Discrete Mathematics 341 (2018) 1732–1739

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Induced-bisecting families of bicolorings for hypergraphs

Niranjan Balachandran [a], Rogers Mathew [b,*], Tapas Kumar Mishra [b], Sudebkumar Prasant Pal [b]

[a] *Department of Mathematics, Indian Institute of Technology, Bombay 400076, India*
[b] *Department of Computer Science and Engineering, Indian Institute of Technology, Kharagpur 721302, India*

ARTICLE INFO

Article history:
Received 3 June 2017
Received in revised form 2 February 2018
Accepted 15 March 2018
Available online 2 April 2018

Keywords:
Separating family
Bisecting families
Hypergraph bicoloring

ABSTRACT

Two n-dimensional vectors A and B, $A, B \in \mathbb{R}^n$, are said to be *trivially orthogonal* if in every coordinate $i \in [n]$, at least one of $A(i)$ or $B(i)$ is zero. Given the n-dimensional Hamming cube $\{0, 1\}^n$, we study the minimum cardinality of a set $\mathcal{V}$ of n-dimensional $\{-1, 0, 1\}$ vectors, each containing exactly d non-zero entries, such that every 'possible' point $A \in \{0, 1\}^n$ in the Hamming cube has some $V \in \mathcal{V}$ which is orthogonal, but not trivially orthogonal, to A. We give asymptotically tight lower and (constructive) upper bounds for such a set $\mathcal{V}$ except for the case where $d \in \Omega(n^{0.5+\epsilon})$ and d is even, for any ϵ, $0 < \epsilon \leq 0.5$.

1. Introduction

Two n-dimensional vectors $A = (A(1), \ldots, A(n))$ and $B = (B(1), \ldots, B(n))$, $A, B \in \mathbb{R}^n$, are said to be *trivially orthogonal* if in every coordinate $i \in [n]$, at least one of $A(i)$ or $B(i)$ is zero. The vectors A and B are *non-trivially orthogonal* if they are orthogonal, but not trivially orthogonal. Consider the following problem: "Given the n-dimensional Hamming cube $\{0, 1\}^n$, what is the minimum cardinality of a subset $\mathcal{V}$ of n-dimensional $\{-1, 0, 1\}$ vectors, each containing exactly d non-zero entries, such that every point $A \in \{0, 1\}^n$ in the Hamming cube has some $V \in \mathcal{V}$ which is non-trivially orthogonal to A?". It is not hard to see that the all-zero vector and the unit vectors $\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$ can never have any non-trivially orthogonal vector in $\{-1, 0, 1\}^n$. Additionally, the all-ones vector $(1, \ldots, 1)$ cannot be non-trivially orthogonal to any vector in $\{-1, 0, 1\}^n$ consisting of exactly d non-zero entries, when d is odd. We call the vectors $(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ (and additionally, $(1, \ldots, 1)$ when d is odd) as *trivial*. Since no n-dimensional $\{-1, 0, 1\}$ vector with exactly one non-zero entry is non-trivially orthogonal to any non-trivial point of the Hamming cube, we assume that $d \geq 2$ in the rest of the paper.

Definition 1. Let $2 \leq d \leq n$, where d and n are integers. We define $\beta^d(n)$ as the minimum cardinality of a subset $\mathcal{V}$ of n-dimensional $\{-1, 0, 1\}$ vectors, each containing exactly d non-zero entries, such that every non-trivial point in the Hamming cube $\{0, 1\}^n$ has a non-trivially orthogonal vector $V \in \mathcal{V}$.

In this paper, we study the problem of estimation of bounds for $\beta^d(n)$.

Before we consider the general version of the problem, we introduce a few definitions. Let $[n]$ denote the finite set $\{1, \ldots, n\}$. For any $S \subset [n]$, let (i) $R_S = (r_1, \ldots, r_n) \in \{-1, 1\}^n$ denote the incidence vector corresponding to S: r_i is 1 if and only if $i \in S$; (ii) $X_S = (x_1, \ldots, x_n) \in \{0, 1\}^n$ denote the incidence vector corresponding to S: x_i is 1 if and only if $i \in S$.

* Corresponding author.
E-mail addresses: niranj@iitb.ac.in (N. Balachandran), rogers@cse.iitkgp.ernet.in (R. Mathew), tkmishra@cse.iitkgp.ernet.in (T.K. Mishra), spp@cse.iitkgp.ernet.in (S.P. Pal).

https://doi.org/10.1016/j.disc.2018.03.015
0012-365X/

The inner product $\langle A, B\rangle$ is the standard dot product $\sum_{i=1}^{n} a_i b_i$, where $A = (a_1, \ldots, a_n)$, and $B = (b_1, \ldots, b_n)$. Let $G([n], E(G))$ denote the hypergraph where $[n]$ is the set of vertices and $E(G)$ is the set of hyperedges. We denote the set of all distinct k-sized subsets of $[n]$ as $\binom{[n]}{k}$ for $k \le n$.

We now define a general version of the aforementioned problem in terms of bicolorings of a hypergraph. Let G be a hypergraph on the vertex set $[n]$. Corresponding to the trivial vectors/points of the Hamming cube, the singleton sets and the empty set (and additionally, the set $[n]$ when d is odd) are the *trivial hyperedges* or *trivial subsets* of $[n]$. Let X^S denote a ± 1 bicoloring of vertices of $S \subseteq [n]$, i.e. $X^S : S \to \{+1, -1\}$, for some $S \subseteq [n]$, $S \ne \phi$. We abuse the notation to denote the subset of vertices colored with $+1$ (resp., -1) with respect to bicoloring X^S as $X^S(+1)$ (resp., $X^S(-1)$), i.e. $X^S(+1) = \{i \in S : X^S(i) = +1\}$ and $X^S(-1) = \{i \in S : X^S(i) = -1\}$ for $S \ne \phi$.

