System of unbiased representatives for a collection of bicolorings

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Abstract

Let $\mathcal{B}$ denote a set of bicolorings of $[n]$, where each bicoloring is a mapping of the points in $[n]$ to $\{-1, +1\}$. For each $B \in \mathcal{B}$, let $Y_B = (B(1), \ldots, B(n))$. For each $A \subseteq [n]$, let $X_A \in \{0, 1\}^n$ denote the incidence vector of A. A non-empty set A is said to be an 'unbiased representative' for a bicoloring $B \in \mathcal{B}$ if $\langle X_A, Y_B \rangle = 0$. Given a set $\mathcal{B}$ of bicolorings, we study the minimum cardinality of a family $\mathcal{A}$ consisting of subsets of $[n]$ such that every bicoloring in $\mathcal{B}$ has an unbiased representative in $\mathcal{A}$.

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1. Introduction

Let $\mathcal{B}$ denote a set of bicolorings of $[n] = \{1, \ldots, n\}$, where each bicoloring $B \in \mathcal{B}$ maps each point $x \in [n]$ to either -1 or +1. Let Y_B denote the n-dimensional vector representing the bicoloring B, i.e. $Y_B = (B(1), \ldots, B(n))$. A non-empty set $A \subseteq [n]$ is said to be an *unbiased representative* for a bicoloring $B \in \mathcal{B}$ if $\langle X_A, Y_B \rangle = 0$, where X_A denotes the 0-1 n-dimensional incidence vector corresponding to A. We call a family $\mathcal{A}$ of subsets of $[n]$ a *system of unbiased representatives* (or 'SUR') for $\mathcal{B}$ if for every bicoloring $B \in \mathcal{B}$, there exists at least one set $A \in \mathcal{A}$ such that $\langle X_A, Y_B \rangle = 0$. Note that the two monochromatic bicolorings can never have any unbiased representatives - we call these bicolorings 'trivial'. Let $\gamma(\mathcal{B})$ denote the minimum cardinality of a system of unbiased representatives for $\mathcal{B}$. We define the maximum of $\gamma(\mathcal{B})$ over all possible families $\mathcal{B}$ of non-trivial bicolorings of $[n]$ as $\gamma(n)$. Note that no singleton set of $[n]$ is a member of any optimal system of unbiased representatives.

Unbiased representatives are useful in testing products such as drugs over a large population where the effectiveness (or side-effect) of a new drug is studied in correlation with a large set of patient attributes such as body weight, height, age, etc. Complementary extremes in the attributes, such as being obese or underweight, tall or short, and young or old, are relevant in such correlation studies. Such studies require patients with complementary ranges of values of a certain attribute to be present in equal (or roughly equal) numbers in the representative group for that attribute – such a group may be deemed to be an unbiased representative for the attribute. However, selecting a separate sample of individuals for each attribute having equal representation of the complementary traits is practically impossible. So, one needs to select a family $\mathcal{A}$ of samples of individuals such that for any attribute B, there exists a sample $A \in \mathcal{A}$ which has an

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equal representation of individuals from the complementary traits of B. It is in the best interest to choose a family $\mathcal{A}$ of such groups of representatives of the smallest possible cardinality. It is not hard to see the direct mapping of this problem to the problem addressed in this paper. In a generic setting, SURs are useful in various applications where a collection of items (like individual patients) have many attributes (like weight, height and age), where the objective is to form a small collection of subsets of items with almost equal representation of opposite or complementary traits for each attribute.

1.1. Definitions and notations

We use 'SUR' to denote the phrase 'system of unbiased representatives'. For integers n and p, let $[n]$ denote the set $\{1, \ldots, n\}$, and $[n \pm p]$ denote the set $\{n - p, n - p + 1, \ldots, n + p\}$. A bicoloring B of $[n]$ is called a k-bicoloring if the number of $+1$'s in B is exactly k. For a bicoloring $B : [n] \to \{-1, 1\}$, we use $B(+1)$ (respectively, $B(-1)$) to denote the set of points receiving color $+1$ (respectively, -1) under B. We use Y_B (X_A) to denote the n-dimensional ± 1 vector (respectively, 0-1 vector) representing the bicoloring B (respectively, $A \subseteq [n]$), i.e. $Y_B = (B(1), \ldots, B(n))$. Note that $\langle Y_B, X_A \rangle = 0$ for some $A \in \binom{[n]}{r}$ implies that that r is even. Throughout the rest of the paper, we consider only the non-trivial bicolorings and assume that every set in a SUR is of even cardinality.

Let $\gamma(\mathcal{B}, r)$ (respectively, $\gamma(\mathcal{B}, [r_1, r_2])$) be the minimum cardinality of a SUR $\mathcal{A}$ for $\mathcal{B}$, where each $A \in \mathcal{A}$ is an r-sized (respectively, at least r_1-sized and at most r_2-sized) subset of $[n]$. We define $\gamma(n, k, r)$ ($\gamma(n, [k_1, k_2], [r_1, r_2])$) as follows.

$$\gamma(n, k, r) = \max_{\mathcal{B}} \gamma(\mathcal{B}, r), \text{where } \mathcal{B} \text{ consists of only } k\text{-bicolorings.}$$

$$\gamma(n, [k_1, k_2], [r_1, r_2]) = \max_{\mathcal{B}} \gamma(\mathcal{B}, [r_1, r_2]), \text{where } B \in \mathcal{B} \text{ has } k_1 \le |B(+1)| \le k_2.$$

Note that $\gamma(\mathcal{B}, [2, n])$ is the same as $\gamma(\mathcal{B})$. Since no singleton set of $[n]$ can be a member of any optimal system of unbiased representative and the monochromatic bicolorings, consisting of exactly zero (or n) $+1$'s, are trivial, $\gamma(n, [1, n-1], [2, n])$ is the same as $\gamma(n)$.

1.2. Relation to existing works

Given a family $\mathcal{F}$ of subsets of $[n]$, finding another family $\mathcal{F}'$ with certain properties in relation with $\mathcal{F}$ has been well investigated. One of the most studied problems in this direction is the computation of *separating families*(see [1]). Let $\mathcal{F}$ consist of pairs $\{i, j\}$, $i, j \in [n]$, $i \ne j$ and $\mathcal{S}$ be another family of subsets on $[n]$. A subset S separates a pair $\{i, j\}$ if $i \in S$ and $j \notin S$ or vice versa. The family $\mathcal{S}$ is a separating family for $\mathcal{F}$ if every pair $\{i, j\} \in \mathcal{F}$ is separated by some $S \in \mathcal{S}$ (see [1, 2, 3, 4, 5] for detailed results and related problems on separating families). Separating families have many applications like 'Wasserman-type' blood tests of large populations, diagnosis and chemical analysis, locating defective items, etc (see [6]). An extension of the separating family problem is the 'test cover' problem: "Given a family $\mathcal{T}$ of subsets of $[n]$, finding a sub-collection $\mathcal{S} \subseteq \mathcal{T}$ of minimum cardinality such that every pair of $[n]$ is separated by some $S \in \mathcal{S}$". The test cover problem is studied in the context of drug testing, biology [7, 8, 9] and pattern recognition [10]. For results and related notions, see [11, 12, 13, 14, 15]. In the above problems, any two sized set $F = \{i, j\}$ can be viewed as a partial bicoloring $\chi : [n] \to \{-1, 0, 1\}$ where $\chi(i) = -1$, $\chi(j) = +1$, and $\chi(p) = 0$ for any $p \in [n] \setminus \{i, j\}$ and a set S covers F if and only if $\langle X_S, Y_\chi \rangle \in \{-1, +1\}$.

An affine hyperplane is a set of vectors $H(a, b) = \{x \in \mathbb{R}^n : \langle a, x \rangle = b\}$, where $a \in \mathbb{R}^n$ is a nonzero vector, $b \in \mathbb{R}$. Covering the $\{0, 1\}^n$ Hamming cube with the minimum number of affine hyperplanes has been well studied - a point $x \in \{0, 1\}^n$ is said to be *covered* by a hyperplane $H(a, b)$ if $\langle a, x \rangle = b$ (see [16, 17, 18]). It is not hard to see that any SUR for the $2^n - 2$ non-trivial bicolorings is a covering for all the points of the $\{-1, 1\}^n$ Hamming cube, except $\{(-1, \ldots, -1), (1, \ldots, 1)\}$, by hyperplanes $H(a, b)$ satisfying (i) $a \in \{0, 1\}^n$ and (ii) $b = 0$.

The problem addressed in this paper can be viewed as an inverse of the problem of *bisecting families*[19]. Let $\mathcal{A}$ be a family of subsets of $[n]$. Another family $\mathcal{B}$ of subsets of $[n]$ is called a *bisecting family* for $\mathcal{A}$, if for

each $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ such that $|A \cap B| \in \{\lceil \frac{|A|}{2} \rceil, \lfloor \frac{|A|}{2} \rfloor\}$. In the bicoloring terminology, let X_A and Y_B denote the $(0,1)$ and $(-1,+1)$ incidence vectors corresponding to sets A and B, respectively, where $X_A(i) = 1$ ($Y_B(i) = 1$) if and only if $i \in A$ (respectively, $i \in B$). If $\mathcal{B}$ is a *bisecting family* for $\mathcal{A}$, then for each $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ such that $|\langle X_A, Y_B \rangle| \le 1$. Apart from the similarity in the formulations, there is no apparent connection between the problems of bisecting families and SURs either in the bounds obtained or the techniques used in obtaining the bounds.

