$ denote the sets of indices of asymptotically stable and unstable vertices (systems in family (1)), respectively.

In the light of Fact 3 and the definition above, we can rephrase Theorem 2.2 in a purely graph theoretic language as:

Theorem 3.1. *Consider the underlying weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ of the switched system (2). The switched system (2) is globally asymptotically stable (GAS) under all switching signals σ, whose corresponding infinite walks (à la Fact 3) W satisfy*

$$\varliminf_{\substack{|W'|\to+\infty\\ W'\leqslant W}} \nu(W') > 0, \tag{9}$$

and

$$\varlimsup_{\substack{|W'|\to+\infty\\ W'\leqslant W}} \Xi(W') < 1, \tag{10}$$

where $\nu(W')$ and $\Xi(W')$ are as defined in (7) and (8), respectively.

Since we are in the discrete-time setting, the association (à la Fact 3) of the length of a walk with time is natural. It is clear that both the set of admissible transitions (related to "directional" connectivity of G) between subsystems, and the properties of the subsystems (captured by the vertex and edge weights of G) play distinct roles in determining whether there exists an infinite walk W that satisfies (10). In this terminology Problem 1 is described as:

Problem 2. *What class of weighted digraphs admits infinite walks satisfying (10)?*

Such infinite walks are called *stabilizing* in the sequel. We shall discuss algorithmic solutions to Problem 2 shortly.

Our algorithmic synthesis mechanism of stabilizing infinite walks will involve concatenations of finite walks satisfying the following property:

Definition 3.2. A walk W on the weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ is *contractive* if

$$\Xi(W) < 1. \tag{11}$$

We provide a sufficient condition for the existence of an infinite walk satisfying (10) in terms of a closed contractive walk (necessarily of finite length) on G.

Lemma 3.3 ([9, Theorem 2(a)])**.** *Consider the underlying weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ of the switched system (2). If there exists a closed contractive walk W on $G(\mathcal{P}, E(\mathcal{P}))$, then the infinite walk obtained by repeating the closed walk W satisfies (10).*

In Lemma 3.3, "repetition" is an iterative process consisting of a mechanism requiring a bounded quantum of memory, to generate finite walks W_k of length $n_k > 0$, $k \in \mathbb{N}$, satisfying the condition that the final vertex of W_{k-1} is identical to the initial vertex of W_k for each k. We build the infinite walk W as the limit of $W_1 W_2 \cdots W_{k-1} W_k$, $k \in \mathbb{N}$. The task of algorithmic synthesis of a closed contractive walk on G is computationally simpler under Lemma 3.3 since the length of the walk is finite. We adopt the following convention: The total number of times a closed walk W visits a vertex $j \in \mathcal{P}$ is the same as the total number of times W visits its outgoing edges. As a result, $\sharp\{j\}_W$ can be replaced by $\displaystyle\sum_{(j,\ell)\in E(\mathcal{P})} \sharp\{j \to \ell\}_W$. Since we are concerned with an infinite walk constructed by repeating the closed contractive walk W indefinitely many times, assuming the above is no loss of generality. Accordingly, condition (11) becomes

$$\Xi'(W) := \sum_{(k,\ell)\in E(\mathcal{P})} \Big(w(k,\ell) + w(k)1_{\mathcal{P}_U}(k) - w(k)1_{\mathcal{P}_S}(k) \Big) \sharp\{k \to \ell\}_W < 0. \tag{12}$$

Remark 1. The mechanism explained above shows that for a walk W generated by concatenating the walks W_1 and W_2 each satisfying the usual contractivity condition (11), we have $\Xi'(W) = \Xi'(W_1) + \Xi'(W_2) < 0$. However, algorithmic synthesis of a closed contractive walk on G is also difficult due to the absence of a bound on the length of the closed walk W. Consequently, the length at which the algorithm that attempts to synthesize a closed contractive walk should terminate must be specified and its a priori selection becomes a heuristic.

A natural alternative to searching for contractive walks over all closed walks (of arbitrary length) is to restrict the walks to those with bounded length, for example, circuits or cycles. We have:

Theorem 3.4 ([8, Theorem 6.13])**.** *Consider the underlying weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ of the switched system (2). The following are equivalent:*
- ∘ *$G(\mathcal{P}, E(\mathcal{P}))$ admits a closed contractive walk.*
- ∘ *$G(\mathcal{P}, E(\mathcal{P}))$ admits a contractive circuit.*
- ∘ *$G(\mathcal{P}, E(\mathcal{P}))$ admits a contractive cycle.*

As a consequence, an infinite walk obtained by repeating one of the above satisfies (10).