Definition 2. Given a hypergraph G, a hyperedge $A \in E(G)$ is said to be *induced-bisected* by a bicoloring X^S of a subset $S \subseteq V(G)$, if $|A \cap X^S(+1)| = |A \cap X^S(-1)| \ne 0$. A set $\mathcal{X} = \{X^{S_1}, \ldots, X^{S_t}\}$ of t bicolorings is called an *induced-bisecting family of order d* for G if

1. each $S_i \subseteq [n]$ has exactly d vertices, $1 \le i \le t$, $2 \le d \le n$, and
2. every non-trivial hyperedge $A \in E(G)$ is induced-bisected by at least one X^{S_i}, $1 \le i \le t$.

Let $\beta^d(G)$ denote the minimum cardinality of an induced-bisecting family of order d for hypergraph G.

From Definitions 1 and 2, it is clear that the maximum of $\beta^d(G)$ over all hypergraphs G on $[n]$ is $\beta^d(n)$.

Example 1. Let $\mathcal{H}$ be the hypergraph with all the $2^n - n - 1$ non-trivial subsets of $[n]$ as hyperedges and let $d = 2$. For any $S \in \binom{[n]}{2}$, let X^S color one point in S with color $+1$ and the other with -1. Observe that $\mathcal{X} = \{X^S | S \in \binom{[n]}{2}\}$ forms an induced-bisecting family of order 2 for $\mathcal{H}$. $\beta^2(\mathcal{H}) \le \binom{n}{2}$. Moreover, this upper bound is also tight: if $X^{\{a,b\}} \notin \mathcal{X}$, $\{a, b\} \in \binom{[n]}{2}$, then the hyperedge $\{a, b\} \in \mathcal{H}$ cannot be induced-bisected.

1.1. Application

In biology, a *character* is a feature of an organism that may be present as various *traits*.[1] A drug company wishes to perform a comparative study of its new drug with an existing drug (these drugs have small half-life[2] periods). Consider a population of n individuals participating in the drug test. Corresponding to various combinations of traits of various characters, there are m groups. For instance, individuals with brown eyes forming one group, tall and female individuals constitute another group. The drug company wishes to determine those traits (or combination of traits) for which the new drug performs better on individuals possessing the traits than the old drug. In order to make such a comparative study 'fair', for each group out of the m aforementioned groups, the company requires that equal number of individuals must receive the new drug and the old drug while comparing the effectiveness of the drugs in correlation of traits of that group. The lab facility available to the company for such a test is limited: at most d individuals can be tested at a time. It is clear that this study may require multiple rounds of testing. In each round, d individuals are tested: each individual receiving exactly one out of the two drugs. Note that each round of testing can be viewed as a bicoloring of d out of n element set (representing the population), where individuals receiving different drugs get different colors in the bicoloring. It is in the best interest of the company to minimize the number of rounds (i.e. bicolorings) for conducting such a 'fair' comparative study.

1.2. Relations to existing work

The problem addressed in this paper can be viewed as a generalization of the problem of *bisecting families* [1]. Let $n \in \mathbb{N}$ and let $\mathcal{A}$ be a family of subsets of $[n]$. Another family $\mathcal{B}$ of subsets of $[n]$ is called a *bisecting family* for $\mathcal{A}$, if for each $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ such that $|A \cap B| \in \{\lceil \frac{|A|}{2} \rceil, \lfloor \frac{|A|}{2} \rfloor\}$. In the bicoloring terminology, letting $S = [n]$, $X^S(+1) = B$, $X^S(-1) = [n] \setminus B$, the bisecting family $\mathcal{B}$ maps to a collection $\mathcal{X}$ of bicolorings such that for each $A \in \mathcal{A}$, there exists a bicoloring $X \in \mathcal{X}$ such that $|A \cap X(+1)| - |A \cap X(-1)| \in \{-1, 0, 1\}$. In [1], the authors define $\beta_{[\pm 1]}(n)$ as the minimum cardinality of a bisecting family for the family consisting of all the non-empty subsets of $[n]$, and they prove that $\beta_{[\pm 1]}(n) = \lceil \frac{n}{2} \rceil$ [1]. Note that when $d = n$ and $\mathcal{A}_e$ denotes the family of non-trivial even subsets of $[n]$, any induced bisecting family of order d for $\mathcal{A}_e$ is a bisecting family for the family consisting of all the non-empty subsets of $[n]$. In other words, $\beta^n(\mathcal{A}_e) = \beta_{[\pm 1]}(n)$. However, observe that when $d = n$, i.e. $S = [n]$, no odd subset of $[n]$ can be induced-bisected: this follows from the fact that for any odd subset A, $|A \cap X^S(+1)| - |A \cap X^S(-1)|$ is odd.

Induced-bisecting families can be viewed as an extension of *separating families* and *test covers*. A subset S separates a pair $\{i, j\}$ if $i \in S$ and $j \notin S$ or vice versa. The family $\mathcal{S}$ is a separating family for $\mathcal{F}$ if every pair $\{i, j\} \in \mathcal{F}$ is separated by some $S \in \mathcal{S}$ (see [14,9,17,6,16] for detailed results and related problems on separating families). An extension of the separating family problem is the 'test cover' problem: "Given a family $\mathcal{F}$ of subsets of $[n]$, finding a sub-collection $\mathcal{T} \subseteq \mathcal{F}$ of

[1] For instance, 'eye color' is a character and, 'blue eye color' and 'brown eye color' are various traits corresponding to this character.

[2] The half-life of a drug is the time taken for the drug concentration in the body to be reduced to one-half.

minimum cardinality such that every pair of $[n]$ is separated by some $S \in \mathcal{T}$" (see [12,8,4,3,2]). The problems of separating families and test covers come under the broader area of combinatorial group testing (see [7]), and are studied in the context of 'Wasserman-type' blood tests of large populations and locating defective items (see [10]), biology [13,18,11] and pattern recognition [5].