1.3. Summary of results

The paper is divided into three logical sections. The first section (Section 2) focuses on obtaining $O(\log |\mathcal{B}|)$ upper bounds for SURs when (i) the collection $\mathcal{B}$ of bicolorings is unrestricted or has minor restrictions, and (ii) the sets in the SURs are unrestricted or have minor restrictions. When $\mathcal{B}$ consists of all the $2^n - 2$ non-monochromatic bicolorings, it is not difficult to show that $\frac{n}{2} \le \gamma(\mathcal{B}, [2, n]) \le n - 1$. Using an application of Combinatorial Nullstellensatz [20], we improve the above lower bound to $n - 1$.

Theorem 1 *Let n be a positive integer and $k \in [n]$. Then, $\gamma(n, [1, n-k], [2, n]) = n - 1$, where $1 \le k \le \lceil \frac{n}{2} \rceil$.*

We relate the problem of SUR to the hitting set problem, which in turn implies relations with 'VC-dimension' provided $\epsilon n \le |B(+1)| \le (1 - \epsilon)n$ for each $B \in \mathcal{B}$. For such families $\mathcal{B}$, this relationship assists in establishing an $O(\log |\mathcal{B}|)$ upper bound for cardinalities of any optimal SUR. Under a similar restriction for each $B \in \mathcal{B}$, if it is mandatory that each set in the SUR is of cardinality exactly r, the best upper bound obtained is large ($\Omega(e^r \log |\mathcal{B}|)$). In order to establish an $\ln |\mathcal{B}|$ upper bound for the size of an optimal SUR under this restriction, we relax the cardinality restriction on the sets in $\mathcal{A}$ and permit some tolerance in $|\langle Y_B, X_A \rangle|$ to obtain the following theorem.

Theorem 2 *Let $r' \in [r \pm \lceil \frac{r}{2} \rceil]$, where $r \ge 8$ is an integer. Let $\mathcal{B}$ denote the set of all bicolorings $B \in \{-1, +1\}^n$, where $||B(+1)| - |B(-1)|| \le d$, for some $d \in \mathbb{N}$. Then, with high probability, one can construct a family $\mathcal{A}$ of cardinality at most $\ln |\mathcal{B}|$ in $O(n|\mathcal{B}| \ln |\mathcal{B}|)$ time consisting of r'-sized subsets such that for every $B \in \mathcal{B}$, there exists a set $A \in \mathcal{A}$ with $|\langle Y_B, X_A \rangle| \le e\sqrt{r} + \frac{dr}{n}$.*

In the second part of the paper (Section 3), we study the SUR problem where each $B \in \mathcal{B}$ is restricted to have exactly k +1's and each set in the SUR is required to be of cardinality exactly r, for some $r, k \in [n]$, $2 \le r \le 2k$. We relate the SUR problem under such restrictions to 'covering' problems, that enables us to use a deterministic algorithm of Lovász [21] and Stein [22] to compute such a SUR in polynomial time. In particular, for sufficiently large values of n, and $k \le \log_4(\log n)^{1-\epsilon}$ for some $0 < \epsilon < 1$, we use a result of Alon et al. [23, Corollary 1.3] to establish the following asymptotically tight bound on $\gamma(n, k, 2k)$.

Theorem 3 *For sufficiently large values of n,*

$$\frac{\binom{n}{k}}{\binom{2k}{k}} \le \gamma(n, k, 2k) \le \frac{\binom{n}{k}}{\binom{2k}{k}}(1 + o(1)),$$

provided $k \le \log_4(\log n)^{1-\epsilon}$, for any $0 < \epsilon < 1$.

The problem of estimation of $\gamma(n, k, r)$ becomes interesting when $k = \frac{n}{2}$ - the reduction to coverings gives a lower and upper bound of $\max\left(\lceil \frac{n}{2r} \rceil, c_1 \sqrt{\frac{r(n-r)}{n}}\right)$ and $O(n\sqrt{\frac{r(n-r)}{n}})$, respectively. For $r = f(n)$, where $f(n)$ is an increasing function in n, this establishes only sub-linear lower bounds for $\gamma(n, \frac{n}{2}, r)$. We use a vector space orthogonality argument combined with a theorem of Keevash and Long [24] to obtain a linear lower bound on $\gamma(n, k, r)$ under certain restrictions on n, k and r.

Theorem 4 *Let $r = 2c$ for any odd integer $c \in \{1, \ldots, \frac{n}{2}\}$. Let k be an even integer, where $\epsilon n < k < (1 - \epsilon)n$ for some $0 < \epsilon < 0.5$. Then, $\gamma(n, k, r) \ge \delta n$, where $\delta = \delta(\epsilon)$ is some real positive constant.*

Combined with an upper bound construction given in Lemma 22, this establishes an asymptotically tight bound for $\gamma(n, \frac{n}{2}, \frac{n}{2})$, when $\frac{n}{2} \equiv 2 \pmod 4$.

In the third part of the paper (Section 4), we obtain the following inapproximability result for computing optimal SURs by using a result of Dinur and Steurer [25] on the inapproximability of the hitting set problem.

Theorem 5 *Let n and m be integers and let $r \leq (1 - \Omega(1))\frac{\ln m}{4}$. Then, no deterministic polynomial time algorithm can approximate the system of unbiased representative problem for a family of m bicolorings on $[n]$ to within a factor $(1 - \Omega(1))\frac{\ln m}{4r}$ of the optimal when each set chosen in the representative family is required to have its cardinality at most r, unless P=NP.*

2. When cardinalities of sets in the 'SUR' are unrestricted or semi-restricted

In this section, we proceed with the natural motivation of identifying various cases where a SUR of small size exists for a given family of bicolorings. More specifically, given a family of bicoloring $\mathcal{B}$, we show that there exists a SUR of size $O(\log |\mathcal{B}|)$ when (a) $|\mathcal{B}|$ is large (Section 2.1), (b) every bicoloring in $\mathcal{B}$ has sufficient number of +1's and -1's, (Section 2.2), and (c) each set in the SUR is sufficiently large but we tolerate a controlled amount of bias (Section 2.3).

2.1. Bounds on $\gamma(n, [k, n-1], [2, n])$

Recall that $\gamma(n) = \max_{\mathcal{B}} \gamma(\mathcal{B}, [2, n])$, where $\gamma(\mathcal{B}, [2, n])$ is the cardinality of an optimal system of unbiased representative for $\mathcal{B}$. Observe that $\gamma(\mathcal{B}_1) \leq \gamma(\mathcal{B}_2)$ when $\mathcal{B}_1 \subseteq \mathcal{B}_2$. So, to establish bounds on $\gamma(n)$, it suffices to consider the set of all the $2^n - 2$ non-monochromatic bicolorings as $\mathcal{B}$ and establish bounds on $\gamma(\mathcal{B}, [2, n])$. We have the following proposition.

Proposition 6 *Let n be a positive integer and $k \in [n]$.*

(i) $\gamma(n, [k, n-1], [2, n]) = \gamma(n, [1, n-k], [2, n])$.

(ii) $\gamma(n, [1, n-k], [2, n]) = \gamma(n, [1, \lfloor \frac{n}{2} \rfloor], [2, n])$, *for any $1 \leq k \leq \lceil \frac{n}{2} \rceil$.*

(iii) $\gamma(n, [1, n-k], [2, n]) \leq n - 1$, *for $1 \leq k \leq n$.*

(iv) $\frac{n}{2} \leq \gamma(n, 1, [2, n]) \leq \gamma(n, [1, n-k], [2, n])$, *for $1 \leq k \leq n - 1$.*

PROOF. (i) For any k-bicoloring B, any unbiased representative A for B is also an unbiased representative for the bicoloring B', where $B'(+1) = B(-1)$ and $B'(-1) = B(+1)$.
(ii) The proof follows from the proof of Statement (i) in Proposition 6.
(iii) Let $\mathcal{B}$ denote the set of all the $2^n - 2$ non-monochromatic bicolorings. It is not hard to see that $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\}$ is an SUR of cardinality $n - 1$ for $\mathcal{B}$.
(iv) Let $\mathcal{B} = \{B | |B(+1)| = 1\}$. So, $|\mathcal{B}| = n$. For any $B \in \mathcal{B}$, if for any $A \subseteq [n]$, $\langle Y_B, X_A \rangle = 0$, then $|A| = 2$. Moreover, for any $A \in \binom{[n]}{2}$, exactly two $B \in \mathcal{B}$ has $\langle Y_B, X_A \rangle = 0$. So, we need at least $\frac{n}{2}$ two sized sets to form a SUR for $\mathcal{B}$. The second inequality follows from containment. □

In the construction leading to the proof of Statement (iii) in Proposition 6, only two-sized sets are used as unbiased representatives. We have a slightly non-trivial construction assuming $n = 2^p$, for some integer p, giving similar bounds in Appendix A.