Given a weighted digraph G, Theorem 3.4 gives a set of necessary and sufficient conditions for the existence of a closed contractive walk in terms of a contractive circuit and a contractive cycle.

Armed with Theorem 3.4, in this article to Problem 2 we provide:

Solution 1. We propose a *linear time* (in the order of the graph) algorithm that constructs a *cycle* of a certain fixed maximal length on G. Under mild assumptions on the connectivity and the weights associated to the vertices and edges of G, we provide strong probabilistic guarantees of the cycle obtained as above being contractive in Theorem 4.2, construct stabilizing infinite walks out of such cycles, and deduce strong assertions about the set of stabilizing infinite walks in Theorem 4.3.

4. Main results.

4.1. **Nicely connected and weighted digraphs.** We begin with the central definition of this work, followed by an elaboration of its connections with system-theoretic ideas. Recall, in this context, that $N^+_{\mathcal{P}_S}(v) := \{u \in \mathcal{P}_S \mid (v,u) \in E\}$ denotes the set of outneighbours of a vertex v in $\mathcal{P}_S$, and $d^+_{\mathcal{P}_S}(v) := \left|N^+_{\mathcal{P}_S}(v)\right|$ gives the outdegree of v in $\mathcal{P}_S$.

Definition 4.1. Let $\Phi : \mathbb{N} \to \mathbb{R}$ be a monotone increasing function. A weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ is said to be

- *nicely connected* if $d^+_{\mathcal{P}_S}(j) \geqslant \lfloor \Phi(|\mathcal{P}_S|) \rfloor$ for all $j \in \mathcal{P}$, and
- *nicely weighted* if the vertex and edge weights on G satisfy the following conditions:
  - ⋄ there exist $\beta, B > 0$ satisfying $0 < \beta < B$ such that the vertex weights $w(j)$ satisfy $0 < w(j) \leqslant B$ and $\mathsf{E}\big[w(j)\big|\{w_i\}_{i\neq j}, \{w(k,\ell)\}_{(k,\ell)\in E(\mathcal{P})}\big] = \beta$ for all $j \in \mathcal{P}$, and
  - ⋄ there exist constants $A > 0$ and $\alpha < \beta$ such that for every $(i,j) \in E(\mathcal{P})$, the edge weight $w(i,j) \in [-A, A]$ and $\mathsf{E}\big[w(i,j)\big|\{w_i\}_{i\in\mathcal{P}}, \{w(k,\ell)\}_{(k,\ell)\neq(i,j)}\big] \leqslant \alpha$.

Remarks 1. Let us provide some insights into Definition 4.1:

- The function Φ in Definition 4.1 serves the purpose of quantifying the 'density' of edges in the digraph G in terms of its order. Intuitively it makes sense that the stronger the connectivity of the vertices of $G(\mathcal{P}, E(\mathcal{P}))$ with the set $\mathcal{P}_S \subset \mathcal{P}$, the more likely is it to find contractive cycles on $G(\mathcal{P}, E(\mathcal{P}))$; a quantitative bound on the error probability of how likely it is to find contractive cycles on such graphs is provided by Theorem 4.2. The higher the growth rate of Φ, the stronger is the probabilistic estimate in Theorem 4.2 below.
- By Definition 4.1, every vertex in a nicely connected digraph has at least $\lfloor \Phi(|\mathcal{P}_S|) \rfloor$ asymptotically stable outneighbours. The condition that the vertex and edge weights $w(j)$ and $w(i,j)$ are uniformly bounded if $G(\mathcal{P}, E(\mathcal{P}))$ is nicely weighted is no loss of generality on account of the graph $G(\mathcal{P}, E(\mathcal{P}))$ being finite. However, it is also possible to consider the case in which the bounds on the weights depend on the size of the graph $G(\mathcal{P}, E(\mathcal{P}))$, as explained in Remark 4 below. We stick to the simpler case in order not to blur the message of our result.
- From a switched systems standpoint, the nicely connected property of $G(\mathcal{P}, E(\mathcal{P}))$ describes the richness of admissible switches among the subsystems. The nicely weighted property, in this context, provides quantitative information about the (in)stability of the subsystems at an abstract level in terms of vertex and edge weights. These weights (scalars) are, of course, obtained from Lyapunov-like functions corresponding to the individual subsystems, the choice of which is certainly not unique.
- Note that the size of the graph $G(\mathcal{P}, E(\mathcal{P}))$ is *not* related to the dimension of the system (2); it reflects the internal structure of (2): if there are a large number of substructures in a system with variable structure, that property is captured by a large order of $G(\mathcal{P}, E(\mathcal{P}))$, and the possible transitions between the various structures are captured by the set of directed edges between the subsystems.
- Definition 4.1 does not require independence of the vertex and edge weights. In view of the association of these weights with properties of (2) as elaborated at the beginning of §3, we see that there is no assumption of λ_k's and $\ln \mu_{ij}$'s being independent. This aspect is especially important because, for instance, given $w(i,j)$ (i.e., $\ln \mu_{ij}$), the weight $w(j,i)$ (i.e., $\ln \mu_{ji}$) cannot be independent of $w(i,j)$. In particular, we do not assume any particular probabilistic model (that may be tuned to specific applications,) for the weights; this ensures maximum generality of our results.