Main result

In this paper, we establish the following theorem.

Theorem 3. *Let $2 \leq d \leq n$, where d and n are integers. Then,* $\frac{2n(n-1)}{d^2} \leq \beta^d(n) \leq \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$. *Moreover,* $\beta^d(n) \geq n-1$, *when d is odd.*

When d is odd, $\beta^d(n) \geq \max(\frac{2n(n-1)}{d^2}, n-1) \geq \frac{\frac{2n(n-1)}{d^2}+n-1}{2}$; so the bounds are tight up to constant factors for all values of n when d is odd. Moreover, the bound is tight up to constant factors when $d \in O(n^{0.5-\epsilon})$ for any ϵ, $0 < \epsilon \leq 0.5$, even if d is even. However, when $d \in \Omega(n^{0.5+\epsilon})$ and d is even, there is a gap between the lower and upper bounds, for any ϵ, $0 \leq \epsilon \leq 0.5$.

2. Lower bounds

Let $\mathcal{H}$ denote the hypergraph consisting of all the non-trivial subsets of $[n]$. Let the set $\mathcal{X} = \{X^{S_1}, \ldots, X^{S_t}\}$ of bicolorings be any optimal induced-bisecting family of order d for $\mathcal{H}$, where $t \in \mathbb{N}$.

Considering only the two sized subsets of $[n]$, we note that every two element hyperedge $\{a, b\}$, $a, b \in [n]$, must lie in some S_i, $S_i \in \{S_1, \ldots, S_t\}$; otherwise, no bicoloring in $\mathcal{X}$ can induced-bisect $\{a, b\}$. So, it follows that $\sum_{X^S \in \mathcal{X}} \binom{d}{2} \geq \binom{n}{2}$, i.e., $\beta^d(n) \geq \frac{n(n-1)}{d(d-1)}$. A constant factor improvement in the lower bound can be obtained by the following observation: the maximum number of two element subsets $\{a, b\}$ that can be induced-bisected by any $X^S \in \mathcal{X}$, $|S| = d$, is $\frac{d^2}{4}$. So, we have the following proposition.

Proposition 4. $\beta^d(n) \geq \frac{2n(n-1)}{d^2}$.

Observe that when d is large, say $d \in \Omega(n^{0.5+\epsilon})$, where $0 < \epsilon \leq 0.5$, Proposition 4 only yields a sublinear lower bound. When d is odd, we can prove a general lower bound of $n-1$ on $\beta^d(n)$ using the following version of Cayley–Bacharach Theorem by Riehl and Graham [15] on the maximum number of common zeros between n quadratics and any polynomial P of smaller degree.

Theorem 5 (*[15]*). *Given the n quadratics in n variables $x_1(x_1-1), \ldots, x_n(x_n-1)$ with 2^n common zeros, the maximum number of those common zeros a polynomial P of degree k can go through without going through them all is $2^n - 2^{n-k}$.*

Lemma 6. $\beta^d(n) \geq n-1$, *when d is odd.*

Proof. Let $\mathcal{B}$ be a minimum-cardinality induced-bisecting family for all the non-trivial subsets $A \subseteq [n]$. Let R_B denote the n-dimensional vector representing the bicoloring $B \in \mathcal{B}$, i.e. $R_B \in \{-1, 0, 1\}^n$ and R_B contains exactly d nonzero entries. Consider the polynomials $M(X)$, $N(X)$, and $P(X)$, $X \in \{0, 1\}^n$.

$$M(X = (x_1, \ldots, x_n)) = \prod_{B \in \mathcal{B}} \langle R_B, X \rangle. \tag{1}$$

$$N(X = (x_1, \ldots, x_n)) = \sum_{i=1}^{n} x_i - 1. \tag{2}$$

$$P(X) = M(X)N(X). \tag{3}$$

Let X_A denote the 0-1 n-dimensional incidence vector corresponding to $A \subseteq [n]$. Note that $M(X_A)$ vanishes for each $A \subseteq [n]$ except (i) the all 1's vector, $(1, \ldots, 1)$, since d is odd, and (ii) possibly the singleton sets. Since $N(X_A)$ vanishes for all singleton sets, $P(X_A)$ vanishes on all subsets $A \subseteq [n]$ except for the set $[n]$ (corresponding to the all 1's vector). Since the degree of P is $|\mathcal{B}| + 1$ and P is non-zero only at $X_A = (1, \ldots, 1)$, using Theorem 5, we have $|\mathcal{B}| \geq n-1$. □

However, when d is even, the above lower bounding technique does not work since the polynomial M may vanish at every point of the Hamming cube $\{0, 1\}^n$. In this case, we can obtain a lower bound of $\Omega(\sqrt{d})$ by considering the maximum number of hyperedges that can be induced-bisected by a single bicoloring.

3. Induced-bisecting families when n is $d+1$

In what follows, we consider the hypergraph $\mathcal{H}$ consisting of all the non-trivial hyperedges of $[n]$, where $n = d+1$ and demonstrate a construction of an induced-bisecting family of order d of cardinality $d+1$.

