Minimal Perturbations for Zero Controllability of Discrete-time Linear Systems: Complexity Analysis

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***Abstract*—This article deals with computational complexity of various problems related to the zero controllability of a discrete-time linear time-invariant system, assuming that purely structural conditions at the level of the connections between the system states and the inputs are known. Given a generically zero controllable system, the following problems are considered: i) determine a minimal set of input-connections whose removal makes the resulting system not generic zero controllability; ii) identify a minimal cost set of input-connections that must be retained from the pre-specified set of input-connections while preserving generic zero controllability property; and iii) given a system that is not generically zero controllable, find a smallest set of state-connections whose removal makes the resulting system generically zero controllable. Problem i) is polynomially solvable. Problems ii) and iii) are NP-hard, and an approximate solution with best approximation ratio is provided for each of them. The results of i) and iii) provide clues to analyze the fragility and hardness of the system structure. Problems ii) is useful to provide insights on understanding and deriving results about the properties of the system.**

***Index Terms*—generic zero controllability, perturbations, feedback arc set.**

I. Introduction

We consider a discrete-time linear system

$$x(t+1) = \bar{A}x(t) + \bar{B}u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^d$ and $u(t) \in \mathbb{R}^m$ are the states and the inputs at time t, and $\bar{A} \in \mathbb{R}^{d\times d}$ and $\bar{B} \in \mathbb{R}^{d\times m}$ are the given system and input matrices respectively.

We say that (1) is *zero controllable* if the states in (1) can be driven from any initial state x_0 at $t = 0$ to the zero state at some finite time $t = \tau$ [1]. Over the past few years there has been a surge in the study of various problems/conditions pertaining to the zero controllability of a given system; necessary and sufficient conditions for zero controllability was investigated for bilinear systems in [2], for fractional discrete-time system in [3], for discrete-time system with randomly jumping system parameters in [4], for discrete time behaviour [5], etc., assuming that the accurate values of the parameters of the system are known.

Since it is usually impossible to get the precise values of the system parameters that governs a system's dynamics, we often rely on structured system theory that only considers the zero/non-zero pattern of entries of system matrices [6] for analysis. In particular, we consider the structural counterpart of zero controllability, namely, generic zero controllability (see Definition II.1) in this article. Specifically, we study the following three optimization problems:

- ($\mathcal{P}_1$): Assuming a generically zero controllable (g.z.c) system (1), identify the minimal number of input-connections[1] whose removal makes the resulting system not g.z.c.
- ($\mathcal{P}_2$): Given a g.z.c system (1), find a set of input-connections of minimal cost to be retained from the available set of input-connections while maintaining generic zero controllability property of the system.
- ($\mathcal{P}_3$): Given a not g.z.c system (1), find a smallest set of connections between the states whose removal makes the resulting system g.z.c.

($\mathcal{P}_1$) is particularly useful to comment on the resilience of a large-scale system (modelled as discrete-time linear systems) under unknown attacks on input-connections/inputs. The removal of these vulnerable connections will not only make the resulting system not zero controllable but also results in uncontrollability of the system; see Remark IV.3 for a technical discussion. For an external adversary, finding such a set of vulnerable connections is of great interest. For a systems operator, this information serves to plan effective strategies to enhance network resilience.

($\mathcal{P}_2$) is essential to study the dynamics of the (nonlinear) large-scale systems around us and derive information about the system. We often study their dynamics via Taylor linearization at the nominal point and represent them in the form of a linear discrete-time model. A linear model is an accurate approximation of the original system when the states stay sufficiently close to the point of linearization, or, by change of coordinates, the states are close to zero [7]. One way to steer the states to zero is to design an input matrix such that the resulting system attains generic zero controllability. In this direction, ($\mathcal{P}_2$) seeks a minimal cost input matrix when the set of inputs and their input-connections are pre-specified with each input-connection being linked to a non-negative cost. The cost arises out of various factors–installation and maintenance cost of the system, etc. By actuating the system by inputs, we ensure that results obtained for the linear system is expected to be valid for the original nonlinear system locally [7].

($\mathcal{P}_3$) is crucial since it provides useful insights regarding the hardness involved in modifying a system structure.

We establish the following results for the three problems:

- ($\mathcal{P}_1$) is solvable in $O(d^2m)$ time (Theorem IV.1).
- ($\mathcal{P}_2$) is NP-hard, even with identical input-connection costs (Theorem IV.4).
- There is a $O(d^2)$ algorithm with a worst-case approximation ratio of $O(\log d)$ for ($\mathcal{P}_2$) (Theorem IV.10 (a) and (c)).[2]

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[1]The connections from the inputs to states are termed as input-connections.

[2]This result is under the assumption that the digraph $G(A)$ associated with A contains cycles. ($\mathcal{P}_2$) is easily solvable when $G(A)$ is acyclic.

- Under mild condition (satisfied by many systems), ($\mathcal{P}_2$) is polynomially solvable (Theorem IV.10(b) & Remark IV.11).
- ($\mathcal{P}_3$) is NP-complete (Theorem IV.15).
- We find an approximate solution of ($\mathcal{P}_3$) (Lemma IV.16).

Related work: A necessary and sufficient condition for generic zero controllability of a discrete-time linear system was introduced in [9]. Recently, finding a sparsest input configuration to guarantee generic zero controllability, when the given input matrix is unconstrained, was considered in [7]. This particular problem is a special case of ($\mathcal{P}_2$), when all the input-connections are allowed between the given d inputs and states and a uniform non-zero cost is assigned to each input-connection. In [7], the author reduced this problem to an instance of the minimum set cover problem. However, the complexity of the problem is not addressed. In contrast, our problem is a vast generalization of that problem.

In the context of ($\mathcal{P}_1$), considerable efforts have been made to examine the consequences of unknown attacks on controllability or observability of a system. For instance, the preservation of observability under sensor failure is studied in [10]. Controllability preservation under failure of agents and connections between agents has been studied in [11], [12]. However, the focus of most of the prior works is to classify agents/connections depending on the effect of their failure on controllability. Recently, the problem of identifying a set of input-connections of minimal cardinality when removed would lead to a system's uncontrollability was studied in [13]. This problem turns out to be computationally difficult and no constant approximation factor solution yet exists for it [13]. Our approach is somewhat different; we inspect the resilience of the system from the perspective of zero controllability.

In the context of ($\mathcal{P}_3$), similar kind of feature that strong couplings between the states usually lead to loss of controllability was observed in [8]. However, in their analysis, the values of the system parameters were assumed to be known.

Moreover, our results are in sharp contrast to the results known in the context of structural controllability of a discrete-time linear system [6]; see Remarks IV.3, IV.12, and IV.17 for further technical details.

To the best of our knowledge, the literature on problems of complexity and approximation of different optimization problems related to generic zero controllability under structural perturbations seems woefully incomplete. We believe that this work is a step in that general direction, and hopefully provokes more investigation of this nature, in the future.

This article unfolds as follows: §II reviews certain graph-theoretic preliminaries. The formal statements of the problems are given in §III, and §IV provides methods/algorithms to solve these problems. In §V, examples are included to illustrate the usefulness of the techniques established in §IV.

II. Background

Notations: We represent the set of positive integers by $\mathbb{N}$ and we let $[r] \coloneqq \{1, 2, \dots, r\}$ for $r \in \mathbb{N}$. The size of a finite set X is denoted by $|X|$. The identity matrix of dimension r is denoted by I_r. $\mathbf{0}_{r\times s}$ is a zero matrix of dimension $r \times s$, where $r, s \in \mathbb{N}$. We define $\mathbb{1}$ associated with the (i, j)-th entry of a matrix $A = [A_{ij}]$ as $\mathbb{1}_{\{A_{ij}\neq 0\}} \coloneqq 1$ if $A_{ij} \neq 0$, and 0 otherwise.