To establish a tight lower bound on $\gamma(n, [1, \lceil \frac{n}{2} \rceil], [2, n])$ ($\gamma(n, [1, n-1], [2, n])$), we make use of Combinatorial Nullstellensatz [26] stated below to establish Lemma 8.

Theorem 7 (Combinatorial Nullstellensatz) *[26] Let $\mathbb{F}$ be a field and $f \in \mathbb{F}[x_1, \ldots, x_n]$ be some polynomial of degree d. Let the coefficients of the term $x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$ be non-zero and $t_1 + \ldots + t_n = d$. If $S_1, \ldots, S_n$ are finite sets with each $|S_i| \geq t_i + 1$, then there exists a point $X \in S_1 \times \cdots \times S_n$ such that $f(X)$ is nonzero.*

Using Theorem 7, we prove the following simple lemma.

Lemma 8 *Let $F \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial and $S_1, \ldots, S_n$ be non-empty subsets of $\mathbb{F}$, for some field $\mathbb{F}$. If F vanishes on all but one point $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n \subseteq \mathbb{F}^n$, then $deg(F) \geq \sum_{i=1}^{n}(|S_i| - 1)$.*

PROOF. For the sake of contradiction, assume that $\deg(F) < \sum_{i=1}^{n}(|S_i| - 1)$. Consider the following polynomials.

$$H_i(x_i) = \prod_{s \in S_i \setminus \{s_i\}} (x_i - s).$$

$$G(x_1, \ldots, x_n) = \prod_{i=1}^{n} H_i(x_i).$$

Note that $\deg(G)$ is $\sum_{i=1}^{n}(|S_i| - 1)$. Let $F(s_1, \ldots, s_n) = c_1$ and $G(s_1, \ldots, s_n) = c_2$. Note that $c_2 \neq 0$ since none of the H_i's vanish at this point. Then, the polynomial $c_2 F - c_1 G$ vanishes on all points of $S_1 \times \cdots \times S_n$. However, $c_2 F - c_1 G$ has degree $\sum_{i=1}^{n}(|S_i| - 1)$: the monomial $x_1^{|S_1|-1} \cdots x_n^{|S_n|-1}$ has $-c_1$ as its coefficient. Using Combinatorial Nullstellensatz [26], there exists at least one point in $S_1 \times \cdots \times S_n$ where $c_2 F - c_1 G$ is non-zero which is a contradiction. $\square$

Proof of Theorem 1

Statement of Theorem 1. Let n be a positive integer and $k \in [n]$. Then, $\gamma(n, [1, n-k], [2, n]) = n - 1$, where $1 \leq k \leq \lceil \frac{n}{2} \rceil$.

PROOF. From Statements (ii) and (iii) of Proposition 6, we know that in order to prove Theorem 1, we only need to establish a lower bound of $n - 1$ for $\gamma(n, [1, n-1], [2, n])$.

Let $\mathcal{B}$ denote the set of all the $2^n - 2$ non-monochromatic bicolorings of $[n]$. Let $\mathcal{A}$ be a SUR of minimum cardinality for $\mathcal{B}$. Let Y_B (X_A) denote the n-dimensional ± 1 vector (respectively, 0-1 vector) representing the bicoloring B (respectively, $A \subseteq [n]$). Consider the polynomial $P(y)$, where $y = (y_1, \ldots, y_n)$ is a vector with each $y_i \in \{-1, +1\}$.

$$P(y) = \prod_{A \in \mathcal{A}} \langle X_A, y \rangle. \tag{1}$$

From the definition of $\mathcal{A}$, $P(Y_B)$ vanishes on all non-trivial bicolorings B of $[n]$. Now, consider the following polynomial $P'(x)$, where $x = (x_1, \ldots, x_n)$ is a vector with each $x_i \in \{0, 1\}$.

$$P'(x = (x_1, \ldots, x_n)) = P(1 - 2x_1, \ldots, 1 - 2x_n)(x_1 + \ldots + x_n - n). \tag{2}$$

$P'(x_1, \ldots, x_n)$ vanishes at every point in $\{0, 1\}^n$ except at the point $(0, \ldots, 0)$: $P(1 - 2x_1, \ldots, 1 - 2x_n)$ vanishes at every point in $\{0, 1\}^n$ except the two points $(0, \ldots, 0)$ and $(1, \ldots, 1)$ and $(x_1 + \ldots + x_n - n)$ vanishes at $(1, \ldots, 1)$. $P'(x)$ has degree at most $deg(P) + 1$. Using Lemma 8 with each $S_i = \{0, 1\}$, $1 \leq i \leq n$, it follows that $deg(P) + 1 \geq deg(P') \geq n$. So, $|\mathcal{A}| = deg(P) \geq n - 1$. $\square$

Remark 1 *Lemma 8 can also be used to obtain an alternative proof of induction base case of the Cayley-Bacharach Theorem by Riehl and Graham [27] (see Appendix B). An alternative proof of the above lower bound can also be obtained using the Cayley-Bacharach Theorem by Riehl and Graham [27].*

Note that in Section 2.1, the underlying set $\mathcal{B}$ of all the non-trivial bicolorings of $[n]$, has cardinality $|\mathcal{B}| = 2^n - 2$. In this case, Theorem 1 establishes that $\gamma(n, [1, n-1], [2, n]) = n - 1 = \Theta(\log |\mathcal{B}|)$. In the following section, we match the $O(\log |\mathcal{B}|)$ upper bound for slightly restricted sets $\mathcal{B}$ of bicolorings.

2.2. Relation to hitting sets for an arbitrary collection of bicolorings

Let $\mathcal{S}$ denote a collection of subsets of $[n]$. A subset $V \subseteq [n]$ is a *hitting set* for $\mathcal{S}$ if for every $S \in \mathcal{S}$, $V \cap S$ is non-empty. Let $H(\mathcal{S})$ denote a minimum cardinality hitting set of $\mathcal{S}$. The decision version of the hitting set problem is: "Given the pair $(\mathcal{S}, [n])$ and an integer k as input, decide whether there exists a hitting set of cardinality at most k for $\mathcal{S}$". We have the following relation between SURs and hitting sets.

Lemma 9 *Let $\mathcal{B} = \{B_0, \ldots, B_{m-1}\} \subseteq \{-1, +1\}^n$ be a family of bicolorings of $[n]$. Construct the family $\mathcal{C} = \{C_1, \ldots, C_{2m}\}$ where $C_{2i+1} = B_i(+1)$ and $C_{2i+2} = B_i(-1)$, for $0 \le i \le m-1$. Let $H = \{h_1, h_2, h_3, \ldots\}$ denote a hitting set for $\mathcal{C}$. Define $\mathcal{A} = \{(h_1, h_q) | h_q \in H, q > 1\}$. Then, $\mathcal{A}$ is a SUR for $\mathcal{B}$ of cardinality $|H| - 1$.*

PROOF. For the sake of contradiction, assume that $B_i \in \mathcal{B}$ has no unbiased representative in $\mathcal{A}$. Assume that $h_1 \in B_i(+1)$. Since H is a hitting set for $\mathcal{C}$, there exists some $h_q \in H$ such that h_q hits C_{2i+2} (and, thereby $B_i(-1)$). Then, the pair (h_1, h_q) is an unbiased representative for B_i, a contradiction to our assumption. So, $h_1 \notin B_i(+1)$. But this implies that $h_1 \in B_i(-1)$. A similar contradiction can be obtained in this case. $\quad\square$

Given a hypergraph $\mathcal{G} = (V, E)$, a subset $A \subseteq V$ is said to be *shattered by $\mathcal{G}$* if for every $B \subseteq A$, there exists an $e \in E$ such that $e \cap A = B$. The VC-dimension of $\mathcal{G}$ is the size of any largest subset of V that is shattered by $\mathcal{G}$. Given that a hypergraph $\mathcal{G}$ has VC-dimension d, Haussler and Welzl [28] gave a method for estimating the minimum cardinality of a hitting set $\tau(\mathcal{G})$. We state an improved bound due to Komlos et al. [29].

Theorem 10 *[30, Corollary 15.6] Let $\mathcal{G} = (V, E)$ be a hypergraph of VC-dimension d. If every hyperedge of $\mathcal{G}$ has at least $\epsilon|V|$ elements for some $\epsilon \le \frac{1}{2}$, then*

$$\tau(\mathcal{G}) \le \frac{d}{\epsilon}(\ln \frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon} + 6).$$

Let $\mathcal{B}$ be restricted to a special family of bicolorings: the number of $+1$'s for each $B \in \mathcal{B}$ lies in the range ϵn and $(1-\epsilon)n$, i.e., $\epsilon n \le |B(+1)| \le (1-\epsilon)n$, for some fixed $0 < \epsilon < \frac{1}{2}$. Construct the family $\mathcal{C}$ as in Lemma 9 and let d be the VC-dimension of $\mathcal{C}$. Note that every $C \in \mathcal{C}$ has size at least ϵn, for some fixed $\epsilon < \frac{1}{2}$. Using Theorem 10, we can get an a hitting set H for $\mathcal{C}$ of cardinality at most $\frac{d}{\epsilon}(\ln \frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon} + 6)$. Using Lemma 9, it follows that we can construct a SUR for $\mathcal{B}$ of cardinality $\frac{d}{\epsilon}(\ln \frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon} + 6) - 1$. Since any family $\mathcal{C}$ of VC-dimension d has cardinality at least 2^d, this establishes an $O(\log |\mathcal{C}|) = O(\log |\mathcal{B}|)$ upper bound for the cardinality of any optimal SUR under no restriction on set sizes. We state the result as a proposition below.