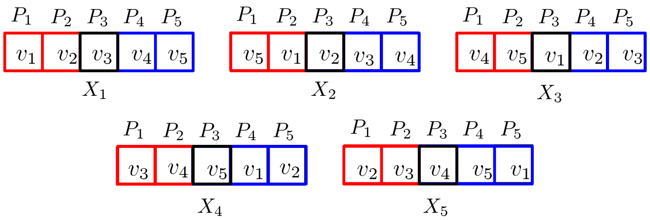

Fig. 1. Vertices in (i) P_1 and P_2 are colored with $+1$, (ii) P_4 and P_5 are colored with -1; the vertex in P_3 remains uncolored. $\mathcal{X} = \{X_1, \ldots, X_5\}$ is an induced bisecting family when $n = d + 1 = 5$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Theorem 7. *Let d be an integer greater than 1. Then, $d \leq \beta^d(d+1) \leq d+1$. Moreover, $\beta^d(d+1) = d+1$, when d is even.*

Proof. We consider the cases when d is even and d is odd separately. We start our analysis with the case when d is even. Let $v_1, \ldots, v_{d+1}$ denote the $d+1$ vertices. Consider a circular clockwise arrangement of $d+1$ slots, namely $P_1, \ldots, P_{d+1}$ in that order. The slots P_1 to $P_{\frac{d}{2}}$ are colored with +1, slots $P_{\frac{d}{2}+2}$ to P_{d+1} are colored with -1, and only slot $P_{\frac{d}{2}+1}$ remains uncolored. Each slot can contain exactly one vertex and each vertex takes the color of the slot it resides in. As for the initial configuration, let $v_i \in P_i$, for $1 \leq i \leq d+1$. This configuration gives the coloring X_1, where (i) $X_1(+1) = \{v_1, \ldots, v_{\frac{d}{2}}\}$, (ii) $X_1(-1) = \{v_{\frac{d}{2}+2}, \ldots, v_{d+1}\}$, and, (iii) the vertex $v_{\frac{d}{2}+1}$ remains uncolored. We obtain the second coloring X_2 from X_1 by one clockwise rotation of the vertices in the circular arrangement. Therefore, we have, $X_2(+1) = \{v_{d+1}, v_1, \ldots, v_{\frac{d}{2}-1}\}$, $X_2(-1) = \{v_{\frac{d}{2}+1}, \ldots, v_d\}$; the vertex $v_{\frac{d}{2}}$ remains uncolored. See Fig. 1 for an illustration. Similarly, repeating the process d times, we obtain the set $\mathcal{X} = \{X_1, \ldots, X_{d+1}\}$ of bicolorings. We have the following observations.

Observation 8. *If $\mathcal{X}$ induced-bisects every non-trivial odd subset of $[d+1]$, then $\mathcal{X}$ induced-bisects every non-trivial even subset of $[d+1]$ as well.*

To prove the observation, consider an even hyperedge $A_e \subset [d+1]$, and let $X \in \mathcal{X}$ be the bicoloring that induced-bisects the odd hyperedge $\bar{A_e} = [d+1] \setminus A_e$. Note that one vertex in $\bar{A_e}$ remains uncolored under X. Otherwise, $\bar{A_e}$ cannot get induced-bisected under X. Since $|X(+1)| = \frac{d}{2}$ and $|\bar{A_e} \cap X(+1)| = \frac{|\bar{A_e}|-1}{2}$, it follows that $|A_e \cap X(+1)| = |X(+1) \setminus (\bar{A_e} \cap X(+1))| = \frac{d}{2} - \frac{|\bar{A_e}|-1}{2}$. Similarly, $|A_e \cap X(-1)| = \frac{d}{2} - \frac{|\bar{A_e}|-1}{2}$. So, A_e is induced-bisected under X. This completes the proof of Observation 8. □

Therefore, it suffices to prove that $\mathcal{X}$ induced-bisects every non-trivial odd subset of $[d+1]$. For the sake of contradiction, assume that A is an odd hyperedge not induced-bisected by $\mathcal{X}$. Let $c_i = |A \cap X_{i+1}(+1)| - |A \cap X_{i+1}(-1)|$, for $0 \leq i \leq d$. All additions/subtractions in the subscript of c are modulo $d+1$. Our assumption implies that $c_i \neq 0$ for all $0 \leq i \leq d$.

Observation 9. *$|c_i - c_{i+1}| \leq 2$, for $0 \leq i \leq d$. Furthermore, if $c_i > c_{i+1}$ and c_i is odd, then $c_i - c_{i+1} = 1$.*

The first part of Observation 9 follows from the construction and we omit the details for brevity. Note that when c_i is odd, the element in $P_{\frac{d}{2}+1}$ cannot belong to the odd hyperedge A. This takes care of the second part of Observation 9. □

Observe that a bicoloring $X_j \in \mathcal{X}$, $1 \leq j \leq d+1$, induced-bisects the odd hyperedge A if and only if c_j is 0. We know that bicoloring X_2 (resp. X_{i+1}) is obtained from X_1 (resp. X_i) by one clockwise rotation of vertices in the circular arrangement. Thus, during the construction of bicolorings X_1 through X_{d+1}, we perform a full rotation of the vertices with respect to their starting arrangement in X_1. So, it follows that there exist i and j such that c_i is positive and c_{i+j} is negative. Combined with the second part of Observation 9, this implies the existence of an index p such that $c_p = 0$. This is a contradiction to the assumption that A is not induced-bisected by $\mathcal{X}$. Therefore, every odd subset of $[d+1]$ is induced-bisected by $\mathcal{X}$, and using Observation 8, the upper bound on $\beta^d(d+1)$ follows.

To see that the upper bound is tight, observe that exactly one d-sized hyperedge can get induced bisected under a single bicoloring — the hyperedge missing the uncolored vertex. This completes the proof of Theorem 7 for even values of d.

Recall that along with the empty set and the singleton sets, the set $[d+1]$ becomes trivial when d is odd. When d is odd, the slots P_1 to $P_{\frac{d+1}{2}-1}$ are colored with $+1$, slots $P_{\frac{d+1}{2}+1}$ to P_{d+1} are colored with -1, and only slot $P_{\frac{d+1}{2}}$ remains uncolored. If we generate the bicolorings $\{X_1, \ldots, X_{d+1}\}$ as in the proof of Theorem 7, by similar arguments, it can be shown that $\{X_1, \ldots, X_{d+1}\}$ is indeed an induced-bisecting family for the hypergraph consisting of all the non-trivial hyperedges (see Appendix for a proof). The fact that $\beta^d(d+1) \geq d$ for odd values of d follows directly from Lemma 6. □

We have the following corollary which establishes an upper bound on the cardinality of any induced-bisecting family for arbitrary values of n.