The linear system (1) is zero controllable (with $\{0\}$ as the stability region) if and only if $\operatorname{rank}\left(\bar{A} - zI_d, \bar{B}\right) = d$ for every $z \in \mathbb{C}\setminus\{0\}$ [1, Theorem 1], where $\mathbb{C}$ represents complex numbers. We sometimes use $(\bar{A}, \bar{B})$ to refer to system (1). In our investigation, the exact values of the entries of $\bar{A}$ and $\bar{B}$ does not matter, but the information about the locations of the fixed zeros in $\bar{A}$ and $\bar{B}$ are crucial. Therefore, let $A \in \{0, \star\}^{d\times d}$ and $B \in \{0, \star\}^{d\times m}$ represent the sparsity matrices of $\bar{A}$ and $\bar{B}$ in (1) where each entry is either a fixed zero or an independent free parameter, denoted by star. A *numerical realization* of (A, B) is a matrix pair obtained by assigning numerical values to the star entries of A and B.

Definition II.1. A pair (A, B) is said to be *generically zero controllable (g.z.c in short)* if at least one numerical realization (A', B') of (A, B) exists that is zero controllable.[3]

Given a linear system (1), we associate the states $x(t) \in \mathbb{R}^d$ and the inputs $u(t) \in \mathbb{R}^m$ to the state vertices $\mathcal{A} = \{x_1, x_2, \dots, x_d\}$ and the input vertices $\mathcal{U} = \{u_1, u_2, \dots, u_m\}$, respectively. Consider $E_A = \{(x_j, x_i) \mid A_{ij} \neq 0\}$ and $E_B = \{(u_j, x_i) \mid B_{ij} \neq 0\}$. Define the digraph $G(A, B) = (\mathcal{A} \sqcup \mathcal{U}, E_A \sqcup E_B)$ associated with the pair (A, B) and $\sqcup$ represents the disjoint union. The edges in E_A and E_B are referred as *state-connections* and *input-connections* respectively. Sometimes we shall also need the digraph $G(A) = (\mathcal{A}, E_A)$.

We refer to [14, Section 3] for the definitions of subgraph, induced subgraph, directed path, cycle, and strongly connected component (SCC) of $G(A)$. Given $G(A, B)$, a state vertex x_i is said to be *reachable* if there exists a directed path from some input u_j to x_i; otherwise, it is *unreachable*. $G(A)$ is *acyclic* if it contains no cycles. A SCC or a cycle of $G(A)$ is reachable if all the state vertices it contains are reachable; otherwise it is unreachable. A SCC is said to be *nontrivial* if it consists of at least one edge among its vertices; otherwise it is said to be *trivial*. It means that a trivial SCC must contain only one state vertex with no edges (including no self loops). A SCC $\mathcal{S}_i$ is said to be *reachable from a SCC* $\mathcal{S}_j$ if there exists a directed path from a vertex $x_k \in \mathcal{S}_j$ to a vertex $x_n \in \mathcal{S}_i$ in $G(A)$. A SCC is said to be *source strongly connected component* (SSCC) of $G(A)$ if there exists no incoming edges from the vertices of other SCCs into any vertex in it.

A pair (A, B) is said to be *irreducible* if and only if every state vertex in $G(A, B)$ is reachable from input set $\mathcal{U}$ [9]. If a pair (A, B) is irreducible then it is also g.z.c. [4] However, when a pair (A, B) is not irreducible then we establish a connection between generic zero controllability and certain properties of $G(A, B)$ by decomposing the vertex set of digraph $G(A) = (\mathcal{A}, E_A)$ into $\mathcal{A} = \mathcal{A}_r \sqcup \mathcal{A}_{urn}$, where $\mathcal{A}_r$ is the set of states that are reachable and $\mathcal{A}_{urn}$ is the set of states that are unreachable.

Definition II.2. The digraph induced by the vertices of $\mathcal{A}_{urn}$ represents the *unreachable part* of the graph $G(A)$ associated with the pair (A, B) denoted by $G_{urn}(A, B) = (\mathcal{A}_{urn}, E_{urn})$, where E_{urn} is the set of edges between the vertices in $\mathcal{A}_{urn}$.

It is easy to see that a pair (A, B) is irreducible iff $G_{urn}(A, B)$ is empty since no state vertex is unreachable.

[3] If one numerical realization is zero controllable then almost all numerical realizations of (A, B) is zero controllable.

[4] A irreducible pair (A, B) has generically $\operatorname{rank}\left(A - zI_d, B\right) = d$ for all $z \in \mathbb{C} \setminus \{0\}$ [15].

The following theorem characterises a generically zero controllable system (A, B) when (A, B) is not irreducible.

Theorem II.3. *[9, Theorem 4.4, p. 93] Let (A, B) be a linear system (1) and $G(A, B) = (\mathcal{A} \sqcup \mathcal{U}, E_A \sqcup E_B)$ be its associated digraph. Let $G_{urn}(A, B)$ represent the unreachable part of $G(A)$ associated with pair (A, B) from Definition II.2. The pair (A, B) is g.z.c if and only if (iff in short) $G_{urn}(A, B)$ does not contain any cycle.*

If $G(A)$ is acyclic then the system is g.z.c even without any input-connections (i.e., $B = \mathbf{0}_{d\times m}$) as $G_{urn}(A, B) = G(A)$. If $G(A)$ contains cycles then we have:

Theorem II.4. *Let (A, B) be a linear system (1) and $G(A, B) = (\mathcal{A} \sqcup \mathcal{U}, E_A \sqcup E_B)$ be its associated digraph. Let $G_{urn}(A, B)$ represent the unreachable part of $G(A)$ associated with pair (A, B) from Definition II.2. Suppose $G(A)$ contains cycles. The pair (A, B) is g.z.c iff each cycle is reachable in $G(A, B)$.*

Proof. Let the pair (A, B) be g.z.c. For an irreducible pair (A, B), all the cycles are automatically reachable. If (A, B) is not irreducible then each cycle is reachable since the unreachable part $G_{urn}(A, B)$ contains no cycles.

Suppose every cycle is reachable in $G(A, B)$. If (A, B) is irreducible then it is automatically g.z.c. If not, then the reachability of all the cycles of $G(A)$ implies that $G_{urn}(A, B)$ has no cycles. By Theorem II.3, the pair (A, B) is g.z.c. $\square$

A simple consequence of the above theorem is as follows.

Corollary II.5. *Let (A, B) be a linear system (1) and suppose $G(A)$ is not acyclic. The pair (A, B) is g.z.c iff each nontrivial SCC is reachable in $G(A, B)$.*

Remark II.6. Given $G(A, B) = (\mathcal{A} \sqcup \mathcal{U}, E_A \sqcup E_B)$, the determination of the unreachable state vertices can be done by using depth-first search [16] in $O(d^2+dm)$ time. The existence of cycles in the associated unreachable part $G_{urn}(A, B)$ (if non-empty) is checked again via depth first search in $O(d^2)$ complexity. Thus, generic zero controllability of a pair (A, B) can be checked in $O(d^2 + dm)$ run time complexity.