4.2. **A cycle-detection algorithm.** We provide the following probabilistic algorithm for detection of cycles in $\mathcal{P}_S$; it will be utilized in Theorem 4.2 below for ***nicely connected and weighted digraphs*** to furnish certain genericity assertions.

Algorithm 1.

Step 1: Set $k = 0$.
Pick $j_k \in \mathcal{P}_S$ uniformly at random.

Step 2: If $N^+_{\mathcal{P}_S}(j_k)\backslash\{j_0, \cdots, j_k\} \neq \emptyset$,
Pick $j_{k+1} \in N^+_{\mathcal{P}_S}(j_k)\backslash\{j_0, \cdots, j_k\}$ uniformly at random.
Set $k = k + 1$.
Go to Step 2.
Else
Pick $j_{k+1} = j_i$ such that $j_i \in N^+_{\mathcal{P}_S}(j_k)$ and $(k - i)$ is maximum.
Go to Step 3.

Step 3: End.

On the digraph $G(\mathcal{P}, E(\mathcal{P}))$ Algorithm 1 generates a walk in the following fashion: At the first step a vertex corresponding to an asymptotically stable system is picked uniformly at random. At every $(k+1)$-th step we identify the subset of outneighbours of the vertex picked at the k-th step such that the vertices correspond to asymptotically stable systems, and they have not been picked till (and including) step k; then a vertex from the above subset is picked uniformly at random. If there is no such outneighbour corresponding to the asymptotically stable vertices that were not picked earlier, the algorithm selects an outneighbour that is at the maximum 'distance' from the current vertex in the already generated sequence, and the algorithm is stopped.

Since we deal with finite digraphs, it is evident that every walk generated by our algorithm is closed. In addition, the mechanism of repeating vertices (as described above) makes this closed walk a cycle. Obviously, the length of the cycles is bounded above by the order of the graph.

Example 1. Consider a nicely connected digraph with the set of vertices

$$\mathcal{P} = \{1, 2, 3, 4, 5\},$$

where

$$\mathcal{P}_S = \{1, 2, 3\} \quad \text{and} \quad \mathcal{P}_U = \{4, 5\},$$

and let the set of directed edges be

$$E(\mathcal{P}) = \big\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 5), (5, 1)\big\}.$$

Let $j_0 = 1 \in \mathcal{P}_S$. Then $j_1 \in \{2, 3\}\backslash\{1\}$. Let $j_1 = 2$. Then $j_2 \in \{1, 3\}\backslash\{1, 2\}$. Consequently, $j_2 = 3$. Now, $\{2\}\backslash\{1, 2, 3\} = \emptyset$. As a result, $j_3 = 2$. So we have obtained the walk

$$1, (1, 2), 2, (2, 3), 3, (3, 2), 2,$$

which contains the cycle $2, (2, 3), 3, (3, 2), 2$.

In the remainder of this section we show that for digraphs satisfying the properties of nice connectivity and weights in Definition 4.1, a cycle obtained from Algorithm 1 satisfies (12) with high probability.

Remark 2. Deterministic algorithms for detecting cycles on weighted digraphs may not be applicable to switched systems whose underlying digraphs are large, especially if their sizes are so large that not all the weights can be kept in memory at once. Such large-scale switched systems are increasingly becoming common in the networked systems, see [11] and the references therein. For such large digraphs our algorithm provides probabilistic guarantees in the spirit of randomized algorithms for detection and synthesis of contractive cycles. In addition, our algorithm has an "online learning" property in the following sense: starting with a rough probabilistic description of the underlying weighted digraph, (i.e., without knowledge of the precise values of the weights,) we explore the digraph and synthesize a cycle during this exploration that is contractive with high probability.