Corollary 10. *Let $\mathcal{H}$ be any hypergraph on vertex set $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ and let $d \leq n-1$. Let $\mathcal{F}$ consist of $(d+1)$-sized subsets of $V(\mathcal{H})$ such that for every $B \in E(\mathcal{H})$, there exists an $A \in \mathcal{F}$ with (i) $|B \cap A| \geq 2$, when d is even; (ii) $2 \leq |B \cap A| \leq d$, when d is odd. Then, we can construct an induced-bisecting family of order d of cardinality $|\mathcal{F}|(d+1)$ for $\mathcal{H}$.*

Proof. For any subset $A \in \mathcal{F}$, using the procedure used in the proof of Theorem 7, we can obtain an induced-bisecting family $\mathcal{X}_A$ for all the non-trivial subsets of A, where $|\mathcal{X}_A| = d+1$. When d is even, $\mathcal{X}_A$ induced-bisects all the $2^{d+1} - (d+1) - 1$ non-empty and non-singleton subsets of A; therefore, each $B \in E(\mathcal{H})$ with $|B \cap A| \geq 2$ is induced-bisected by $\mathcal{X}_A$. When d is odd, $\mathcal{X}_A$ induced-bisects all but the empty set, the singleton sets, and A; so, each $B \in E(\mathcal{H})$ with $2 \leq |B \cap A| \leq d$ is induced-bisected by $\mathcal{X}_A$. Repeating the process for each $A \in \mathcal{F}$, we get an induced-bisecting family of cardinality $|\mathcal{F}|(d+1)$ for $\mathcal{H}$. □

Theorem 7 provides evidence for the following property (which is described in Corollary 11) of the odd subsets under any circular permutation of odd number of elements which may be of independent interest. For any circular permutation σ of $[n]$, $a, b \in [n]$, let $dist_\sigma(a, b)$ denote the clockwise distance between a and b with respect to σ, which is one more than the number of elements residing between a and b in the permutation σ in the clockwise direction.

Corollary 11. *Consider any circular permutation σ of $[n]$, where n is odd. For any odd k-sized subset $A \subseteq [n]$, let $(a_0, \ldots, a_{k-1})$ be the ordering of A with respect to σ. Then, there exists an index $i \in \{0, \ldots, k-1\}$ such that $dist_\sigma(a_i, a_{i+\lfloor \frac{k}{2} \rfloor}) < \frac{n}{2}$ and $dist_\sigma(a_{i+\lfloor \frac{k}{2} \rfloor+1}, a_i) < \frac{n}{2}$, where summation in the subscript of a is modulo k.*

Proof. Consider a circular clockwise arrangement of n slots, namely $P_1, \ldots, P_n$ in that order. Put vertex $\sigma(i)$ in P_i. Now, following the procedure outlined in the proof of Theorem 7, obtain a bicoloring that bisects A. Pick the uncolored vertex residing in slot $P_{\lceil \frac{n}{2} \rceil}$ with respect to the bicoloring X. Observe that this vertex satisfies the desired property. □

4. Upper bounds for $\beta^d(n)$ and proof of Theorem 3

From Proposition 4, we know that $\beta^d(n) \geq \frac{2n(n-1)}{d^2}$. In this section, we prove an upper bound of $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$ for $\beta^d(n)$.

4.1. A deterministic construction of induced-bisecting families

Lemma 12. $\beta^d(n) \leq \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$.

Before proceeding to the proof of the above lemma, we give few definitions that simplify the proof considerably. Let d be a positive even integer. Let $\mathcal{S}(n, d) = \{P_1, \ldots, P_{\lceil \frac{2n}{d} \rceil}\}$ denote a partition of $[n]$, where each $P \in \mathcal{S}(n, d) \setminus \{P_{\lceil \frac{2n}{d} \rceil}\}$ is of cardinality exactly $\frac{d}{2}$, and $|P_{\lceil \frac{2n}{d} \rceil}| \leq \frac{d}{2}$. Let $P^1_{\lceil \frac{2n}{d} \rceil} = P_{\lceil \frac{2n}{d} \rceil} \cup Q_1$, $P^2_{\lceil \frac{2n}{d} \rceil} = P_{\lceil \frac{2n}{d} \rceil} \cup Q_2$, where Q_i denotes a fixed $(\frac{d}{2} - |P_{\lceil \frac{2n}{d} \rceil}|)$-sized subset of P_i. For an even d, we define $\mathcal{P}(n, d)$, $\mathcal{D}(n, d)$ and $\mathcal{B}(n, d)$ as follows.

Definition of $\mathcal{P}(n, d)$

$$\mathcal{P}(n, d) = \begin{cases} \mathcal{S}(n, d), \text{ if } \frac{d}{2} \text{ divides } n \\ \mathcal{S}(n, d) \setminus \{P_{\lceil \frac{2n}{d} \rceil}\} \cup \{P^1_{\lceil \frac{2n}{d} \rceil}, P^2_{\lceil \frac{2n}{d} \rceil}\}, \text{ otherwise.} \end{cases} \quad (4)$$

Definition of $\mathcal{B}(n, d)$

$\frac{d}{2}$ divides n. For each $i, j \in \left[\frac{2n}{d}\right]$, $i < j$, let $B_{i,j} : P_i \cup P_j \to \{+1, -1\}$ denote a bicoloring, where

$$B_{i,j}(x) = \begin{cases} +1, \text{ if } x \in P_i \\ -1, \text{ if } x \in P_j. \end{cases}$$

Let $\mathcal{B}(n, d) = \{B_{i,j} | i, j \in \left[\frac{2n}{d}\right], i < j\}$ denote the set of bicolorings.