III. Problem formulation

We now introduce several norms and operations that will arise in the rest of the article. For a matrix $N \in \{0, \star\}^{r\times s}$, where $r, s \geq 1$ are positive integers, we let: $\|N\|_0$ denote the number of non-zero entries in the matrix N. Suppose $w_{ij} \geq 0$ be a cost corresponding to each $N_{ij} \neq 0$. We define $\|N\|_w := \sum_{i=1}^{r}\sum_{j=1}^{s} w_{ij}\mathbb{1}_{\{N_{ij}\neq 0\}}$, where $\mathbb{1}$ is defined in §II. Given $N \in \{0, \star\}^{r\times s}$ and $M \in \{0, \star\}^{r\times s}$,

- $M \subset N$ if $M_{ij} = \star$ implies $N_{ij} = \star$.
- The operation $N \ominus M \in \{0, \star\}^{r\times s}$ with $M \subset N$ is:

$$[N \ominus M]_{ij} := \begin{cases} \star & \text{if } N_{ij} = \star \text{ and } M_{ij} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We formally state the three optimization problems of interest:

(a) Given $A \in \{0, \star\}^{d\times d}$ and $B \in \{0, \star\}^{d\times m}$. Let the pair (A, B) be g.z.c. Solve

$$\begin{aligned} &\underset{B^u \subset B}{\text{minimize}} \quad \|B^u\|_0 \\ &\text{such that} \quad (A, B \ominus B^u) \text{ is not g.z.c.} \end{aligned} \qquad (\mathcal{P}_1)$$

(b) Given $A \in \{0, \star\}^{d\times d}$ and $B \in \{0, \star\}^{d\times m}$. Let the pair (A, B) be g.z.c. Recall $G(A, B) = (\mathcal{A} \sqcup \mathcal{U}, E_A \sqcup E_B)$. Define

$$\mathcal{K} := \Big\{ B' \,\Big|\, B' \subset B \text{ and } (A, B') \text{ is g.z.c} \Big\}. \qquad (2)$$

By assumption, $\mathcal{K}$ always contains at least one element since the given pair (A, B) is g.z.c. Let $w_{ij} \geq 0$ denote the cost of the input-connection $(u_j, x_i) \in E_B$ in $G(A, B)$. Solve

$$\underset{B' \in \mathcal{K}}{\text{minimize}} \quad \|B'\|_w \,. \qquad (\mathcal{P}_2)$$

Special case of ($\mathcal{P}_2$)*:* Assume that a fixed and non-zero cost is assigned to each input-connection in $G(A, B)$. In this case ($\mathcal{P}_2$) reduces to:

$$\underset{B' \in \mathcal{K}}{\text{minimize}} \quad \|B'\|_0 \,. \qquad (\mathcal{P}_2')$$

(c) Let $A \in \{0, \star\}^{d\times d}$ and $B \in \{0, \star\}^{d\times m}$. We assume that the pair (A, B) is not g.z.c. Solve

$$\begin{aligned} &\underset{A_s \subset A}{\text{minimize}} \quad \|A_s\|_0 \\ &\text{such that} \quad (A \ominus A_s, B) \text{ is g.z.c.} \end{aligned} \qquad (\mathcal{P}_3)$$

Remark III.1. Problem ($\mathcal{P}_1$) is trivially invalid if the graph $G(A)$ contains no cycles since the system remains g.z.c even after removal of all the input-connections. *Therefore, for* ($\mathcal{P}_1$)*, we assume that the given pair (A, B) is such that the digraph $G(A)$ contains at least one cycle.*

IV. Main Results

A. Problem ($\mathcal{P}_1$) is polynomial time solvable

Recall that $G(A) = (\mathcal{A}, E_A)$ and $G(A, B) = (\mathcal{A} \sqcup \mathcal{U}, E_A \sqcup E_B)$. We assume that the given pair (A, B) is g.z.c such that $G(A)$ contains at least one cycle. By Corollary II.5, each nontrivial SCC of $G(A)$ are reachable in $G(A, B)$. To solve ($\mathcal{P}_1$), we take the collection of nontrivial SCCs $\{\mathcal{S}_j\}_{j=1}^{q}$ of $G(A)$. We say that a SCC $\mathcal{S}_j$ is *reachable from an input vertex* u_i (denoted by$u_i \rightarrow \mathcal{S}_j$) if there exists a directed path from u_i to some $x_k \in \mathcal{S}_j$. We employ Algorithm 1 to find the minimum number of input-connections $T_{del}(A, B)$ whose removal makes at least one nontrivial SCC of $G(A)$ unreachable from the input set $\mathcal{U}$, and consequently makes the resulting system not g.z.c (by Corollary II.5).

Algorithm 1: Procedure to determine $T_{del}(A, B)$.

Input: $G(A, B) = (\mathcal{A} \sqcup \mathcal{U}, E_A \sqcup E_B)$
Output: $T_{del}(A, B)$ to solve ($\mathcal{P}_1$)

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Identify the nontrivial SCCs {S_j}_{j=1}^q of G(A).
Replace each e_i = (u_j, x_k) ∈ E_B by creating an input
   vertex u'_i connected via a directed edge to x_k.
Define L_j = 0 for all j ∈ [q]
for each nontrivial SCC S_j, j ∈ [q]
for i = 1, 2, ..., |E_B|
  if u'_i → S_j
     L_j = L_j + 1
  end
end for
end for
Define T_del(A, B) := min_{1≤j≤q} L_j.
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Theorem IV.1. *Let the pair (A, B) be a g.z.c linear system (1) and $G(A, B)$ be its associated digraph. Suppose $G(A)$ (associated with A) contains at least one cycle. Then $T_{del}(A, B)$ obtained by Algorithm 1 solves ($\mathcal{P}_1$) in $O(d^2m)$ time.*

Proof. Without loss of generality, assume that L_j associated with the nontrivial SCC S_j has the least value among all $\{L_i\}_{i=1}^{q}$ in Algorithm 1, i.e., $T_{del}(A, B) = L_j$. Then, we obtain B^u as follows:

$$B^u_{r\ell} \leftarrow \begin{cases} \star & \text{if } u_i' \rightarrow \mathcal{S}_j \text{ and } e_i = (u_\ell, x_r) \in E_B, \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

Note that $T_{del}(A,B) = \|B^u\|_0$. Clearly, $(A, B \ominus B^u)$ is not g.z.c since $\mathcal{S}_j$ is unreachable in $G(A, B \ominus B^u)$. To prove optimality of B^u, we proceed by contradiction. Let $B' \subset B$ be a feasible solution of $(\mathcal{P}_1)$, i. e., $(A, B \ominus B')$ is not g.z.c such that $\|B'\|_0 < \|B^u\|_0$. By Corollary II.5, $G_{urn}(A, B \ominus B')$ contains at least one nontrivial SSC of $G(A)$, say $\mathcal{S}_k$. Then the number of non-zero entries in B' must be at least equal to L_k (computed in Algorithm 1) since L_k is the minimum number of input-connections essential to ensure the reachability of $\mathcal{S}_k$. Therefore, $\|B'\|_0 \geq L_k \geq T_{del}(A,B) = \|B^u\|_0$, which is a contradiction. Therefore, it follows that B^u (from (3)) and hence, $T_{del}(A,B)$ solves $(\mathcal{P}_1)$.