Proposition 11 *Let $0 \le \epsilon \le \frac{1}{2}$ be a constant. Let $\mathcal{B}$ be a family of bicolorings, where $\epsilon n \le |B(+1)| \le (1-\epsilon)n$, for each $B \in \mathcal{B}$. Let $\mathcal{C}$ be the family constructed from $\mathcal{B}$ as in Lemma 9. Let d be the VC-dimension of $\mathcal{C}$. Then, $\gamma(\mathcal{B}) \le \frac{d}{\epsilon}(\ln \frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon} + 6) - 1$.*

In both Section 2.1 and 2.2, the $O(\log |\mathcal{B}|)$ cardinality SURs contained sets of small sizes (2-sized sets) as well. In what follows, we study the problem of SURs made of large cardinality sets. In order to obtain a similar $O(\log |\mathcal{B}|)$ bound for such a SUR, we inevitably introduce some error in the representation.

2.3. Probabilistic construction of SURs consisting of large sets with limited bias

To begin with, in Proposition 12, we give a construction of SURs made of large sets of fixed size using a straightforward application of probabilistic method.

Proposition 12 *Let $\mathcal{B}$ denote a set of bicolorings, where the number of $+1$'s in each $B \in \mathcal{B}$ lies in the range $\{\alpha n, \alpha n + 1, \ldots, (1-\alpha)n\}$ for some $0 < \alpha < \frac{1}{2}$. Then,*

$$\gamma(\mathcal{B}, r) \le \frac{e^r}{2^r (\alpha(1-\alpha))^{\frac{r}{2}}} \ln(|\mathcal{B}|). \tag{3}$$

PROOF. Choose an r element set $A = \{x_1, \ldots, x_r\}$ from $\binom{[n]}{r}$ uniformly at random. The probability that a fixed bicoloring $B \in \mathcal{B}$ has $\langle Y_B, X_A \rangle = 0$ is at least $\frac{\binom{\alpha n}{r/2}\binom{(1-\alpha)n}{r/2}}{\binom{n}{r}} \geq \frac{(\frac{\alpha n}{r/2})^{\frac{r}{2}}(\frac{(1-\alpha)n}{r/2})^{\frac{r}{2}}}{(\frac{en}{r})^r} = (\frac{2}{e})^r (\alpha(1-\alpha))^{\frac{r}{2}}$ (this probability is minimized when B contains exactly αn +1's and $(1-\alpha)n$ -1's or vice versa). So, the probability that a fixed bicoloring $B \in \mathcal{B}$ does not have $\langle Y_B, X_A \rangle = 0$ is at most

$$ 1 - \left(\frac{2}{e}\right)^r (\alpha(1-\alpha))^{\frac{r}{2}} < e^{-(\frac{2}{e})^r (\alpha(1-\alpha))^{\frac{r}{2}}}. $$

Let $\mathcal{A}$ be constructed by choosing t r-element sets into $\mathcal{A}$ independently, where each r-element set is chosen from $\binom{[n]}{r}$ uniformly at random as described above. Using union bound, the probability that some $B \in \mathcal{B}$ has $\langle Y_B, X_A \rangle \neq 0$ for all $A \in \mathcal{A}$, is $|\mathcal{B}|(e^{-(\frac{2}{e})^r (\alpha(1-\alpha))^{\frac{r}{2}}})^t$. Enforcing the above probability to be strictly less than 1, we get an upper bound of $\frac{e^r}{2^r(\alpha(1-\alpha))^{\frac{r}{2}}} \ln(|\mathcal{B}|)$ for a minimum cardinality SUR for $\mathcal{B}$. $\square$

Using Proposition 13, the case when $k = \frac{n}{2}$ and $r = 2$ yields a asymptotically tight example for this upper bound. When $\alpha = \frac{1}{2} - \epsilon$, for some $0 \leq \epsilon < \frac{1}{2}$, Inequality 3 becomes

$$ |\mathcal{A}| \leq \frac{e^r}{(1 - 4\epsilon^2)^{\frac{r}{2}}} \ln(|\mathcal{B}|). \tag{4} $$

Using the fact that $(1 - \frac{1}{m+1})^m \geq \frac{1}{e}$, the right hand term is at most $e^{r + (\frac{4\epsilon^2}{1-4\epsilon^2})\frac{r}{2}} \ln |\mathcal{B}|$. Therefore, when $r \in O(1)$, we have an $O(\ln |\mathcal{B}|)$ upper bound for any optimal SUR consisting of r sized sets for $\mathcal{B}$. However, if r is any increasing function in n, the upper bound given by Proposition 12 is large (even if $\epsilon = \frac{1}{n}$, the expression on the RHS of 4 is $\Omega(e^r \ln |\mathcal{B}|)$). In order to obtain an $O(\ln(|\mathcal{B}|))$ upper bound for $|\mathcal{A}|$ where the sets in $\mathcal{A}$ are large, one may permit some bias in representation. Let $\mathcal{B}$ denote the set of all bicolorings $B \in \{-1, +1\}^n$, where $|B(+1) - B(-1)| \leq d$, for some $d \in \mathbb{N}$. Our problem is to find a small sized family $\mathcal{A}$ for $\mathcal{B}$ such that

1. each $A \in \mathcal{A}$ is reasonably large;

2. for every $B \in \mathcal{B}$, there exists a set $A \in \mathcal{A}$ such that $|\langle Y_B, X_A \rangle| \leq \Delta$, where $\Delta = \Delta(r, d, n)$ is as small as possible.

Proof of Theorem 2

Statement of Theorem 2. Let $r' \in [r \pm \lceil \frac{r}{2} \rceil]$, where $r \geq 8$ is an integer. Let $\mathcal{B}$ denote the set of all bicolorings $B \in \{-1, +1\}^n$, where $||B(+1)| - |B(-1)|| \leq d$, for some $d \in \mathbb{N}$. Then, with high probability, one can construct a family $\mathcal{A}$ of cardinality at most $\ln |\mathcal{B}|$ in $O(n|\mathcal{B}| \ln |\mathcal{B}|)$ time consisting of r'-sized subsets such that for every $B \in \mathcal{B}$, there exists a set $A \in \mathcal{A}$ with $|\langle Y_B, X_A \rangle| \leq e\sqrt{r} + \frac{dr}{n}$.

PROOF. We construct a set $A \subset [n]$ of size $r' \in [r \pm \lceil \frac{r}{2} \rceil]$ by picking each element of $[n]$ into A independently with probability $\frac{r}{n}$. Let $X_A = (a_1, \ldots, a_n)$ denote the corresponding random vector where each $a_i \in \{0, 1\}$. Note that $|A| = \sum_{i=1}^{n} a_i$. So, using linearity of expectation, $(\mu =)\mathbb{E}[|A|] = \sum_{i=1}^{n} \mathbb{E}[a_i] = r$. Moreover, since a_i's are independent, $Var[|A|] = \sum_{i=1}^{n} Var[a_i] = r(1 - \frac{r}{n})$. So, using the following form of Chernoff's bound $P(|X - \mu| > \Delta\mu) < (\frac{e^\Delta}{(1+\Delta)^{(1+\Delta)}})^\mu + (\frac{e^{-\Delta}}{(1-\Delta)^{(1-\Delta)}})^\mu$, we get,

$$ P\left(|\sum_{i=1}^{n} a_i - r| > 0.5r\right) < 0.72, \text{ for } r \geq 8. \tag{5} $$

So, we can sample a family $\mathcal{A}$ of cardinality t (t to be chosen later) consisting of sets of size $r' \in [r \pm \frac{r}{2}]$.