$\frac{d}{2}$ does not divide n. For each $i, j \in \left[\lceil \frac{2n}{d} \rceil - 1\right]$, $i < j$, let $B_{i,j} : P_i \cup P_j \to \{+1, -1\}$ denote a bicoloring, where

$$B_{i,j}(x) = \begin{cases} +1, \text{ if } x \in P_i \\ -1, \text{ if } x \in P_j. \end{cases}$$

Define the maps $B_{1,\lceil \frac{2n}{d} \rceil} : P_1 \cup P^2_{\lceil \frac{2n}{d} \rceil} \to \{-1, 1\}$ and $B_{i,\lceil \frac{2n}{d} \rceil} : P_i \cup P^1_{\lceil \frac{2n}{d} \rceil} \to \{-1, 1\}$, for $2 \leq i \leq \lceil \frac{2n}{d} \rceil - 1$ as follows:

$$B_{1,\lceil \frac{2n}{d} \rceil}(x) = \begin{cases} +1, \text{ if } x \in P_1 \\ -1, \text{ if } x \in P^2_{\lceil \frac{2n}{d} \rceil}. \end{cases}$$

$$B_{i,\lceil \frac{2n}{d} \rceil}(x) = \begin{cases} +1, & \text{if } x \in P_i \\ -1, & \text{if } x \in P^1_{\lceil \frac{2n}{d} \rceil} \end{cases}, \text{ for } 2 \le i \le \left\lceil \frac{2n}{d} \right\rceil - 1.$$

Let $\mathcal{B}(n,d) = \{B_{i,j} | i, j \in [\lceil \frac{2n}{d} \rceil], i < j\}$ denote the set of bicolorings.

Definition of $\mathcal{D}(n,d)$

$\mathcal{D}(n,d) = \{D_k | D_k = P_{2k-1} \cup P_{2k}, k \in [\lceil \frac{n}{d} \rceil - 1]\} \cup \{D_{\lceil \frac{n}{d} \rceil}\}$, where

$$D_{\lceil \frac{n}{d} \rceil} = \begin{cases} P_{\frac{2n}{d}-1} \cup P_{\frac{2n}{d}}, & \text{if } \frac{d}{2} \text{ divides } n \\ P_1 \cup P^2_{\lceil \frac{2n}{d} \rceil}, & \text{if } \frac{d}{2} \text{ does not divide } n \text{ and } \lceil \frac{2n}{d} \rceil \text{ is odd} \\ P_{\lceil \frac{2n}{d} \rceil - 1} \cup P^2_{\lceil \frac{2n}{d} \rceil}, & \text{if } \frac{d}{2} \text{ does not divide } n \text{ and } \lceil \frac{2n}{d} \rceil \text{ is even.} \end{cases} \tag{5}$$

Proof. If $d = n-1$, the statement of the lemma follows directly from Theorem 7. So, we assume that $d < n-1$ in the rest of the proof. We prove this lemma considering the exhaustive cases based on whether d is even or odd, separately.

Case 1. d is even

Let $\mathcal{P} = \mathcal{P}(n,d)$, $\mathcal{B} = \mathcal{B}(n,d)$ and $\mathcal{D} = \mathcal{D}(n,d)$.

Observation 13. *For any $C \subseteq [n]$, $|C| \ge 2$, if $|C \cap P| \le 1$, for all $P \in \mathcal{P}$, then C is induced-bisected by at least one $B \in \mathcal{B}$.*

For any $C \subseteq [n]$, $|C| \ge 2$, it follows from the premise that there exist $P_i, P_j \in \mathcal{P}$, $i < j$, such that $|C \cap P_i| = |C \cap P_j| = 1$. C is induced-bisected by the bicoloring $B_{i,j}$, thus completing the proof of Observation 13.

Let $\mathcal{C}$ denote the family of all the subsets of $[n]$ that are not induced-bisected by any $B \in \mathcal{B}$. Rephrasing Observation 13, for each $C \in \mathcal{C}$, there exists a $P \in \mathcal{P}$ (and thus, a $D \in \mathcal{D}$) such that $|C \cap P| \ge 2$ (respectively, $|C \cap D| \ge 2$). Let $\mathcal{D}' = \{D \cup \{j\} | j \in [n] \setminus D, D \in \mathcal{D}\}$. Recall that $|D| = d$, where d is an even integer less than $n-1$. So, each $D' \in \mathcal{D}'$ is a $(d+1)$-sized set. Using Corollary 10, every $C \in \mathcal{C}$ can be induced-bisected using $|\mathcal{D}|(d+1)$ bicolorings. Therefore, we have, $\beta^d(n) \le |\mathcal{B}| + |\mathcal{D}|(d+1) = \binom{\lceil \frac{2n}{d} \rceil}{2} + \lceil \frac{n}{d} \rceil (d+1)$, when d is even.

Case 2. d is odd

Let $\mathcal{P} = \mathcal{P}(n-1, d-1)$, $\mathcal{B} = \mathcal{B}(n-1, d-1)$ and $\mathcal{D} = \mathcal{D}(n-1, d-1)$. Since $d-1$ is even, $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{D}$ are well defined. We extend the domain of each $B \in \mathcal{B}$ to $domain(B) \cup \{n\}$, and assign a $+1$ color to n in each B. Now, each $B \in \mathcal{B}$ colors exactly d elements of $[n]$.

Observation 14. *For any $C \subseteq [n]$ with $|C| \ge 2$, if $n \notin C$ and $|C \cap P| \le 1$ for all $P \in \mathcal{P}$, then C is induced-bisected by at least one $B \in \mathcal{B}$.*

The proof of this observation is exactly the same as the proof of Observation 13.

Let $\mathcal{C}$ denote the family of all the subsets of $[n]$ that are not induced-bisected by any $B \in \mathcal{B}$. For any $D \subseteq [n]$, let $\max(D)$ denote the maximum integer in the set D. Let $\mathcal{D}' = \{D \cup \{n\} \cup \{\max(D) + 1\} | D \in \mathcal{D}\}$, where the addition is modulo $n-1$.