The SCCs of $G(A)$ can be determined by using depth-first search twice in time $O(|\mathcal{A}| + |E_A|)$ [16]. Here $|\mathcal{A}| = d$ and $|E_A| \leq d^2$. The collection of nontrivial SCCs from them can be identified in linear time since it involves to check whether a SCC has at least one edge or not. Hence, the nontrivial SCCs are obtained in $O(d^2)$ time (Step 1). The complexity of Step 2 is linear in number of edges in E_B, i.e., $O(|E_B|)$. The overall complexity of Steps 3-9 is $O(q\,|E_B|)$. Since $|E_B| \leq dm$ and $q \leq d$, Algorithm 1 has $O(d^2 m)$ run complexity. $\square$

Remark IV.2. Let $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_{q_1}\}$ be the collection of nontrivial SSCCs (if present) of $G(A)$, subset of the set of nontrivial SCCs of $G(A)$.[5] If rest of the nontrivial SCCs are reachable from some nontrivial SSCCs, then $(\mathcal{P}_1)$ can be solved easily. The generic zero controllability of the pair (A,B) and the definition of SSCC ensure that every nontrivial SSCC $\mathcal{N}_i$ has at least one input-connection from some input vertex $u_j \in \mathcal{U}$ to a state vertex $x_k \in \mathcal{N}_i$ in $G(A,B)$. Each one of them in $\mathcal{N}$ has directed edges from only input vertices. Thus, for each nontrivial SSCC $\mathcal{N}_i$, if all the associated input-connections are removed, then $\mathcal{N}_i$ becomes unreachable from $\mathcal{U}$. It follows that $T_{del}(A,B) := \min_{1 \leq i \leq q_1} \deg^-(\mathcal{N}_i)$, where $\deg^-(\mathcal{N}_i)$ and q_1 denote the number of input-connections of $G(A,B)$ entering $\mathcal{N}_i$ and the number of nontrivial SSCCs.

Remark IV.3. The notions of structural controllability and generic zero controllability are closely related to each other. Generic zero controllability depends on the connectivity of the system, whereas structural controllability emphasizes on the irreducibility and the generic rank of the system. Over the past few decades, structural controllability has been the subject of a lot of interest, see e.g., [6], [17]. Consider a structurally controllable discrete-time system (A,B) such that $G(A)$ contains cycles. Since $G_{urn}(A,B)$ is empty, the pair (A,B) is also generically zero controllable. If removal of a subset of input-connections violates the generic zero controllability of the system then it also makes the resulting system structurally uncontrollable. However, the converse may not be true. Given a structurally controllable system, the problem $(\mathcal{P})$ of identifying a subset of input-connections of minimum cardilnality, whose removal would result in a structurally uncontrollable system is NP-hard [13]. In contrast, we demonstrate that $(\mathcal{P}_1)$ is polynomially solvable.[6]

[5] $\mathcal{N} \subset \{\mathcal{S}_j\}_{j=1}^{q}$. However, we use a different notation for them to distinguish between them from the rest of the nontrivial SCCs of $G(A)$.

[6] $(\mathcal{P})$ is valid without imposing any assumption on the system whereas $(\mathcal{P}_1)$ is valid when $G(A)$ contains cycles. Therefore, $(\mathcal{P})$ is NP-hard and $(\mathcal{P}_1)$ is solvable w.r.t. the conditions where they are valid.

B. *NP-hardness of Problem $(\mathcal{P}_2)$ and Approximation*

Before moving to $(\mathcal{P}_2)$, we describe the weighted set cover problem (WSCP in short), a well-studied NP-hard problem [16] that will be used later. Let U be a universe containing N elements, i.e., $U = [N]$ with a collection of r sets $P = \{Z_i\}_{i=1}^{r}$ such that each $Z_i \subset U$ and $\bigcup_{i=1}^{r} Z_i = U$. A non-negative cost function $c : P \to \mathbb{R}^+$ assigns cost to each set in P. The objective is to find an $L^* \subset P$ such that $\bigcup_{Z_i \in L^*} Z_i = U$ and $\sum_{Z_i \in L^*} c(Z_i) \leq \sum_{Z_i \in \tilde{L}} c(Z_i)$ for any $\tilde{L}$ that covers U, i.e, $\bigcup_{Z_i \in \tilde{L}} Z_i = U$. Each set Z_i is an element of P. There exist greedy algorithms to find an approximate solution to the WSCP with an approximation ratio of $(1 + \log N)$ [19].

Theorem IV.4. *$(\mathcal{P}'_2)$ is NP-hard.*

Proof. Given $A \in \{0, \star\}^{d \times d}$ and $B = I_d \in \{0, \star\}^{d \times d}$, the problem of finding the minimum number of input-connections required to preserve generic zero controllability is NP-hard [18, Theorem 5].[7] This implies that $(\mathcal{P}'_2)$ is also NP-hard since parameters A and $B = I_d$ is a special instance of $(\mathcal{P}'_2)$. $\square$

Since $(\mathcal{P}'_2)$ is a special case of $(\mathcal{P}_2)$, it follows that $(\mathcal{P}_2)$ is also NP-hard. However, $(\mathcal{P}_2)$ is trivially solvable when $G(A)$ is acyclic with optimal solution $B^* = \mathbf{0}_{d \times m}$. Therefore, NP-hardness arises due to existence of cycles in $G(A)$. In fact, it infers from the proof of Theorem 5 in [18] that $(\mathcal{P}'_2)$ remains NP-hard when $G(A)$ contains cycles since the instance constructed in the proof belongs to this class. *Hereafter, we assume that $G(A)$ contains at least one cycle* and find an approximate solution of $(\mathcal{P}_2)$ via the following two subproblems:

Part (a): Recall that $\mathcal{N} = \{\mathcal{N}_i\}_{i=1}^{q_1}$ denotes the collection of nontrivial SSCCs of $G(A)$. We have the following problem:

Problem IV.5. *Given (A,B), find $B_e^* \in \arg\min_{B' \subset B} \|B'\|_w$ such that each nontrivial SSCCs in $\mathcal{N}$ is reachable in $G(A, B_e^*)$.*

Algorithm 2: Algorithm to find B_e^*

Input: $A \in \{0, \star\}^{d \times d}$, $B \in \{0, \star\}^{d \times m}$
Output: A matrix $B_e^* \subset B$

Find the nontrivial SSCCs $\mathcal{N} = \{\mathcal{N}_i\}_{i=1}^{q_1}$ of $G(A)$
if $\mathcal{N} = \emptyset$ then $B_e^* = \mathbf{0}_{d \times m}$
else
let $D \leftarrow \emptyset$
for each $\mathcal{N}_j$ **do**
  choose an input-connection of the least cost of the form $(u_k, x_\ell) \in E_B$ where $x_\ell \in \mathcal{N}_j$ and $u_k \in \mathcal{U}$ among all existing input-connections associated with $\mathcal{N}_j$.
$D \leftarrow D \cup (u_k, x_\ell)$
end **for**
Define: B_e^* as $[B_e^*]_{\ell k} = \star$ if $e = (u_k, x_\ell) \in D$, and 0 otherwise.
end

Clearly, the procedure involved in Algorithm 2 guarantees that an input-connection of the least cost is selected for each

[7] The structural controllability of a discrete-time linear systems with delays in behaviour sense is dealt in [18]. It is proved that a discrete-time system (1) is structurally controllable in behaviour sense iff it is g.z.c. A special instance of $(\mathcal{P}'_2)$ with A and $B = I_d$ is shown to be NP-hard. In the proof, the delays are assumed to be zero. Hence, the system constructed in the proof is a discrete-time linear system.

nontrivial SSCCs in $\mathcal{N}$ (if non-empty). Thus, we obtain an optimal B_e^* that solves Problem IV.5. In Algorithm 2, finding the nontrivial SCCs and their partial order to identify the nontrivial SSCCs among them involves $O(d^2)$ computations (Step 1). The steps 5-7 takes time linear in $|E_B|$. Since $|E_B| \leq dm$, the complexity of Algorithm 2 to find a solution of Problem IV.5 is $O(\max\{d^2, dm\})$. This completes Part (a).