4.3. **Contractive cycles and density of stabilizing infinite walks.** Our first main result is the following proposition that shows that on a nicely connected digraph, a cycle obtained from Algorithm 1 is of length at least $\lfloor \Phi(|\mathcal{P}_S|) \rfloor$.

Proposition 1. *If the weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ is nicely connected, then Algorithm 1 synthesizes a cycle W on $G(\mathcal{P}, E(\mathcal{P}))$ such that all the vertices in W are from $\mathcal{P}_S$ and the length of W is at least $\lfloor \Phi(|\mathcal{P}_S|) \rfloor$.*

Proof. Let

$$W' = j_0, (j_0, j_1), j_1, \cdots, j_{k-1}, (j_{k-1}, j_k), j_k, (j_k, j_i), j_i$$

be a walk obtained from Algorithm 1. Consider the sub-walk

$$W = j_i, (j_i, j_{i+1}), j_{i+1}, \cdots, j_{k-1}, (j_{k-1}, j_k), j_k, (j_k, j_i), j_i,$$

which is a cycle by construction. By Algorithm 1, all the vertices of W are in $\mathcal{P}_S$. We claim that $|W| \geqslant \lfloor \Phi(|\mathcal{P}_S|) \rfloor$. Assume, if possible, that $|W| < \lfloor \Phi(|\mathcal{P}_S|) \rfloor$. But

$$\begin{aligned} |W| &= |j_i, (j_i, j_{i+1}), j_{i+1}, \cdots, j_{k-1}, (j_{k-1}, j_k), j_k| \\ &\quad + |j_k, (j_k, j_i), j_i| \\ &= |j_i, (j_i, j_{i+1}), j_{i+1}, \cdots, j_{k-1}, (j_{k-1}, j_k), j_k| + 1. \end{aligned}$$

By hypothesis, $d^+_{\mathcal{P}_S} \geqslant \lfloor \Phi(|\mathcal{P}_S|) \rfloor$, which implies that

$$\left| N^+_{\mathcal{P}_S} \right| \geqslant \lfloor \Phi(|\mathcal{P}_S|) \rfloor. \tag{13}$$

By the choice of j_i in Algorithm 1,

$$\{j_0, j_1, \cdots, j_{i-1}\} \notin N^+_{\mathcal{P}_S}(j_k). \tag{14}$$

From (13) and (14), it follows that

$$|\{j_i, j_{i+1}, \cdots, j_k\}| \geqslant \left| N^+_{\mathcal{P}_S}(j_k) \right|.$$

But

$$\left| N^+_{\mathcal{P}_S} \right| \geqslant \lfloor \Phi(|\mathcal{P}_S|) \rfloor,$$

which implies that

$$|\{j_i, j_{i+1}, \cdots, j_k\}| \geqslant \lfloor \Phi(|\mathcal{P}_S|) \rfloor,$$

and therefore we reach a contradiction. Consequently, $|W| \geqslant \lfloor \Phi(|\mathcal{P}_S|) \rfloor + 1$. □

The nicely connected property of a digraph, and consequently the lower bound on the length of a cycle obtained from our Algorithm as ascertained by Proposition 1, will be useful in providing a probabilistic estimate of how likely it is that the above cycle is contractive. Of course this likelihood depends on the set of vertex and edge weights associated to the digraph. This matter is addressed in the remainder of this section.

Our second main result is the following theorem:

Theorem 4.2. *Consider the switched system* (2) *and the underlying weighted digraph* $G(\mathcal{P}, E(\mathcal{P}))$ *as described in* §3. *Suppose that* $G(\mathcal{P}, E(\mathcal{P}))$ *is nicely connected and nicely weighted. Then a cycle of length at least* $\lfloor \Phi(|\mathcal{P}_S|)\rfloor$ *on* $G(\mathcal{P}, E(\mathcal{P}))$ *obtained from Algorithm* 1 *is contractive with probability at least* $1-\exp\left(-\frac{1}{2}\left(\frac{\alpha-\beta}{A+B}\right)^2 \lfloor \Phi(|\mathcal{P}_S|)\rfloor\right)$.