Let $B \in \mathcal{B}$ be a bicoloring, where $|B(+1)| - |B(-1)| = d_1$, where $-d \leq d_1 \leq d$. Let $Y_B = (b_1, \ldots, b_n)$ denote the corresponding bit vector, where each $b_i \in \{-1, +1\}$. Let $Y = \langle Y_B, X_A \rangle$. Since $Y = \sum_{i=1}^{n} a_i b_i$,

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Y becomes a random variable ($a_i b_i$ can take values from $\{-1, 0, 1\}$ and are independent). So, $\mathbb{E}[Y] = \sum_{i=1}^{n} b_i \mathbb{E}[a_i] = \frac{d_1 r}{n}$. It follows that $Var[Y] = \sum_{i=1}^{n} b_i^2 Var[a_i] = r(1 - \frac{r}{n})$. So, using Chebyshev's inequality, we get, $P\left(|Y - \frac{d_1 r}{n}| \geq e\sqrt{r}\right) \leq \frac{1}{e^2}(1 - \frac{r}{n}) < \frac{1}{e^2}$. That is, the probability that $|\langle Y_B, X_A \rangle| > \frac{d_1 r}{n} + e\sqrt{r}$ is at most $\frac{1}{e^2}$. Let E denote the bad event that some $B \in \mathcal{B}$ has $|\langle Y_B, X_A \rangle| > \frac{dr}{n} + e\sqrt{r}$ for all $A \in \mathcal{A}$. Using union bound, $P(E) \leq |\mathcal{B}|(\frac{1}{e^2})^t$. Setting this 'failure' probability $|\mathcal{B}|(\frac{1}{e^2})^t$ to at most $\frac{1}{2}$, we get, $t \geq \ln|\mathcal{B}|$.

Independently choose $100t$ subsets of $[n]$ (call this collection $\mathcal{D}$), where each $D \in \mathcal{D}$ is constructed by picking an element of $[n]$ independently with probability $\frac{r}{n}$. Let $\mathcal{C} \subseteq \mathcal{D}$ be the sub-collection of r'-sized subsets in $\mathcal{D}$, where $r' \in [r \pm \frac{r}{2}]$. Note that for a fixed $A \in \mathcal{D}$ with $X_A = (a_1, \ldots, a_n)$, the probability that $A \notin \mathcal{C}$ is same as $P(|\sum_{i=1}^{n} a_i - r| > 0.5r)$, which by Equation 5 is at most 0.72 for $r \geq 8$. So, $E[|\mathcal{C}|] \geq 28t$. Since $Var[|\mathcal{C}|] \leq 25t$, using Chebyshev's inequality, with high probability, $|\mathcal{C}| \geq 10t$. Partition $\mathcal{C}$ into t-sized sets. From the discussions in the previous paragraph, it follows that with high probability, one of the parts will form our desired family $\mathcal{A}$ that is a SUR (with restricted error) for $\mathcal{B}$.

It takes $\mathcal{O}(n \ln|\mathcal{B}|)$ time to sample $\mathcal{D}$, filter out $\mathcal{C}$ and partitioning $\mathcal{C}$ into t-sized sets. It takes $\mathcal{O}(n|\mathcal{B}| \ln|\mathcal{B}|)$ time to check whether a particular $\ln|\mathcal{B}|$ sized collection forms a SUR for $\mathcal{B}$. □

Comparison between Theorem 2 and Proposition 12: Expressing d in Theorem 2 in terms of α in Proposition 12, $(1 - 2\alpha)n = d$. So, $\epsilon = \frac{1}{2} - \alpha = \frac{d}{2n}$. Substituting this value of ϵ in Inequality 4, we get a SUR of cardinality $\Omega(e^r \ln|\mathcal{B}|)$ with no error for $\mathcal{B}$.

3. When cardinalities of sets in the 'SUR' and +1's in the bicolorings are restricted: Estimating $\gamma(n, k, r)$

In this section, we study the SUR problem where each $B \in \mathcal{B}$ is restricted to have exactly k +1's and each set in the SUR is required to be of cardinality exactly r, for some $r, k \in [n]$, $2 \leq r \leq 2k$. For any k-bicoloring B of $[n]$, and any $A \subseteq [n]$, if A is an unbiased representative for B, then $2 \leq |A| \leq 2k$. Recall that $\gamma(n, k, r) = \gamma(\mathcal{B}, r)$, where (i) $\mathcal{B}$ is the collection of the $\binom{[n]}{k}$ distinct k-bicolorings, (ii) $\gamma(\mathcal{B}, r)$ is the cardinality of an optimal SUR $\mathcal{A}$ for $\mathcal{B}$, and, (iii) each $A \in \mathcal{A}$ has cardinality exactly r. We have the following propositions.

Proposition 13 $\max(\lceil \frac{n-k}{r} \rceil, \lceil \frac{k}{r} \rceil) \leq \gamma(n, k, r)$.

PROOF. Consider the case when $k \leq \lfloor \frac{n}{2} \rfloor$. Given a SUR $\mathcal{A}$ of cardinality $\lfloor \frac{n-k}{r} \rfloor$ consisting of r-sized subsets, there exists a k-sized subset (say, S) of $[n]$ that is completely disjoint from the union of these r-sized subsets. The bicoloring with the points in S colored +1 and the points in $[n] \setminus S$ colored -1 does not have any unbiased representative in $\mathcal{A}$. □

Proposition 14 $\frac{2}{r(r-1)}\gamma(n, k-1, r-2) \leq \gamma(n, k, r) \leq (n-r+1)\gamma(n, k-1, r-2)$, *for* $r \geq 4$.

PROOF. Let $\mathcal{B}_i$ denote the set of all the bicolorings consisting of exactly i +1's, for $i \in \{k, k-1\}$. Let $\mathcal{A}_{r-2}$ denote a family of $(r-2)$-sized subsets that is an optimal unbiased representative family for $\mathcal{B}_{k-1}$. For any $A \in \mathcal{A}_{r-2}$, let $\bar{A} = [n] \setminus A = \{x_1, \ldots, x_{n-r+2}\}$. For each $A \in \mathcal{A}_{r-2}$, we construct $(n-r+1)$ r-sized subsets as follows: $A^1 = A \cup \{x_1, x_2\}$, $A^2 = A \cup \{x_1, x_3\}$, $\cdots$, $A^{n-r+1} = A \cup \{x_1, x_{n-r+2}\}$. Let $\mathcal{A}_r = \cup_{A \in \mathcal{A}_{r-2}}\{A^1, \cdots, A^{n-r+1}\}$. To see that $\mathcal{A}_r$ is a system of unbiased representative for $\mathcal{B}_k$, consider any $B \in \mathcal{B}_k$ and a bicoloring $B' \in \mathcal{B}_{k-1}$ such that $B'(+1) \subset B_k(+1)$. Let $A' \in \mathcal{A}_{r-2}$ has $\langle Y_{B'}, X_{A'} \rangle = 0$. From the construction, it follows that there is at least one $A \in \{A'^1, \cdots, A'^{n-r+1}\}$ such that $\langle Y_B, X_A \rangle = 0$.

For the lower bound, consider a SUR $\mathcal{A}$ for $\mathcal{B}_k$ of size $\gamma(n, k, r)$. For each $A \in \mathcal{A}$, let $\mathcal{F}_A$ denote the family of $\binom{r}{r-2}$ distinct $(r-2)$-sized subsets of A. Then, $\mathcal{A}' = \cup_{A \in \mathcal{A}}\mathcal{F}_A$ is an unbiased representative family for $\mathcal{B}_{k-1}$ where each set in the family is of size exactly $(r-2)$. □

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A single r-sized set can be an unbiased representative for at most $\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}$ distinct bicolorings consisting of k +1's. Using this fact, we have the following lower bound.

$$\gamma(n,k,r) \geq \frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}}. \tag{6}$$

To establish an upper bound, we reduce this problem to a covering problem and then make use of a result by Lovász and Stein [21, 22].

Definition 15 *Given a family $\mathcal{F}$ of subsets of some finite set X, the cover number $Cov(\mathcal{F})$ of $\mathcal{F}$ is the minimum number of members of $\mathcal{F}$ whose union includes all the points in X.*

Theorem 16 *(Lovász-Stein Theorem)[21, 22, 31] If each member of $\mathcal{F}$ covers at most a elements and each element in X is covered by at least v members of $\mathcal{F}$, then*

$$Cov(\mathcal{F}) \leq \frac{|\mathcal{F}|}{v}(1+\ln a).$$

In the proof of the above Lovász-Stein Theorem, one considers a $|X| \times |\mathcal{F}|$ matrix $M = (a_{i,j})$, where $a_{i,j}$ is 1 if and only if ith element in X belongs to jth set in $\mathcal{F}$. Then, one chooses a subset of columns C of size at most $\frac{|\mathcal{F}|}{v}(1+\ln a)$ starting with an empty set C and adding to it a column containing most number of 1's and then deleting all rows containing an 1 in that column. This process is repeated until all rows are exhausted. Using the Lovász-Stein result, we have the following theorem.

Theorem 17 *Let n be an integer, $r,k \in [n]$, $2 \leq r \leq 2k$ and r is even. Then,*

$$\frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}} \leq \gamma(n,k,r) \leq \frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}} \left(1 + 0.7r + \ln\left(\binom{n-r}{k-\frac{r}{2}}\right)\right).$$

PROOF. Consider the following construction of a uniform family of subsets based on the $\binom{n}{[k]}$ distinct k-bicolorings and $\binom{n}{r}$ distinct r-sized subsets of $[n]$.