Part (b): Let $W = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p\}$ denote the set of those nontrivial SCCs of $G(A)$ that are unreachable from the vertices of any nontrivial SSCC in $\mathcal{N}$. If $\mathcal{N} = \emptyset$ (from Part (a)) then all the nontrivial SCCs of $G(A)$ are assumed to be unreachable from $\mathcal{N}$. In other words, all the nontrivial SCCs of $G(A)$ belongs to set W. Thus, if $\mathcal{N} = \emptyset$ then $W \neq \emptyset$ since $G(A)$ contains at least one cycle $\mathcal{C}$. The nontrivial SCC containing $\mathcal{C}$ belong to W.

Problem IV.6. *Given (A, B), find $B_n^* \in \arg\min_{B_n \subset B} \|B_n\|_w$ such that each nontrivial SCC in W is reachable in $G(A, B_n^*)$.*

Clearly, Problem IV.6 is trivially solvable when $W = \emptyset$, i.e., if $W = \emptyset$ then $B_n^* = \mathbf{0}_{d\times m}$. Thus, for further analysis in Part (b), we assume that $W \neq \emptyset$. Let $M = \{x_i\}_{i=1}^n$ denote the set of those state vertices of $\mathcal{A}$ such that for each $x_i \in M$ there exists an $u_j \in \mathcal{U}$ with $(u_j, x_i) \in E_B$. We define the *cost of actuation* of each $x_i \in M$ as $c_i := \min_{(\cdot, x_i)\in E_B} w_{i,\cdot}$, i.e., the cost of the input-connection with end vertex x_i that has the least cost among the set of all input-connections associated with x_i. Define a variable $F : M \to \mathcal{U}$ for each state vertex $x_i \in M$ as: $F(x_i) = u_k$ if $(u_k, x_i) \in E_B$ has the cost c_i. For each $x_i \in M$, if more than one input-connection has cost c_i then any one of them is selected to define F. Note that the vector F can be obtained in linear complexity of $|E_B|$, i.e., $O(dm)$. A nontrivial SCC $\mathcal{R}_j$ is *reachable from a state vertex* x_i (denoted by $x_i \to \mathcal{R}_j$) if x_i has a directed path to some $x_k \in \mathcal{R}_j$ or $x_i \in \mathcal{R}_j$. We reduce Problem IV.6 to weighted set cover problem (WSCP in short) via Algorithm 3.

Algorithm 3: Reduce Problem IV.6 to a WSCP

Input: $G(A, B)$, $M = \{x_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$, and F
Output: The input matrix $B_n(L) \in \{0, \star\}^{d\times m}$

1 Find the nontrivial SCCs unreachable from the nontrivial SSCCs in $\mathcal{N}$, $W = \{\mathcal{R}_j\}_{j=1}^p$
2 The weighted set cover problem is defined as follows:
3 $U = \{1, 2, \ldots, p\}$ as Universe
4 Sets $Z_i = \{j \mid j \in [p]$ and $x_i \to \mathcal{R}_j\}$ for $i \in [n]$
5 cost associated with each $Z_i = c_i$ for $i \in [n]$
6 Given a cover L such that $\cup_{Z_i \in L} Z_i = U$, define:
7 cost of cover L, $c(L) = \sum_{Z_i \in L} c_i$
8 State vertices selected under L, $M(L) \leftarrow \{x_i \mid Z_i \in L\}$
9 Define:
$$[B_n(L)]_{ij} \leftarrow \begin{cases} \star & \text{if } x_i \in M(L) \text{ and } F(x_i) = u_j, \\ 0 & \text{otherwise.} \end{cases}$$

The generic zero controllability of the given pair (A, B) ensures that a solution of the WSCP exists since all the nontrivial SCCs are reachable in $G(A, B)$. In Algorithm 3, given $G(A)$, we determine the nontrivial SCCs in the set W (Step 1). The set cover problem is constructed as follows: The universe U contains the indices of the nontrivial SCCs in W(Step 1). Each Z_i (associated with $x_i \in M$) contains the indices of those nontrivial SCCs $\mathcal{R}_j \in W$ that are reachable from the state vertex x_i. Each set Z_i is assigned a cost of actuation c_i (discussed earlier) in Step 5. Given a solution L to the WSCP, we define its associated cost $c(L)$ (Step 7). The set of state vertices selected under L (given by $M(L)$) and the obtained input matrix $B_n(L)$ are defined in Step 8 and 9, respectively. An optimal solution to Problem IV.6 is taken as B_n^* and an optimal solution to the WSCP in Algorithm 3 is considered as L^*. We prove that an ϵ-approximation algorithm for the WSCP obtained via Algorithm 3 gives such an algorithm to Problem IV.6 too.

Lemma IV.7. *Given $G(A, B)$, $G(A)$, and the corresponding weighted set cover problem (WSCP) obtained using Algorithm 3. Then, for any $\epsilon \geq 1$, if L is an ϵ-optimal solution to the WSCP, then $B_n(L)$ is an ϵ-optimal solution to Problem IV.6.*

Proof. The proof of the lemma has two steps: (i) we prove that an optimal solution L^* to the WSCP gives an optimal solution $B_n(L^*)$ to Problem IV.6 and (ii) we show that if $c(L) \leq \epsilon c(L^*)$ then $\|B_n(L)\|_w \leq \epsilon \|B_n^*\|_w$.

For (i), given a feasible solution L^* to the WSCP, the construction of $B_n(L^*)$ via Algorithm 3 confirms that each nontrivial SCC in W is reachable from some input in $G(A, B_n(L^*))$. Thus, $B_n(L^*)$ is a feasible solution to Problem IV.6. Recall that for a state vertex $x_i \in M$ with cost of actuation $c_i = \min_{(\cdot,x_i)\in E_B} w_{i,\cdot}$, we have $F(x_i) = u_j$ if the input-connection (u_j, x_i) has the least cost of c_i (considered as the cost of set Z_i in Algorithm 3). By steps 5,7,8, and 9

$$\begin{aligned} \|B_n(L^*)\|_w &= \sum_{i=1}^{d}\sum_{j=1}^{m} w_{ij} \mathbb{1}_{\{[B_n(L^*)]_{ij}\neq 0\}} = \sum_{\substack{\{x_i \in M(L^*)\\ F(x_i)=u_j\}}} w_{ij} \\ &= \sum_{x_i \in M(L^*)} c_i = \sum_{Z_i \in L^*} c_i = c(L^*). \end{aligned} \tag{4}$$

To prove optimality we proceed by contradiction. Suppose there exists another feasible solution to Problem IV.6, $B' \subset B$ such that $\|B'\|_w < \|B_n(L^*)\|_w$. Let $K = \{x_i \,|\, B'_{ij} = \star$ for some $u_j \in \mathcal{U}\}$. Clearly, $K \subset M$. Each $x_i \in K$ has a cost of actuation c_i. By definition of c_i, it follows that

$$\sum_{x_i \in K} c_i = \sum_{\substack{\{x_i \in K\\ F(x_i)=u_\ell\}}} w_{i\ell} \leq \sum_{i=1}^{d}\sum_{j=1}^{m} w_{ij} \mathbb{1}_{\{B'_{ij}\neq 0\}} = \|B'\|_w \,.$$