Proof. Since the given weighted digraph $G(\mathcal{P}, E(\mathcal{P}))$ is nicely connected, by Lemma 1 there exists a cycle on $G(\mathcal{P}, E(\mathcal{P}))$ with all vertices of the cycle being in $\mathcal{P}_S$ and the length of the cycle is at least $\lfloor \Phi(|\mathcal{P}_S|)\rfloor$. Such a cycle can be constructed by Algorithm 1.

Consider a cycle $W = j_0, (j_0, j_1), j_1, \cdots, j_{n-1}, (j_{n-1}, j_0), j_0$ of length exactly $\lfloor \Phi(|\mathcal{P}_S|)\rfloor = n$ (say). Since $\{j_0, j_1, \cdots, j_{n-1}\} \in \mathcal{P}_S$, (12) can be written as

$$\sum_{k=1}^{n} w(j_{k-1}, j_k) - \sum_{k=0}^{n} w(j_k) < 0.$$

Let

$$X_n \coloneqq \sum_{k=1}^{n} w(j_{k-1}, j_k) - \sum_{k=0}^{n} w(j_k). \tag{15}$$

Define the filtration $(\mathfrak{F}_m)_{m=0}^n$ by

$$\mathfrak{F}_m = \sigma\big\{w(j_{k-1}, j_k), w(j_\ell) \,\big|\, k = 1, \cdots, m,\ \ell = 0, \cdots, m\big\}.$$

Since G is nicely weighted,

$$\begin{aligned} \mathsf{E}^{\mathfrak{F}_{k-1}}[X_k] &= X_{k-1} + \mathsf{E}^{\mathfrak{F}_{k-1}}[w(j_{k-1}, j_k) - w(j_k)] \\ &\leqslant X_{k-1} + \alpha - \beta \\ &< X_{k-1} \quad \text{since } \alpha < \beta \text{ by Definition 4.1.} \end{aligned} \tag{16}$$

Let $(X_m)_{m=0}^n \coloneqq (\xi_0 + M_m + A_m)_{m=0}^n$ denote the a.s. unique Doob decomposition [3, Theorem 5.2.10] of the process $(X_m)_{m=0}^n$ with $(M_m)_{m=0}^n$ as the martingale and $(A_m)_{m=0}^n$ as the compensator. In other words, with $M_0 \coloneqq 0$ and $A_0 \coloneqq 0$, we have $\xi_0 \coloneqq X_0$, and for $m = 1, \ldots, n$,

$$M_m \coloneqq \sum_{k=1}^{m} \big(X_k - \mathsf{E}^{\mathfrak{F}_{k-1}}[X_k]\big), \quad A_m \coloneqq \sum_{k=1}^{m} \big(\mathsf{E}^{\mathfrak{F}_{k-1}}[X_k] - X_{k-1}\big).$$

The inequality (16) shows that $(X_k)_{k=0}^n$ is an $(\mathfrak{F}_k)_{k=0}^n$ strict supermartingale; the compensator process $(A_k)_{k=0}^n$ is, therefore, strictly decreasing.

The definition of ξ_0 shows that $\xi_0 \leqslant 0$, and from (16) we get $A_n \leqslant (\alpha - \beta)n$. Since

$$\begin{aligned} \mathsf{P}(X_n > 0) &= \mathsf{P}(\xi_0 + M_n + A_n > 0) \\ &\leqslant \mathsf{P}(M_n + A_n > 0) \\ &\leqslant \mathsf{P}(M_n > -n(\alpha - \beta)) \\ &= \mathsf{P}\bigg(\frac{M_n}{A+B} > \frac{-n(\alpha-\beta)}{A+B}\bigg), \end{aligned}$$

we apply Azuma's inequality [18, p. 92] to the zero-mean martingale process $(M_m)_{m=0}^n$ to get

$$\mathsf{P}(X_n > 0) \leqslant \mathsf{P}\bigg(\frac{M_n}{A+B} > \Big(\frac{-(\alpha-\beta)\sqrt{n}}{A+B}\Big)\sqrt{n}\bigg)$$

$$\leqslant \exp\left(-\frac{1}{2}\left(\frac{(\alpha-\beta)\sqrt{n}}{A+B}\right)^2\right),$$

which gives the estimate in the theorem. ☐

Remark 3. Theorem 4.2 asserts that a cycle obtained via Algorithm 1 is contractive with high probability provided $|\mathcal{P}_S|$ is large. Consequently, repeating such a cycle derived from Algorithm 1 generates an infinite walk W that, in view of Lemma 3.3, satisfies (10). This in turn identifies a class of switched systems (whose underlying weighted digraph G is nicely connected and nicely weighted) that admits switching signals satisfying the conditions proposed in [9, Theorem 1] with overwhelming probability.