Construction 1 *Corresponding to each distinct k-bicoloring B in $\binom{[n]}{k}$, we add a point v_B to X. Corresponding to each distinct r-sized subset A in $\binom{[n]}{r}$, we add a set e_A to $\mathcal{F}$, where e_A is the collections of all v_B's such that $\langle X_A, Y_B \rangle = 0$. So, e_A 'covers' v_B if and only if $v_B \in e_A$.*

So, $|X| = \binom{n}{k}$, $|\mathcal{F}| = \binom{n}{r}$. Clearly, $a = \binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}$, $v = \binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}}$. It follows from the construction that $\gamma(n,k,r) = Cov(\mathcal{F})$. So, from Theorem 16, we have

$$\gamma(n,k,r) \leq \frac{\binom{n}{r}}{\binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}}} \left(1 + \ln\left(\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}\right)\right). \tag{7}$$

Double counting (B, A) pairs, where B is a k-bicoloring and A is a r-sized subset that covers B, we get

$$\binom{n}{k}\binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}} = \binom{n}{r}\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}. \tag{8}$$

Note that this equation can also be obtained as a result of direct simplification of both LHS and RHS. Combining Inequalities 7 and 8, and from Inequality 6, Theorem 17 follows. $\square$

Since Lovász-Stein method is deterministic and constructive, the above reduction gives a deterministic polynomial time algorithm for obtaining a SUR. Moreover, from Theorem 17, it follows that $\gamma(n,k,r)$ is $O(k \ln n)$ approximable $(k + 0.2r + (k - \frac{r}{2}) \ln(\frac{n-r}{k-\frac{r}{2}})$ using the approximation $\binom{n}{k} \leq (\frac{en}{k})^k)$ and when $k = \frac{r}{2}$, the approximation factor becomes $O(r)\,(1 + 0.7r)$. However, if $k \leq \log_4(\log n)^{1-\epsilon}$ and $r = 2k$, for some $0 < \epsilon < 1$, then this upper bound can be improved further.

3.1. Tight upper bounds under more restrictions

From Construction 1, it is clear that the approximation factor for $\gamma(n,k,r)$ in Theorem 17 comes as a consequence of the approximation factor for the cover number given by Lovász-Stein Theorem. So, tighter bounds for the cover number should translate into tighter bounds for $\gamma(n,k,r)$. Let $v(B,D)$ denote the number of r-sized sets that are unbiased representatives for both B and D, for any pair (B,D) of k-bicolorings, where $B \neq D$. Let $v_{pair} = \max\limits_{\substack{B,D \in \binom{[n]}{k}, \\ B \neq D}} v(B,D)$. The Rödl nibble method [32, 33] establishes asymptotically tight bounds for the cover number provided the uniformity a of the family $\mathcal{F}$ in Construction 1 is fixed, $v \to \infty$, and $v_{pair} \in o(v)$. The following theorem by Alon et al. [23, Corollary 1.3] relaxed the condition for a.

Theorem 18 *[23] Let $\mathcal{H}$ be a a-uniform v-regular hypergraph with maximum codegree v_{pair}. If*

$$a \in o\left(\log\left(\frac{v}{v_{pair} \log(v_{pair} + 1)}\right)\right),$$

then there exists a cover of size $\frac{n}{a}(1 + o(1))$.

From Construction 1, we know that $a = \binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}$, $v = \binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}}$. In the estimation of $\gamma(n,k,r)$, if $k \leq \log_4(\log n)^{1-\epsilon}$ and $r = 2k$, for any $0 < \epsilon < 1$, using Construction 1, it follows that $a < 2^r = 2^{2k} \leq (\log n)^{1-\epsilon}$. We need to show that $(\log n)^{1-\epsilon} \in o\left(\log\left(\frac{v}{v_{pair} \log(v_{pair}+1)}\right)\right)$ in order to use Theorem 18 and prove Theorem 3.

Lemma 19 $(\log n)^{1-\epsilon} \in o\left(\log\left(\frac{v}{v_{pair} \log(v_{pair}+1)}\right)\right)$, *when $r = 2k$ and $k \leq \log_4(\log n)^{1-\epsilon}$, for any $0 < \epsilon < 1$.*

PROOF. In order to prove the lemma, it is important to note that $v(B,D)$ depends intrinsically on the cardinality of $B(+1) \cap D(+1)$. Let S be some r-sized subset of $[n]$. Let $i_B = S \cap (B(+1) \setminus D(+1))$, $i_D = S \cap (D(+1) \setminus B(+1))$, $j_{BD} = S \cap (B(+1) \cap D(+1))$ and $j_{\overline{BD}} = S \cap ([n] \setminus (B(+1) \cup D(+1)))$ (see Figure 1). So, $S = i_B \cup i_D \cup j_{BD} \cup j_{\overline{BD}}$. If S is an unbiased representative for B, then $|i_B| + |j_{BD}| = |i_D| + |j_{\overline{BD}}| = \frac{r}{2}$. If S is an unbiased representative of D, then $|i_D| + |j_{BD}| = |i_B| + |j_{\overline{BD}}| = \frac{r}{2}$. Therefore, if S is an unbiased representative of both B and D, then (i) $|i_B| = |i_D|$ $(= i$, say), (ii) $|j_{BD}| = |j_{\overline{BD}}|$ $(= j$, say), and (iii) $2i + 2j = r = 2k$. Let $x = |B(+1) \cap D(+1)|$. Since $|B(+1)| = |D(+1)| = k$, applying Condition (iii), we get $x = j$ and $k - x = i$. In other words, if S is an unbiased representative of cardinality $r = 2k$ for both the k-bicolorings B and D, $B(+1) \cup D(+1) \subseteq S$. Consequently, $v(B,D) = \binom{x}{x}\binom{n-2k+x}{x}\left(\binom{k-x}{k-x}\right)^2 = \binom{n-2k+x}{x}$.

The general formula for $v(B,D)$ is given by $v(B,D) = \sum\limits_{\substack{i,j:j\leq x, \\ i \leq k-x, \\ i+j=\frac{r}{2}}} \binom{x}{j}\binom{n-2k+x}{j}\left(\binom{k-x}{i}\right)^2$.

So, when $x = k-1$, $v(B,D) = \binom{n-k-1}{k-1}$; when $x = k-2$, $v(B,D) = \binom{n-k-2}{k-2}$, etc. Therefore, $\frac{v(B,D)}{v(B',D')} = \Omega(\frac{n}{k})$ if $|B(+1) \cap D(+1)| = k-1$ and $|B'(+1) \cap D'(+1)| \leq k-2$. So, $v_{pair} = v(B,D)$, when $|B(+1) \cap D(+1)| = k-1$ provided $r = 2k$. Thus, $v_{pair} = \binom{n-k-1}{\frac{r}{2}-1}$, when $r = 2k$. Computing $\frac{v_{pair}}{v}$,

$$\frac{v_{pair}}{v} = \frac{\binom{n-k-1}{\frac{r}{2}-1}}{\binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}}} = \frac{r}{2(n-k)}. \tag{9}$$
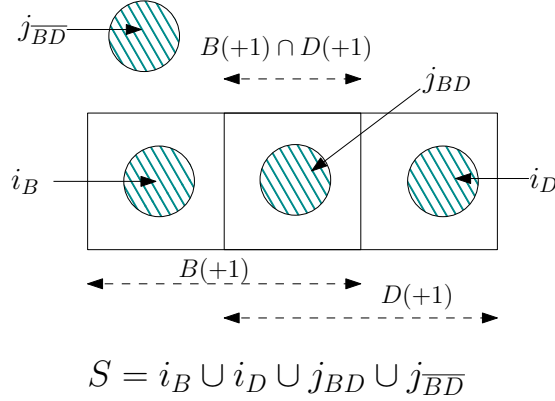
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Figure 1: S is some r-sized subset of $[n]$. Let $i_B = S \cap (B(+1) \setminus D(+1))$, $i_D = S \cap (D(+1) \setminus B(+1))$, $j_{BD} = S \cap (B(+1) \cap D(+1))$ and $j_{\overline{BD}} = S \cap ([n] \setminus (B(+1) \cup D(+1)))$. So, $S = i_B \cup i_D \cup j_{BD} \cup j_{\overline{BD}}$. If S is an unbiased representative for B, then $|i_B| + |j_{BD}| = |i_D| + |j_{\overline{BD}}|$. If S is an unbiased representative of D, then $|i_D| + |j_{BD}| = |i_B| + |j_{\overline{BD}}|$. So, if S is an unbiased representative of both B and D, then $|i_B| = |i_D|$ and $|j_{BD}| = |j_{\overline{BD}}|$.

$$\log \left(\frac{v}{v_{pair} \log(v_{pair}+1)} \right) = \log \left(\frac{2(n-k)}{r \log(\binom{n-k-1}{\frac{r}{2}-1}+1)} \right) \geq \frac{1}{2} \log n. \text{ So, } (\log n)^{1-\epsilon} \in o\left(\log \left(\frac{v}{v_{pair} \log(v_{pair}+1)} \right) \right), \text{ when}$$
$n \to \infty$. $\qquad\qquad\square$

Proof of Theorem 3

Statement of Theorem 3. For sufficiently large values of n, $\frac{\binom{n}{k}}{\binom{2k}{k}} \leq \gamma(n,k,2k) \leq \frac{\binom{n}{k}}{\binom{2k}{k}}(1 + o(1))$, provided $k \leq \log_4(\log n)^{1-\epsilon}$, for any $0 < \epsilon < 1$.