Corresponding to K, define Z_K=$\{Z_i \mid x_i \in K\}$ containing the sets associated with the state vertices in K. Clearly, Z_K satisfies the condition $\cup_{Z_i \in Z_K} Z_i = U$ since B' is a feasible solution of Problem IV.6 and the nontrivial SCCs of W are reachable from the vertices in K, obtained by using B'. Also,

$$c(Z_K) = \sum_{Z_i \in Z_K} c_i = \sum_{x_i \in K} c_i \leq \|B'\|_w < \|B_n(L^*)\|_w = c(L^*).$$

This leads to a contradiction and completes the proof of (i). By Algorithm 3, it is easy to see that if L is a feasible solution to the WSCP then $B_n(L)$ is a feasible solution to Problem IV.6 and $c(L) = \|B_n(L)\|_w$ (by employing similar steps as in (4)). By (4), $c(L^*) = \|B_n(L^*)\|_w = \|B_n^*\|_w$. So, if $c(L) \leq \epsilon c(L^*)$ then $\|B_n(L)\|_w \leq \epsilon \|B_n^*\|_w$. This proves part (ii). $\square$

In Algorithm 3, we first find the successors of the vertices in the nontrivial SSCCs in $\mathcal{N}$ by using depth-first search in $O(d^2)$ time and remove them from the graph. The nontrivial SSCs of the resultant graph are the SSCs in W and finding them takes at most $O(d^2)$ time. Since the size of U and $P = \{Z_i\}_{i=1}^{n}$ are bounded by d, the sets can be determined in $O(d^2)$ time. Thus, the reduction to the WSCP in Algorithm 3 takes $O(d^2)$ computations. Also, given a cover L, the states selected under L (Step 8) and $B_n(L)$ (Step 9) can be obtained in linear time.

Using the Lemma IV.7, we have the following theorem.

Theorem IV.8. *Suppose (A, B) be a g.z.c. linear system (1). Then, there exists a polynomial time algorithm that approximates Problem IV.6 to a factor of $1 + \log d$, where d denotes the number of states.*

Proof. Algorithm 3 transforms Problem IV.6 to WSCP in polynomial time. From Lemma IV.7, a polynomial time ϵ-optimal algorithm for the WSCP provides a polynomial time ϵ-optimal algorithm for Problem IV.6. The size of the constructed universe U in Algorithm 3 is p, where p is the number of nontrivial SCCs in the set W. Since $p \leq d$, the greedy algorithm given in [19, p. 234] for solving the WSCP gives an $(1 + \log d)$-optimal solution to Problem IV.6. □

Algorithm 3 transforms an instance of Problem IV.6 to an instance of the WSCP. By using greedy approximation algorithm of [19], we solve a WSCP to get an approximate solution to Problem IV.6. This completes Part (b).

Remark IV.9. We assume that $G(A)$ contains at least one cycle. In our analysis the computation of the set W depends on the existence/non-existence of $\mathcal{N}$. Then we have the following cases: if $\mathcal{N} \neq \emptyset$ then either (a) $W \neq \emptyset$ or (b) $W = \emptyset$; (c) if $\mathcal{N} = \emptyset$ then $W \neq \emptyset$. The presence of cycle(s) ensures that one case among them is always satisfied.

Next, we give Algorithm 4 and main theroem to solve $(\mathcal{P}_2)$ by using Part (a), Part (b), and Remark IV.9.

Algorithm 4: Algorithm to solve $(\mathcal{P}_2)$

Input: $A \in \{0, \star\}^{d\times d}$, $B \in \{0, \star\}^{d\times m}$
Output: The input matrix $B_a \in \{0, \star\}^{d\times m}$

1 Find an optimal solution of Problem IV.5 from Part (a), say B_e^*
2 Find an approximate solution of Problem IV.6 from Part (b), say B_n if $W \neq \emptyset$ else set $B_n = \mathbf{0}_{d\times m}$
3 $B_a \leftarrow B_e^* \sqcup B_n$

Theorem IV.10. *Suppose that the given pair (A, B) be a g.z.c linear system (1) such that $G(A)$ contains at least one cycle. Let $\mathcal{N} = \{\mathcal{N}_j\}_{j=1}^{q_1}$ be the collection of the nontrivial SSCCs and $W = \{\mathcal{R}_i\}_{i=1}^{p}$ be the set of nontrivial SCCs unreachable from the vertices of any nontrivial SSCC in $\mathcal{N}$ of $G(A)$. Assume B^* be an optimal solution of $(\mathcal{P}_2)$. Let B_a be the output of Algorithm 4. Then we have the following cases:*

(a) *if $\mathcal{N} \neq \emptyset$ and $W \neq \emptyset$ then $B_a \in \mathcal{K}$ is an $O(\log d)$-optimal solution of $(\mathcal{P}_2)$.*
(b) *if $\mathcal{N} \neq \emptyset$ and $W = \emptyset$ then $B_a \in \mathcal{K}$ and $\|B_a\|_w = \|B^*\|_w$.*
(c) *if $\mathcal{N} = \emptyset$ and $W \neq \emptyset$ then $B_a \in \mathcal{K}$ is an $O(\log d)$-optimal solution of $(\mathcal{P}_2)$.*

The overall complexity of Algorithm 4 is $O(\max\{d^2, dm\})$.

Proof. **Case (a)**: Consider $B_a = B_e^* \sqcup B_n$. B_e^* ensures that all the nontrivial SSCCs in $\mathcal{N}$ are reachable in $G(A, B_e)$. Among the rest of the nontrivial SCCs, those reachable from some vertex of a nontrivial SSCC are also reachable in $G(A, B_e)$. Next, consider the nontrivial SCCs in W unreachable from $\mathcal{N}$. The input matrix B_n confirms that these nontrivial SCCs are reachable from $\mathcal{U}$ in $G(A, B_n)$. Therefore, $B_a = B_e^* \sqcup B_n$ ensures that every nontrivial SSC is reachable in $G(A, B_a)$. By Corollary II.5, $B_a \in \mathcal{K}$. Let B_n^* be an optimal solution to Problem IV.6. By Theorem IV.8, $\|B_n\|_w \leq \epsilon \|B_n^*\|_w$ with $\epsilon = 1 + \log d$. Let B^* be a solution of $(\mathcal{P}_2)$. It is easy to see that B^* satisfies the following inequalities

$$\|B^*\|_w \geq \|B_e^*\|_w \text{ and } \|B^*\|_w \geq \|B_n^*\|_w .$$

$$\text{Then, } 2\|B^*\|_w \geq \|B_e^*\|_w + \|B_n^*\|_w .$$

$$2\|B^*\|_w \geq \|B_e^*\|_w + \frac{\|B_n\|_w}{\epsilon} \geq \frac{\|B_e^*\|_w + \|B_n\|_w}{\epsilon} = \frac{\|B_a\|_w}{\epsilon}$$

$$2\epsilon\|B^*\|_w \geq \|B_a\|_w , \text{ where } \epsilon = 1 + \log d \geq 1.$$

Case (b): Here $B_n = \mathbf{0}_{d\times m}$ and $B_a = B_e^*$. Each nontrivial SCCs of $G(A)$ is reachable in $G(A, B_a)$. Therefore, $B_a \in \mathcal{K}$. For an optimal solution B^*, it is sufficient to ensure the reachability of every nontrivial SSCCs as $W = \emptyset$. Thus, $\|B^*\|_w = \|B_e^*\|_w = \|B_a\|_w$ and $(\mathcal{P}_2)$ is solved optimally.
Case (c): Here $B_e^* = \mathbf{0}_{d\times m}$ and $B_a = B_n$. Since $\mathcal{N} = \emptyset$, every nontrivial SCC of $G(A)$ is in W and is reachable in $G(A, B_a)$. Thus, $B_a \in \mathcal{K}$ and $\|B_a\|_w = \|B_n\|_w \leq \epsilon\|B_n^*\|_w = \|B^*\|_w$.