Remark 4. The primary engine leading to the estimate in Theorem 4.2 is Azuma's inequality. Our assumption of a uniform bound for the weights due to G being nicely weighted led to a uniform bound on the martingale increments $(M_m - M_{m-1})_{m=1}^n$ in the proof of Theorem 4.2, and our estimate followed at once from Azuma's inequality. A more general version of Azuma's inequality may be employed in an identical fashion to cater to vertex- and edge-dependent weights, leading to a possibly sharper bound. The numerical value of the confidence with which a contractive cycle may be found, however, depends on the size of $\mathcal{P}_S$ and the ability of the function Φ in Definition 4.1 to dominate the accumulation of the weights along the martingale increments.

Theorem 4.2 gives a recipe — via Algorithm 1 — for constructing a contractive cycle with high probability. Armed with this recipe, infinite stabilizing walks, i.e., stabilizing switching signals for (2), can be easily constructed. One such simple construction consists of concatenating a contractive cycle indefinitely many times. However, stabilizing infinite walks consisting of repetitions of contractive cycles are not the only infinite walks on $G(\mathcal{P}, E(\mathcal{P}))$; indeed, under mild conditions, there exist uncountably many stabilizing infinite walks on the digraph. Our final result establishes these facts:

Theorem 4.3. *Consider the switched system* (2) *and the underlying weighted digraph* $G(\mathcal{P}, E(\mathcal{P}))$ *as described in §3. Suppose that* $G(\mathcal{P}, E(\mathcal{P}))$ *is nicely connected and nicely weighted.*

- *With probability at least* $1-\exp\Bigl(-\frac{1}{2}\bigl(\frac{\alpha-\beta}{A+B}\bigr)^2\lfloor\Phi(|\mathcal{P}_S|)\rfloor\Bigr)$ *there exists a stabilizing infinite walk on* $G(\mathcal{P}, E(\mathcal{P}))$*.*
- *If, in addition,* $G(\mathcal{P}, E(\mathcal{P}))$ *has a strongly connected subgraph that contains* $\mathcal{P}_S$ *as a strict subset, then there exists an uncountable family of infinite stabilizing walks on* $G(\mathcal{P}, E(\mathcal{P}))$ *with probability at least* $1-\exp\Bigl(-\frac{1}{2}\bigl(\frac{\alpha-\beta}{A+B}\bigr)^2\lfloor\Phi(|\mathcal{P}_S|)\rfloor\Bigr)$*.*

Proof. Concerning the first assertion, observe that cycles constructed via Algorithm 1 are contractive with probability at least $1-\exp\left(-\frac{1}{2}\left(\frac{\alpha-\beta}{A+B}\right)^2\lfloor\Phi(|\mathcal{P}_S|)\rfloor\right)$, and repeating such a cycle W generates an infinite stabilizing walk in view of Lemma 3.3.

To see the second assertion, we first generate a cycle $W \subset \mathcal{P}_S$ via Algorithm 1; W is contractive with probability at least $1-\exp\left(-\frac{1}{2}\left(\frac{\alpha-\beta}{A+B}\right)^2\lfloor\Phi(|\mathcal{P}_S|)\rfloor\right)$. We then follow the steps below (see Figure 1):