Observation 15. *Let $\mathcal{D}' = \{D'_1, D'_2, \ldots, D'_{\lceil \frac{n-1}{d-1} \rceil}\}$ be the family of subsets constructed as above. Then, $|D'_i \cap D'_{i+1}| = 2$, if $1 \le i \le \lceil \frac{n-1}{d-1} \rceil - 1$, and $|D'_{\lceil \frac{n-1}{d-1} \rceil} \cap D'_1| \ge 2$.*

Recall that each $D \in \mathcal{D}$ is a $(d-1)$-sized subset of $[n-1]$, where d is an odd integer less than $n-1$. So, each $D' \in \mathcal{D}'$ is a $(d+1)$-sized set. From Observation 14, it follows that for each $C \in \mathcal{C}$, there exists at least one $D' \in \mathcal{D}'$ such that $|C \cap D'| \ge 2$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the family of subsets of $[n]$ such that for each $C' \in \mathcal{C}'$, there exists some $D' \in \mathcal{D}'$ such that $2 \le |C \cap D'| \le d$. Using Corollary 10, we can obtain an induced-bisecting family for members of $\mathcal{C}'$ of cardinality $|\mathcal{D}|(d+1)$. So, it follows that any $C \in \mathcal{C} \setminus \mathcal{C}'$ must contain one or more elements from $\{D'_1, D'_2, \ldots, D'_{\lceil \frac{n-1}{d-1} \rceil}\}$ as its subsets.

For any $C \in \mathcal{C} \setminus \mathcal{C}'$, if $D'_i \subseteq C$, then $D'_{i+1} \subseteq C$: otherwise, from Observation 15, $2 \le |C \cap D'_{i+1}| \le d$ and from definition of $\mathcal{C}'$, $C \in \mathcal{C}'$. So, it follows that $\mathcal{C} \setminus \mathcal{C}' = \{[n]\}$, and $[n]$ is a trivial set when d is odd. Therefore, the cardinality of the induced-bisecting family for $[n]$ when d is odd is at most $|\mathcal{B}| + |\mathcal{D}|(d+1) = \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$. □

4.2. Proof of Theorem 3

Statement. *Let $2 \le d \le n$, where d and n are integers. Then, $\frac{2n(n-1)}{d^2} \le \beta^d(n) \le \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$. Moreover, $\beta^d(n) \ge n-1$, when d is odd.*

Proof. Theorem 3 follows from Proposition 4, Lemmas 6 and 12. □

Remark 1. By removing some duplicate bicolorings, we can actually improve the upper bound for $\beta^d(n)$ from $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$ to $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil d$.

Theorem 3 asserts an upper bound of $O(n)$ on $\beta^d(n)$ when $d \in \Omega(\sqrt{n})$. Let $k(G)$ denote the minimum cardinality of any hyperedge of the hypergraph G, i.e., $k(G) = \min_{e \in E(G)} |e|$. For any hypergraph G, the upper bound for $\beta^d(G)$ can be improved to $O(n)$ even if $d \in o(\sqrt{n})$ provided $(d-1)k(G) > n-1$ in the following way. Since $(d-1)k(G) > n-1$, every hyperedge is large enough so that the family $\mathcal{D}'$ constructed in all the cases of proof of Lemma 12 satisfies the conditions of the family requirements of Corollary 11. Therefore, the set of bicolorings given by $\mathcal{B} = \mathcal{B}(n, d)$ (or $\mathcal{B}(n-1, d-1)$) can be completely avoided. Thus, we have the following theorem.

Theorem 16. *For any hypergraph G, let $k(G) = \min_{e \in E(G)} |e|$. If $(d-1)k(G) > n-1$, then $\beta^d(G) \leq \lceil \frac{n-1}{d-1} \rceil (d+1)$.*

Remark 2. The proof of Theorem 3 is algorithmic: it yields an induced bisecting family of cardinality at most $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$ cardinality with a running time of $O(\frac{n^2}{d^2} + n)$. Observe that the running time of our algorithm is asymptotically equivalent to the cardinality of the family of bicolorings it outputs. Therefore, the asymptotic running time of our algorithm is optimal whenever it outputs an asymptotically optimal solution. Recall that Theorem 3 asserts tight bounds for $\beta^d(n)$ except for the case where d is even and $d \in \Omega(n^{0.5+\epsilon})$, for any $0 < \epsilon \leq 0.5$.

We note that if $d = O(1)$, then Theorem 3 asserts that $\beta^d(n) = \theta(n^2)$. However, the corresponding coefficients are not the same: the lower bound has the coefficient $\frac{2}{d^2}$ whereas the upper bound has the coefficient $\frac{2}{(d-1)^2}$. It would be interesting to determine the exact coefficient in this case. Moreover, when d is even and $d \in \Omega(n^{0.5+\epsilon})$, for any $0 \leq \epsilon \leq 0.5$, we have an upper bound of $O(n)$ on $\beta^d(n)$; the lower bound for this case is $o(n)$. We believe that $\beta^d(n)$ is more close to the upper bound and tightening of the bound for $\beta^d(n)$ in this case remains open.

Acknowledgments

The authors thank the anonymous referees for their valuable comments and suggestions.

The research of the first author is supported by grant 12IRCCSG016, IRCC, IIT Bombay. The research of the third author is supported by the doctoral fellowship program of Ministry of Human Resources and Development, Govt. of India.