Step 1 requires $O(\max\{d^2, dm\})$ computations. The formulation of the WSCP by Algorithm 3 and the greedy selection for finding an approximate solution to Problem IV.6 (when $W \neq \emptyset$) requires $O(d^2)$ and $O(d)$ complexity [19], respectively. Hence, Algorithm 4 takes $O(\max\{d^2, dm\})$ time. □

Remark IV.11. Consider *structurally cyclic system* (A, B) where $G(A)$ is spanned by a disjoint union of cycles of state vertices. In several systems, it happens that the dynamics of a state depends on its immediate past, depicted as a self loop. These systems also belongs to the class of structurally cyclic systems. In structurally cyclic systems, every state vertex is contained in a cycle. This implies that each state vertex must be reachable for ensuring generic zero controllability and every SSCC of $G(A)$ is nontrivial. Thus, $\mathcal{N} \neq \emptyset$. By Definition of SSCC, all the states are reachable iff all the (nontrivial) SSCCs are reachable. This implies that $W = \emptyset$. Hence, $(\mathcal{P}_2)$ is solved optimally for this class (Theorem IV.10 case (b)).

Remark IV.12. The problem of determining the minimal number of states to be actuated to ensure structural controllability is polynomial solvable [20], [21]. However, $(\mathcal{P}_2)$ is NP-hard.

C. Complexity of Problem $(\mathcal{P}_3)$ and Approximation

By assumption, the given pair (A, B) is not g.z.c. If a pair (A, B) is not g.z.c then it is reducible. Then, by Theorem II.3, $G(A)$ contains at least one cycle in the associated unreachable part $G_{urn}(A, B)$ of the pair (A, B). Consider the definition:

Definition IV.13 ([22]). A *minimum feedback arc set* of a digraph G is a set of edges of minimum cardinality whose removal make the resultant graph free of cycles, i.e., breaks the cycles present in the graph G.

We have the following proposition that will play a key role.

Proposition IV.14. *Let (A, B) be a not g.z.c system and $G_{urn}(A, B)$ be its associated unreachable part (Definition II.2). The cardinality of an optimal solution of $(\mathcal{P}_3)$ is equal to cardinality of a minimum feedback arc set of $G_{urn}(A, B)$.*

Proof. Let E_{opt} be a minimum feedback arc set of $G_{urn}(A, B)$ of cardinality k. Let A_{opt} be the matrix associated with E_{opt} such that $[A_{opt}]_{ij} = \star$ if $(x_j, x_i) \in E_{opt}$, and 0 otherwise. Note that the set of unreachable states remain unchanged even after removal of the state-connections in E_{opt} in the resulting digraph $G(A \ominus A_{opt}, B)$. Thus, the digraph induced by these unreachable states, i.e., $G_{urn}(A \ominus A_{opt}, B)$ is acyclic. By Theorem II.3, $(A \ominus A_{opt}, B)$ is g.z.c. This shows that a feasible solution to $(\mathcal{P}_3)$ of cardinality k exists. Next, we prove that every feasible solution of $(\mathcal{P}_3)$ has cardinality at least k which completes the proof. Suppose $\hat{A} \subset A$ be such that $(A \ominus \hat{A}, B)$ is g.z.c. Let $\hat{E}$ be the edges associated with the non-zero entries of $\hat{A}$. Clearly, the set of states unreachable in $G(A, B)$ are also unreachable in $G(A \ominus \hat{A}, B)$. The existence of no cycle in digraph $G_{urn}(A \ominus \hat{A}, B)$ implies that there exists an $E_1 \subset \hat{E}$ whose deletion breaks the cycles in $G_{urn}(A, B)$, i.e., E_1 is a feedback arc set of $G_{urn}(A, B)$. It means that $\left\|\hat{A}\right\|_0 = \left|\hat{E}\right| \geq |E_1| \geq |E_{opt}| = k$, and the assertion follows. $\square$

We show that $(\mathcal{P}_3)$ is NP-complete by using the *minimum feedback arc set problem (MFAP)*. For a digraph G and a positive integer k, it is NP-complete to decide whether the cardinality of a minimum feedback arc set is at most k [22].

Theorem IV.15. *$(\mathcal{P}_3)$ is NP-complete.* [8]

Proof. Since generic zero controllability is verifiable in polynomial time (See Remark II.6), the decision version of $(\mathcal{P}_3)$ is in NP. To prove the NP-hardness, we construct an instance of $(\mathcal{P}_3)$ from the instance of the MFAP, i.e., a digraph G containing cycles.[9] Let $G = (V, E)$ be a digraph with $V = \{x_1, x_2, \ldots, x_d\}$ and $e_{ij} = (x_j, x_i) \in E$. Construct a pair (A, B), where $A \in \{0, \star\}^{(d+1)\times(d+1)}$ and $B \in \{0, \star\}^{(d+1)\times 1}$:

$$A_{ij} \leftarrow \begin{cases} \star & \text{if } e_{ij} \in E, \\ 0 & \text{otherwise.} \end{cases} \quad B_{i1} \leftarrow \begin{cases} \star & \text{if } i = d+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $G(A) = (\mathcal{A}, E_A)$, where $\mathcal{A} = \{x_1, x_2, \ldots, x_d, x_{d+1}\}$ and $E_A = E$. Notice that $G_{urn}(A, B) = (\mathcal{A} \setminus x_{d+1}, E_A)$ is simply the input digraph G. The constructed pair (A, B) is not g.z.c since $G_{urn}(A, B)$ has cycles (by Theorem II.3).

Consequently, by Proposition IV.14, the cardinality of an optimal solution of $(\mathcal{P}_3)$ is less than a given number k, if and only if the minimum feedback arc set of G has cardinality below k. Since the reduction can be performed in polynomial time, and the MFAP is NP-complete, the assertion follows. $\square$

Next, we give an approximation of $(\mathcal{P}_3)$ by using the MFAP.

Lemma IV.16. *Let (A, B) be a not g.z.c linear system (1) and $G(A, B)$ be its associated digraph. Let $G_{urn}(A, B) = (\mathcal{A}_{urn}, E_{urn})$ be the associated unreachable part of $G(A)$ defined in Definition II.2. Then, for any $\epsilon \geq 1$, if $E' \subset E_{urn}$ be an ϵ-optimal solution of the MFAP on $G_{urn}(A, B)$ then A' such that $[A']_{ij} = \star$ if $(x_j, x_i) \in E'$, and 0 otherwise, is an ϵ-optimal solution of $(\mathcal{P}_3)$.*

Proof. i) It follows from the proof of Proposition IV.14 that if $E_{opt} \subset E_{urn}$ is an optimal solution of the MFAP on $G_{urn}(A, B)$ then A_{opt} is an optimal solution of $(\mathcal{P}_3)$ with $|E_{opt}| = \|A_{opt}\|_0$. ii) Observe that if E' is a feasible solution of the MFAP on $G_{urn}(A, B)$ then A' is a feasible solution to $(\mathcal{P}_3)$ with $\|A'\|_0 = |E'|$. Thus, given $|E'| \leq \epsilon |E_{opt}|$ implies that $\|A'\|_0 \leq \epsilon \|A_{opt}\|_0$, and completes the proof. $\square$

The best known approximation algorithm for solving the MFAP for a digraph $G = (V, E)$ has a non-constant ratio of $O(\log |V| \log\log |V|)$ [23, Corollory 6, p. 164]. From Lemma IV.16, by finding an approximate solution of MFAP on $G_{urn}(A, B)$ for a given pair (A, B), we obtain an A' that solves $(\mathcal{P}_3)$ approximately. Therefore, we get an $O(\log |\mathcal{A}_{urn}| \log\log |\mathcal{A}_{urn}|)$ solution A' to $(\mathcal{P}_3)$.