- We partition the graph $G = G(\mathcal{P}, E(\mathcal{P}))$ into its strongly connected components $\{\mathcal{C}'_i\}_{i=1}^{\ell}$, where ℓ is a positive integer equal to the number of strongly connected components of G. This decomposition can be achieved in linear (in the size and order of G) time using standard algorithms [2, §22.5] derived from depth-first search.
- We construct a digraph $G' = G'(V', E')$ as follows: the set V' consists of the family $\{\mathcal{C}'_i\}_{i=1}^{\ell}$, and the set of directed edges E' contains every pair $(\mathcal{C}'_i, \mathcal{C}'_j)$ such that there exists a directed edge from any vertex of $\mathcal{C}'_i$ into $\mathcal{C}'_j$ in $E(\mathcal{P})$. The graph G' is acyclic by construction.
- We perform a topological sort [2, pp. 613-614] on the acyclic digraph G', leading to an ordered family $(\mathcal{C}_1, \ldots, \mathcal{C}_\ell)$ of the vertices of G' (here the sets $\{\mathcal{C}_i\}_{i=1}^{\ell}$ and $\{\mathcal{C}'_j\}_{j=1}^{\ell}$ are identical, of course). By construction, each $\mathcal{C}_i$ is a strongly connected subgraph of G, and there is no directed edge from $\mathcal{C}_{i+1}$ into $\mathcal{C}_i$ for any $i = 1, \ldots, \ell - 1$ in $E(\mathcal{P})$.
- Since G is nicely connected and there can exist no directed edge from $\mathcal{C}_{i+1}$ into $\mathcal{C}_i$ for each $i = 1, \ldots, \ell - 1$, it follows that $\mathcal{P}_S \subset \mathcal{C}_\ell$. Moreover, since $\mathcal{P}_S \subsetneq \mathcal{C}_\ell$ by hypothesis, there exists $w \in \mathcal{C}_\ell \setminus \mathcal{P}_S$.

Pick two vertices $v, v' \in W$. By strong connectivity of $\mathcal{C}_\ell$, there exists a directed path from v to w and another directed path from w to v'; we denote the concatenation of these two paths, in that order, by W'. Let the shortest (as measured with respect to the undirected graph obtained from G by eliminating the directions from the edges) directed segment of W from v' to v be denoted by W''. The concatenation of the directed paths W' and W'', in that order, is a (finite) closed walk starting from $v \in W$; we denote this closed walk by $\widetilde{W}$. By definition of $\mathcal{C}_\ell$ it follows that $\widetilde{W} \subset \mathcal{C}_\ell$, but $\widetilde{W}$ is not necessarily contractive. However, since $\Xi'(W) < 0$, it follows that there exists $m \in \mathbb{N}$ such that $m\Xi'(W) + \Xi'(\widetilde{W}) < 0$, where Ξ' is the function defined in (12). Let the closed walk generated by traversing m times the walk W starting at v, followed by a single traversal of $\widetilde{W}$ and terminating at v be denoted by $\overline{W}$. Consider the set $\mathcal{S}$ of infinite walks on G, starting from v and obtained by concatenating $\overline{W}$ and W in arbitrary orders. Elementary calculations show that (9) and (10) hold for each such infinite walk, showing in turn that the corresponding switching signals are stabilizing. It remains to show that the set $\mathcal{S}$ is also uncountable, for which it suffices to recall that the set of words of infinite length constructed out of two distinct letters is uncountable. □

Remark 5. Moving away from the graph-theoretic terminology, we note that in the context of (2):

- The first part of Theorem 4.3 asserts that with high probability there exists a stabilizing periodic switching signal during which only subsystems from $\mathcal{P}_S$ are activated.
- The second part of Theorem 4.3 asserts that with high probability uncountably many stabilizing switching signals exist.
- The proof of the second part will demonstrate two features:
  - that these stabilizing switching signals may be aperiodic, i.e., Theorem 4.3 does *not* restrict attention to just periodic switching signals with the stabilizing property; and
  - that the stabilizing switching signals in Theorem 4.3 may venture out of $\mathcal{P}_S$ into $\mathcal{P}_U$.