Appendix. Proof of Theorem 7 when *d* is odd

Statement. *$d \leq \beta^d(d+1) \leq d+1$, d is an odd integer.*

Proof. As in the proof of Theorem 7, the slots P_1 to $P_{\frac{d+1}{2}-1}$ are colored with $+1$, slots $P_{\frac{d+1}{2}+1}$ to P_{d+1} are colored with -1, and only slot $P_{\frac{d+1}{2}}$ remains uncolored. Note that along with the empty set and the singleton sets, the set $[d+1]$ becomes trivial under this restriction. Each slot can contain exactly one vertex and each vertex takes the color of the slot it resides in. As the initial configuration, let $v_i \in P_i$, for $1 \leq i \leq d+1$. This configuration gives the coloring X_1, where (i) $X_1(+1) = \{v_1, \ldots, v_{\frac{d+1}{2}-1}\}$, (ii) $X_1(-1) = \{v_{\frac{d+1}{2}+1}, \ldots, v_{d+1}\}$, and, (iii) the vertex $v_{\frac{d+1}{2}+1}$ remains uncolored. We obtain the second coloring X_2 from X_1 by one clockwise rotation of the vertices in the circular arrangement. Therefore, we have, $X_2(+1) = \{v_{d+1}, v_1, \ldots, v_{\frac{d+1}{2}-2}\}$, $X_2(-1) = \{v_{\frac{d+1}{2}}, \ldots, v_d\}$; the vertex $v_{\frac{d+1}{2}-1}$ remains uncolored. Similarly, repeating the rotation d times, we obtain the set $\mathcal{X} = \{X_1, \ldots, X_{d+1}\}$ of bicolorings.

The proof for $\mathcal{X}$ being an induced-bisecting family for any odd hyperedge $A_o \subsetneq [d+1]$ is exactly similar to that given in the proof of Theorem 7. So, we consider only the even hyperedges. Let $c_i = |A \cap X_{i+1}(+1)| - |A \cap X_{i+1}(-1)|$, for $0 \leq i \leq d$. All additions/subtractions in the subscript of c are modulo $d+1$. For the sake of contradiction, assume that A is an even hyperedge not induced-bisected by $\mathcal{X}$. If we can show that some c_j, $0 \leq j \leq d$, is zero, then we get the desired contradiction.

Observation 17. *$|c_i - c_{i+1}| \leq 2$, for $0 \leq i \leq d$.*

The proof of Observation 17 follows from the construction. Consider the sequence $(c_i, c_{i+1}, \ldots, c_{i+d+1})$, where $c_i \leq c_j$, $j \in \{i+1, \ldots, i+d+1\}$, and the addition is modulo $d+1$. Since there is a full rotation of the vertex set with respect to the slots, it follows that (i) $c_i \leq 0$, and (ii) there exists another index j such that c_j is positive. From Observation 17, it follows that if none of the c_j, $j \in \{0, \ldots, d\}$, is zero, there exists an index p such that $c_p = -1$ and $c_{p+1} = 1$. Note that $c_p = -1$ asserts that $A \cap P_{\frac{d+1}{2}}$ is non-empty. However, under this configuration, c_{p+1} can never become 1. This yields the desired contradiction. □

References

[1] Niranjan Balachandran, Rogers Mathew, Tapas Kumar Mishra, Sudebkumar Prasant Pal, Bisecting and D-secting families for set systems, Discrete Appl. Math. (2017).

[2] Manu Basavaraju, Mathew C. Francis, M.S. Ramanujan, Saket Saurabh, Partially polynomial kernels for set cover and test cover, SIAM J. Discrete Math. 30 (3) (2016) 1401–1423.

[3] Robert Crowston, Gregory Gutin, Mark Jones, Saket Saurabh, Anders Yeo, Parameterized study of the test cover problem, in: International Symposium on Mathematical Foundations of Computer Science, Springer, 2012, pp. 283–295.

[4] Koen M.J. De Bontridder, B.J. Lageweg, Jan K. Lenstra, James B. Orlin, Leen Stougie, Branch-and-bound algorithms for the test cover problem, in: European Symposium on Algorithms, Springer, 2002, pp. 223–233.

[5] Pierre A. Devijver, Josef Kittler, Pattern recognition: A statistical approach, Vol. 761, Prentice-Hall London, 1982.

[6] T.J. Dickson, On a problem concerning separating systems of a finite set, J. Combin. Theory 7 (1969) 191–196.

[7] Dingzhu Du, Frank K. Hwang, Frank Hwang, Combinatorial group testing and its applications, Vol. 12, World Scientific, 2000.

[8] Bjarni V. Halldórsson, Magnús M. Halldórsson, R. Ravi, On the approximability of the minimum test collection problem, in: European Symposium on Algorithms, Springer, 2001, pp. 158–169.

[9] G. Katona, On separating systems of a finite set, J. Combin. Theory I (1966) 174–194.

[10] G. Katona, Chapter 23 - combinatorial search problems, in: Jagdish N. Srivastava (Ed.), A Survey of Combinatorial Theory, North-Holland, 1973, pp. 285–308.

[11] S.P. Lapage, Shoshana Bascomb, W.R. Willcox, M.A. Curtis, Identification of bacteria by computer: general aspects and perspectives, Microbiology 77 (2) (1973) 273–290.

[12] Bernard M.E. Moret, Henry D. Shapiro, On minimizing a set of tests, SIAM J. Sci. Stat. Comput. 6 (4) (1985) 983–1003.

[13] R.W. Payne, D.A. Preece, Identification keys and diagnostic tables: A review, J. Roy. Statist. Soc. Ser. A (Gen.) (1980) 253–292.

[14] A. Rényi, On random generating elements of a finite Boolean algebra, Acta Sci. Math. (Szeged) 22 (1–2) (1961) 75–81.

[15] Emily Riehl, E. Graham Evans Jr., On the intersections of polynomials and the Cayley–Bacharach theorem, J. Pure Appl. Algebra 183 (1–3) (2003) 293–298.

[16] J. Spencer, Minimal completely separating systems, J. Combin. Theory 8 (1970) 446–447.

[17] I. Wegener, On separating systems whose elements are sets of at most k elements, Discrete Math. 28 (1979) 219–222.

[18] W.R. Willcox, S.P. Lapage, Automatic construction of diagnostic tables, Comput. J. 15 (3) (1972) 263–267.