PROOF. From Lemma 19, and using Theorem 18 to obtain coverings, the proof follows. $\qquad\square$

3.2. $\gamma(n,k,r)$, when $k = n/2$

Let $\mathcal{B}$ denote the set of all $\binom{n}{\frac{n}{2}}$ distinct $\frac{n}{2}$-bicolorings. It is not hard to see that $\mathcal{A} = \{\{1,2\}, \{1,3\}, \ldots, \{1, \frac{n}{2}+1\}\}$ is a SUR of cardinality $\frac{n}{2}$ for $\mathcal{B}$. Together with Proposition 13, this establishes $\frac{n}{4} \leq \gamma(n, \frac{n}{2}, 2) \leq \frac{n}{2}$. It is easy to see that $\gamma(n, \frac{n}{2}, n) = 1$. For arbitrary values of r, from Theorem 17 and Proposition 13, we have,

$$\max \left(\left\lceil \frac{n}{2r} \right\rceil, c_1 \sqrt{\frac{r(n-r)}{n}} \right) \leq \gamma(n, \frac{n}{2}, r) \leq c_2 n \sqrt{\frac{r(n-r)}{n}}, \text{ where } c_1 \text{ and } c_2 \text{ are constants.} \qquad (10)$$

When $r = \frac{n}{2}$, this establishes a lower bound and upper bound of $\Omega(\sqrt{n})$ and $O(n\sqrt{n})$, respectively. In general, when $r = f(n)$ is an increasing function in n, this establishes sub-linear lower bounds for $\gamma(n, \frac{n}{2}, r)$.

We use an extension of a theorem of Frankl and Rödl [34] given by Keevash and Long [24] to obtain a linear lower bound on $\gamma(n,k,r)$ under certain restrictions on k and r. Let $\mathcal{D} \subseteq [q]^n$ be a q-ary code. For any $x, y \in \mathcal{D}$, the Hamming distance between x and y is the number of indices where $x(i) \neq y(i)$, for $1 \leq i \leq n$. The code $\mathcal{D}$ is called d-*avoiding* if the Hamming distance between no pair of code-words in $\mathcal{D}$ is d. The following upper bound for d-avoiding codes is given in [24].

Theorem 20 *[24] Let $\mathcal{D} \subseteq [q]^n$ and let ϵ satisfy $0 < \epsilon < \frac{1}{2}$. Suppose that $\epsilon n < d < (1 - \epsilon)n$ and d is even if $q = 2$. If $\mathcal{D}$ is d-avoiding, then $|\mathcal{D}| \leq q^{(1-\delta)n}$, for some positive constant $\delta = \delta(\epsilon)$.*

We have the following lower bound for $\gamma(n,k,r)$, when $r = 2c$ for any odd integer $c \in \{1, \ldots, \frac{n}{2}\}$ and $\epsilon n < k < (1 - \epsilon)n$, for some $0 < \epsilon < 0.5$.

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Proof of Theorem 4

Statement of Theorem 4. Let $r = 2c$ for any odd integer $c \in \{1, \ldots, \frac{n}{2}\}$. Let k be an even integer, where $\epsilon n < k < (1-\epsilon)n$ for some $0 < \epsilon < 0.5$. Then, $\gamma(n, k, r) \geq \delta n$, where $\delta = \delta(\epsilon)$ is some real positive constant.

PROOF. Let $\mathcal{B} = \{B_1, \ldots, B_{\binom{n}{k}}\}$ denote the set of all the bicolorings of $[n]$ consisting of exactly k $+1$'s. We construct a family $\mathcal{C} = \{C_1, \ldots, C_{\binom{n}{k}}\}$, where $C_i = B_i(+1)$. Let $\mathcal{A}$ be a SUR for $\mathcal{B}$, where each $A \in \mathcal{A}$ has cardinality exactly $2c$ for some odd number $c \in [n]$. Note that $\langle Y_{B_i}, X_A \rangle = 0$ implies that $\langle X_{C_i}, X_A \rangle = c$, where X_{C_i} denotes the 0-1 incidence vector corresponding to the set C_i. Let $V \subset \{0,1\}^n$ denote the vector space spanned by the vectors X_A's, $A \in \mathcal{A}$, over $\mathbb{F}_2$. Let $V^\perp \subset \{0,1\}^n$ denote the subspace orthogonal to V. Since $\mathcal{A}$ is a SUR for $\mathcal{B}$, it follows that for every C_i, there exists a set $A \in \mathcal{A}$ such that $\langle X_{C_i}, X_A \rangle = 1 (\mod 2)$ (since c is odd). Therefore, $X_{C_i} \notin V^\perp$, for all $X_{C_i} \in \mathcal{C} = \binom{[n]}{k}$. In other words, $V^\perp$ does not contain any vector consisting of exactly k ones. Moreover, observe that for any $x, y \in V^\perp$, the number of ones in $x+y$ is same as the Hamming distance between x and y. Thus, $V^\perp$ is k-avoiding. Since $\epsilon n < k < (1-\epsilon)n$ and k is even, from Theorem 20, it follows that there exists a positive constant $\delta = \delta(\epsilon)$ such that $|V^\perp| \leq 2^{n(1-\delta)}$. So, dimension of $V^\perp$ is at most $n(1 - \delta)$. Therefore, it follows that dimension of V is at least δn. □

Corollary 21 $\gamma(n, \frac{n}{2}, r) \geq \delta n$ provided $\frac{n}{2}$ is even and $\frac{r}{2}$ is odd, for some $0 < \delta < 1$.

Let $\frac{n}{2}$ be even and $\frac{r}{2}$ be odd. From Inequality 10, we have $\gamma(n, \frac{n}{2}, r) \in O(n\sqrt{r})$. When r is a constant, using Corollary 21, this upper bound is asymptotically tight. However, for larger values of r, there can be a large gap (up to $O(\sqrt{n})$ when $r \in \Omega(n)$) between the upper and the lower bound. In what follows, we address the problem for a special case when $r = \frac{n}{2}$ and establish a better upper bound of $\frac{n}{2}$ on $\gamma(n, \frac{n}{2}, \frac{n}{2})$.

Lemma 22 $\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$, where $\frac{n}{2}$ is any even integer.

PROOF. Let $\mathcal{B}$ denote the set of all the bicolorings with equal number of $+1$'s and -1's. Let $A_1 = \{1, 2, \ldots, \frac{n}{2}\}$, $A_2 = \{2, 3, \ldots, \frac{n}{2} + 1\}, \ldots, A_{\frac{n}{2}} = \{\frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1\}$. Let $c_i(B) = \langle Y_B, X_{A_i} \rangle$. For any $B \in \mathcal{B}$, it is not hard to see that each $c_i(B)$ is even and $|c_i(B) - c_{i+1}(B)| \in \{0, 2\}$. Since the bicolorings consist of equal number of $+1$'s and -1's, $c_{\frac{n}{2}}(B) \leq -c_1(B) + 2$ if $c_1(B) \geq 0$, and $c_{\frac{n}{2}}(B) \geq -c_1(B) - 2$ if $c_1(B) < 0$. In particular, we have $c_1(B) c_{\frac{n}{2}}(B) \leq 0$. Since $|c_i(B) - c_{i+1}(B)| \in \{0, 2\}$, this implies the existence of an index i such that $c_i(B) = \langle Y_B, X_{A_i} \rangle = 0$. This concludes the proof that $\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. □

From Corollary 21 and Lemma 22, we have the following theorem.

Theorem 23 $\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. Moreover, $\gamma(n, \frac{n}{2}, \frac{n}{2}) \geq \delta n$ if $n/2$ is even and $n/4$ is odd, for some $0 < \delta < 1$.

4. Inapproximability of the SUR problem

Dinur and Steurer [25] established the following theorem on the inapproximability of the set cover problem.

Theorem 24 *[25] For every $\alpha > 0$, it is impossible to approximate set cover to within $(1 - \alpha) \ln n$, where n is the size of the instance. The reduction runs in time $\mathcal{O}(2^{n^{\frac{1}{\alpha}}})$.*

Given a hypergraph G with n vertices and m hyperedges, using the duality of set cover and hitting set problems, Theorem 24 implies that for every $\alpha > 0$, it is impossible to approximate hitting sets to within $(1 - \alpha) \ln m$ of the optimal solution.

Using Theorem 24, we establish a hardness result of the hitting set problem for a special family of subsets.

Definition 25 *A family $\mathcal{F}$ of subsets of $[n]$ is complement closed on $[n]$ if for all $F \in \mathcal{F}$, $[n] \setminus F \in \mathcal{F}$.*

Proposition 26 *Let n and m be integers. No deterministic polynomial time algorithm can approximate the hitting set problem for a complement closed family consisting of m distinct subsets of $[n]$ to within a factor of $(1 - \Omega(1))\frac{\ln m}{4}$ of the optimal, unless P=NP.*

PROOF. For the sake of contradiction, assume that there exists an algorithm ALG that approximates the hitting set for complement closed families on $[n]$ to within a factor of $(1 - \Omega(1))\frac{\ln m}{4}$ of the optimal. We obtain a contradiction to this assumption by the following reduction from the general hitting set problem.