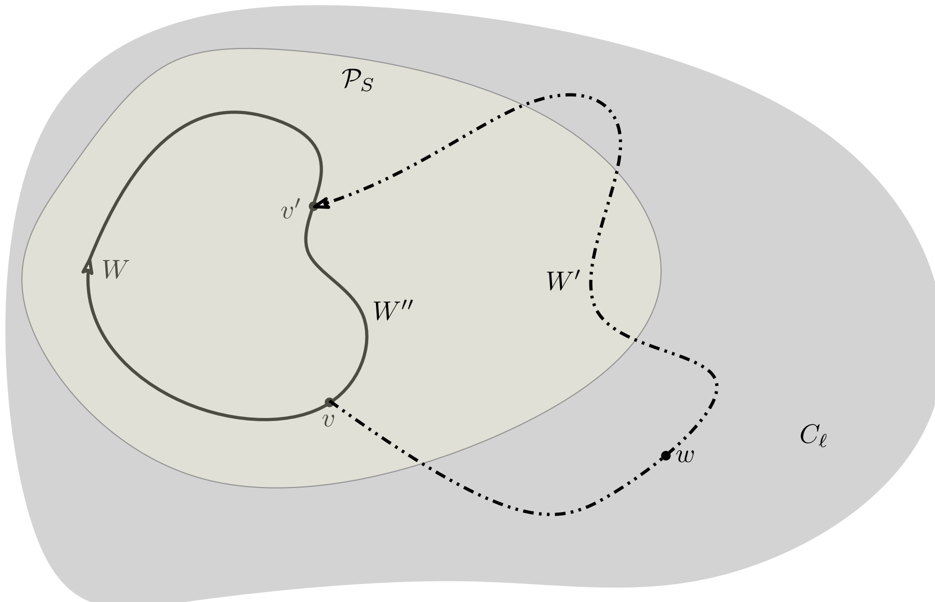

FIGURE 1. An illustration of the steps in the Proof of Theorem 4.3.

Theorem 4.2 and Theorem 4.3 provide the details of Solution 1.

To summarize, in this article we tackled the question of how likely is it for a "generic" switched system to admit a switching signal that satisfies the conditions proposed in [9, Theorem 1]. We associated a weighted digraph to a switched systems. A switched system admitting the stabilizing switching signals under consideration was identified in terms of connectivity and vertex and edge weights of this underlying weighted digraph. The connectivity associates to the admissible switches, and the weights associates (at a level of abstraction in terms of Lyapunov functions) to the (in)stability of the individual subsystems and the gain/loss caused by switching.

5. Numerical example.

In this section we provide a numerical example to demonstrate our Theorem 4.2. Consider a nicely connected and nicely weighted digraph G with

- $|\mathcal{P}_S| = 1000$, $\Phi(r) = \frac{1}{10}\sqrt{r}$, $d^+(j) = \lfloor \Phi(|\mathcal{P}_S|) \rfloor$ for all $j \in \mathcal{P}$, and
- $A = 2.5$, $B = 5$, $\alpha = 0$ and $\beta = 2.5$.

We extract and fix a cycle W obtained from Algorithm 1 on $\mathcal{P}_S \subset \mathcal{P}$. The vertex and edge weights on W are sampled uniformly at random 1000 times from the intervals as stipulated in Definition 4.1. We calculate X_n defined in (15) empirically for n being the length of the cycle W.

The above experiment is repeated for cycles of different length n obtained from Algorithm 1 with uniformly randomly selected initial vertex. We plot the empirical probability of $\{X_n < 0\}$ vs length n of the cycle in Figure 2.

Observe that the synthesis of a contractive cycle from Algorithm 1 does not require a priori knowledge of the vertex and edge weights of G. It is evident from this example as we first fix a cycle W and then select weights from a specified

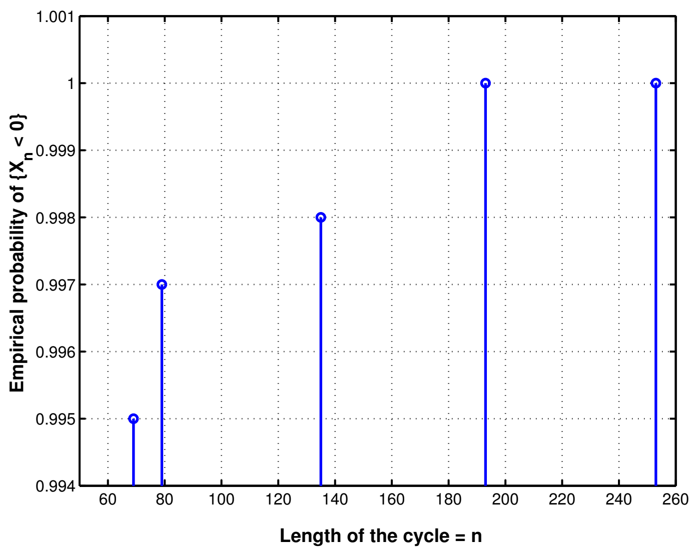

FIGURE 2. Plot for the empirical probability of a cycle being contractive against its length n with $\Phi(r) = \dfrac{1}{10}\sqrt{r}$.

interval. This is not the case with deterministic negative cycle synthesis algorithms, which require complete knowledge of the vertex and edge weights of G prior to their application. In addition, the weights are sampled uniformly at random 1000 times and we find high empirical probability for $\{X_n < 0\}$. This highlights the feature that even if the systems in the given family are prone to evolve over time, our algorithm provides uniform probabilistic guarantees.

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