Remark IV.17. It is well-known that the deletion of state-connections can never strengthen the structural controllability of a system [6], [13]. In fact, given a structurally controllable system, determining a minimal set of state-connections whose removal makes the resulting system structurally uncontrollable was investigated in [13] and shown to be NP-complete. Thus, $(\mathcal{P}_3)$ may appear counter-intuitive at first. However, we demonstrate that by appropriately removing the state-connections the resulting system attains generic zero controllability.

V. Illustrative example

Example 1: We consider a multi-agent network to demonstrate the applicability of $(\mathcal{P}_1)$. Let the communication graph of the agents be $G = (V, E)$, where V represents the set of agents and E be the communication links between them. Let $N^-(x_i)$ be the in-neighbours of agent x_i. Each agent is assumed to be single integrator and communicate with itself. An agent x_i updates its states as

$$x_i(t+1) = a_{ii}x_i(t) + \sum_{x_j \in N^-(x_i)} a_{ij}x_j(t), \tag{5}$$

where $x_i(t)$ is its state at time t. In this setting, a set of the states, say $\mathcal{J}$, are selected to be driven by individual external inputs. By taking into consideration the external inputs, we rewrite the system in compact form as:

$$x(t+1) = Ax(t) + I_{\mathcal{J}}u(t), \tag{6}$$

where $I_{\mathcal{J}}$ is obtained from the identity matrix by retaining the columns corresponding to states connected to inputs. In our analysis, we focus only on the zero/non-zero structure of $A \in \{0, \star\}^{d\times d}$ and $I_{\mathcal{J}} \in \{0, \star\}^{d\times|\mathcal{J}|}$. Observe that A has a zero-free diagonal.

We consider a set of eight agents connected to each other via a communication graph shown in Figure. 1. Every state (agent) linked to an input is enclosed in a blue box. Here

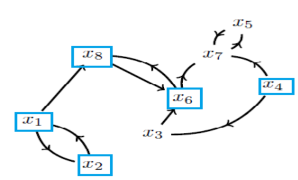

Fig. 1: The communication graph of a multi-agent system. Each agent has a self-loop and are not depicted here.

$\mathcal{J} = \{x_1, x_2, x_4, x_6, x_8\}$. Observe that the pair $(A, I_{\mathcal{J}})$ is g.z.c since it is irreducible. Every state vertex (agent) has a self loop. Under the condition that there exists a self loop at

[8]A problem is NP-complete if it is NP-hard and its decision version lies in NP.

[9] The hardness of the MFAP can be shown by using the vertex cover problem [16] where the constructed instance G of the MFAP in the proof has cycles. Thus, the problem remains NP-complete even when the digraph G in every instance has cycles. The hardness arises due to existence of cycles.

each state, a pair (A, B) is structurally controllable iff it is irreducible [24, Theorem 2]. Therefore, $(A, I_{\mathcal{J}})$ is structurally controllable. We determine the collection of nontrivial SCCs as $\{x_1, x_2\}$, $\{x_3\}$, $\{x_4\}$, $\{x_5, x_7\}$, and $\{x_8\}$. Among them, the nontrivial SSCCs are $\mathcal{N}_1 = \{x_1, x_2\}$ and $\mathcal{N}_2 = \{x_4\}$, collected in $\mathcal{N}$, with $\deg^-(\mathcal{N}_1) = 2$ and $\deg^-(\mathcal{N}_2) = 1$. Since the rest of the nontrivial SCCs are reachable from at least one of the SSCCs in $\mathcal{N}$, we use Remark IV.2 to solve $(\mathcal{P}_1)$. Thus, $T_{del}(A, I_{\mathcal{J}}) = \min_{1\le i\le 2} \deg^-(\mathcal{N}_i) = 1$ and I^u that solves $(\mathcal{P}_1)$ is such that $I^u_{44} = \star$, and 0 otherwise. Clearly, $(A, I_{\mathcal{J}} \ominus I^u)$ is not g.z.c since $G_{urn}(A, I_{\mathcal{J}} \ominus I^u)$ contains a cycle/nontrivial SCC at $\{x_4\}$. It follows that $(A, I_{\mathcal{J}} \ominus I^u)$ is not structurally controllable (see Remark IV.3). Thus, if the input-connection/input corresponding to I^u of cardinality one is removed then the resulting system is not g.z.c and consequently, makes the system structurally uncontrollable.

Example 2: We illustrate the results obtained in §IV-B to solve $(\mathcal{P}_2)$ on an electric power grid. In particular, we consider the IEEE-5 bus system. It is linearized around a nominal point and represented as a discrete-time system by using the modelling given in [25]. It comprises of three generators and two loads linked to each other through transmission lines. The digraph $G(A)$ of the model is depicted in Fig. 2 containing 18 states. The states x_1, x_4, and x_7 denote the frequencies of generators 1-3. In the generators 1-3, the states x_2, x_5, and x_8 represent the mechanical power of the turbine, and x_3, x_6, and x_9 are their valve opening. For the loads 1 and 2, x_{10} and x_{13} denote their frequency measured locally and the real energy consumed by them is given by states x_{11} and x_{13}, respectively. The injected/received power variables to/from the network exhibit the connections between the components. The dynamics of these power variables depends on the frequency of the neighbouring bus components. For the generators and the loads, the injected/received power variables are denoted by x_{14}, x_{15}, x_{16}, x_{17}, and x_{18}, respectively.

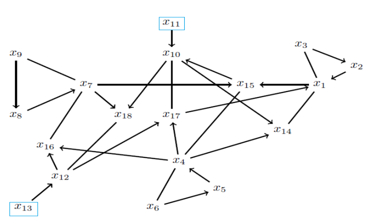

Fig. 2: The digraph $G(A)$ of the IEEE-5 bus system is shown. Every state vertex has a self-loop and are not depicted to avoid clutter. An undirected edge represents bidirectional edges.

We assume that the initial input matrix is $B = I_d \in \{0, \star\}^{d\times d}$ (where $d = 18$) and an uniform cost is given to every input-connection. Clearly, the given pair (A, I_d) is g.z.c since $G_{urn}(A, I_d)$ is empty. The nontrivial SSCCs of $G(A)$ are $\mathcal{N}_1 = \{x_{11}\}$ and $\mathcal{N}_2 = \{x_{13}\}$ (shown in blue boxes in Fig. 2). Notice that all the state vertices are reachable from either $\mathcal{N}_1$ or $\mathcal{N}_2$. Thus, the set W defined in Part (b) of §IV-B is empty and the conditions of case (b) of Theorem IV.10 are satisfied. In this case, we solve $(\mathcal{P}_2)$ easily by using Algorithm 4 and obtain $B^* = B^*_e$ with $B^*_{11,11} = \star$, $B^*_{13,13} = \star$, and 0 otherwise, as a solution of $(\mathcal{P}_2)$.

Acknowledgement

The authors are highly indebted to Prof. Debasish Chatterjee for his valuable guidance and inputs throughout the work.

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