Given a pair $(\mathcal{S}', [n])$ as input to the general hitting set problem, we extend the universe to $[n + 1]$ by adding the element $n + 1$. We construct $\mathcal{S}$ as follows: $\mathcal{S} = \mathcal{S}' \cup \{[n + 1] \setminus S | S \in \mathcal{S}'\}$. Observe that $\mathcal{S}$ is a complement closed family and $|\mathcal{S}| \leq 2|\mathcal{S}'| = 2m$. Let $OPT(\mathcal{S})$ ($OPT(\mathcal{S}')$) denote an optimal solution to the hitting set problem on $\mathcal{S}$ (respectively, $\mathcal{S}'$). Let $ALG(\mathcal{S})$ denote a hitting set output by ALG on $\mathcal{S}$ as input.

Observe that

$$|OPT(\mathcal{S}')| \leq |OPT(\mathcal{S})| \leq |OPT(\mathcal{S}')| + 1 \leq 2|OPT(\mathcal{S}')|. \tag{11}$$

From our assumption, we know that $|OPT(\mathcal{S})| \leq |ALG(\mathcal{S})| \leq (1-\Omega(1))\frac{\ln(2m)}{4}|OPT(\mathcal{S})| < (1-\Omega(1))\frac{\ln m}{2}|OPT(\mathcal{S})|$. Note that $ALG(\mathcal{S})$ is a valid hitting set for $\mathcal{S}'$. So, $|OPT(\mathcal{S}')| \leq |OPT(\mathcal{S})| \leq |ALG(\mathcal{S})| \leq (1-\Omega(1))\frac{\ln m}{2}|OPT(\mathcal{S})| < (1-\Omega(1))\frac{\ln m}{2} \cdot 2|OPT(\mathcal{S}')| = (1-\Omega(1))\ln m|OPT(\mathcal{S}')|$. Therefore, ALG is a $(1-\Omega(1))\ln m$ factor approximation algorithm for the general hitting set problem. By duality of set cover and hitting sets, this establishes a $(1 - \Omega(1))\ln n$ factor approximation algorithm for the set cover problem. However, from Theorem 24, we know it is impossible unless P=NP. □

We use Proposition 26 to establish the following hardness result for the system of unbiased representative problem.

Proof of Theorem 5

Statement of Theorem 5. Let n and m be integers and let $r \leq (1 - \Omega(1))\frac{\ln m}{4}$. Then, no deterministic polynomial time algorithm can approximate the system of unbiased representative problem for a family of m bicolorings on $[n]$ to within a factor $(1-\Omega(1))\frac{\ln m}{4r}$ of the optimal when each set chosen in the representative family is required to have its cardinality at most r, unless P=NP.

PROOF. We prove Theorem 5 by a reduction from an instance of the hitting set problem on complement closed familes. Let $\mathcal{S}$ be a complement closed family on $[n]$. From $\mathcal{S}$, we construct a family $\mathcal{B}$ of bicolorings on $[n]$ in the following way: $\mathcal{B} = \{B|B(+1) = S, B(-1) = [n] \setminus S, S \in \mathcal{S}\}$. We chose $\mathcal{S}$ to be complement closed so that from the construction of $\mathcal{B}$ and using Lemma 9, we have $|OPT_{\text{SUR}}(\mathcal{B})| \leq |OPT_{\text{HIT}}(\mathcal{S})| - 1$. For the sake of contradiction, assume that there exists an algorithm ALG that approximates the system of unbiased representative problem for any family of bicolorings on $[n]$ to within a factor f of the optimal, where $1 \leq f \leq (1 - \Omega(1))\frac{\ln m}{4r}$ and each set in the SUR is required to have its cardinality at most r. Let $OPT_{\text{HIT}}(\mathcal{S})$ ($OPT_{\text{SUR}}(\mathcal{B})$) denote an optimal solution to the hitting set problem (respectively, the system of unbiased representative problem) on $\mathcal{S}$ (respectively, $\mathcal{B}$). Let $ALG(\mathcal{B})$ denote a SUR outputted by ALG with $\mathcal{B}$ as its input. Then, executing ALG on $\mathcal{B}$ as input, we obtain a SUR $\mathcal{A}$ for $\mathcal{B}$ such that (i) $2 \leq |A| \leq r$ for each $A \in \mathcal{A}$, (ii) $|ALG(\mathcal{B})| = |\mathcal{A}| \leq f \cdot |OPT_{\text{SUR}}(\mathcal{B})|$, for some $1 \leq f \leq (1 - \Omega(1))\frac{\ln m}{4r}$. Let $V = \cup_{A \in \mathcal{A}} A$. It follows that $|V| \leq r|\mathcal{A}|$ and V is a hitting set for $\mathcal{S}$.

Since $|OPT_{\text{SUR}}(\mathcal{B})| \leq |OPT_{\text{HIT}}(\mathcal{S})| - 1$,

$$|OPT_{\text{HIT}}(\mathcal{S})| \leq |V| \leq r \cdot |ALG(\mathcal{B})| \leq r \cdot f \cdot |OPT_{\text{SUR}}(\mathcal{B})| < r \cdot f \cdot |OPT_{\text{HIT}}(\mathcal{S})|.$$

So, ALG is a $(r \cdot f)$-factor approximation algorithm for computing hitting set of $\mathcal{S}$. Since $1 \leq f \leq (1 - \Omega(1))\frac{\ln m}{4r}$, this is a contradiction to Proposition 26. □

Remark 2 *Consider the case when the family $\mathcal{B}$ is restricted to a special family of bicolorings, where the number of +1's (or -1's) for each $B \in \mathcal{B}$ is exactly one, i.e. $|B(+1)| = 1$ (or $|B(-1)| = 1$). Then, the problem of system of unbiased representatives reduces to an edge cover problem [35, 36] on a complete graph G, where for each $B \in \mathcal{B}$, a vertex $v_{B(+1)}$ (respectively, $v_{B(-1)}$) is added to $V(G)$. So, this reduction makes the SUR problem polynomial time solvable for such families of bicolorings.*

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Appendix A. An alternative construction of an SUR

Assume that $n = 2^p$, for some integer p. Let $\mathcal{A}_2 = \{\{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\}\}$: a partition of $[n]$ into two-sized sets. Let $\mathcal{A}_4 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \ldots, \{n-3, n-2, n-1, n\}\}$: a partition of $[n]$ into four-sized sets taken in that order. Similarly, repeating the construction for $p-2$ more steps, we obtain a sequence of partitions of $[n]$, $\mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_n$, where $\mathcal{A}_i$ is a partition of $[n]$ into i-sized $\frac{n}{i}$ parts, i.e., $\mathcal{A}_i = \{\{1, \ldots, i\}, \{i + 1, \ldots, 2i\}, \ldots, \{n-i+1, \ldots, n\}\}$. Let $\mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_4 \cup \cdots \cup \mathcal{A}_n$. It follows that $|\mathcal{A}| = 2^{p-1} + 2^{p-2} + \ldots + 1 = 2^p - 1 = n - 1$. To see that this is indeed a SUR for the set of all the $2^n - 2$ non-monochromatic bicolorings, let $B \in \{-1, 1\}^n$ denote any non-trivial bicoloring of $[n]$. Without loss of generality, assume that $|B(+1)| \le |B(-1)|$. Let i $(2 \le i \le n)$ be the minimum index such that there exists an $A \in \mathcal{A}_i$ with $A \setminus B(+1) \ne \emptyset$ and $A \cap B(+1) \ne \emptyset$. Observe that such an index exists since $|B(+1)| \le |B(-1)|$ and B is non-monochromatic. From construction of $\mathcal{A}_i$ and assumption on i, it follows that there exists consecutive parts $A_1, A_2 \in \mathcal{A}_{\frac{i}{2}}$ with $A_1 \subseteq B(+1)$, $A_2 \cap B(+1) = \emptyset$, and $A = A_1 \cup A_2$. So, it follows that A is an unbiased representative for B.

Appendix B. Proof of induction base case of Theorem 27

Theorem 27 *[27] Given the n quadratics in n variables $x_1(x_1 - 1), \ldots, x_n(x_n - 1)$ with 2^n common zeros, the maximum number of those common zeros a polynomial P of degree k can go through without going through them all is $2^n - 2^{n-k}$.*

PROOF. The proof is by induction on n. When $k = 0$, we have nothing to prove. So, we consider all the degree k polynomials P on $k+1$ variables as the base case. For the sake of contradiction, assume that P is a polynomial of degree k on $k+1$ variable and it misses only one common zero of $x_1(1-x_1), \ldots, x_{k+1}(1-x_{k+1})$. Then, using Lemma 8, it follows that degree of P must be $k + 1$, which is a contradiction. This completes the proof of the induction base case. The rest of the proof is exactly the same as given in [27]. □