Addendum to “On hierarchically closed fractional intersecting families” (Electron. J. Comb. 30 (2023), no. 4, #P4.37)

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Abstract

Theorem 4(2) in [1] says that any hierarchically r-bisection closed family $\mathcal{F}$ over $[n]$ (for $r \geq 3$) that attains equality in the bound

$$|\mathcal{F}| \leq \lfloor 3n/2 \rfloor - 2 \tag{$*$}$$

is the family $\mathcal{F}_{\max}$ of Example 1, up to permutations of $[n]$. In the proof of Theorem 4(2), we merely wrote that, “The proof of the upper bound ($*$) shows that if $\mathcal{F}$ is an extremal r-bisection closed family, then $S_{\text{nor}} = \{2, 4\}$.” However, the details require some filling in, which we do so in this addendum.

To show that there is a unique extremal family $\mathcal{F}$ (up to permutations of $[n]$) that attains the bound ($*$), we first show that, among the families satisfying $\mathcal{F} = \mathcal{F}^*$, the extremal ones have size $\lfloor 3n/2 \rfloor - 3$. So, assume that $\mathcal{F} = \mathcal{F}^*$ is extremal over $[n]$. Claims 32–35 hold for any such $\mathcal{F}$. We recall a couple of these claims here:

Claim 33. $S_{nor} \supseteq \{2, 4\}$.

Claim 34. *If there exists $a \in \mathsf{Pet}(\mathcal{F}(2)) \cap A$ for some $A \in \mathcal{F}(\geq 4)$, then $A \in \mathcal{F}(4)$ and $a \in \mathsf{Pet}(A)$.*

An additional hypothesis was introduced in:

Claim 36. *Let $\mathcal{F}$ be an extremal family for which $|\mathcal{F}(2)|$ is maximum. Then for each $B \in \mathcal{F}(4)$, $|\mathsf{Pet}(B) \cap \mathsf{Pet}(\mathcal{F}(2))| \in \{0, 2\}$.*

Using Claim 36 we showed that if $\mathcal{F}$ is any extremal family *for which $|\mathcal{F}(2)|$ is maximum*, then $S_{\text{nor}} = \{2, 4\}$. This was used to establish that $|\mathcal{F}^*| \leq \lfloor 3n/2 \rfloor - 3$ for any r-bisection closed family $\mathcal{F}$ over $[n]$, as well as the following (weaker) uniqueness result (cf. [1, Theorem 4(2)]):

Lemma 37. *Let $\mathcal{F}$ be an extremal r-bisection closed family over $[n]$ for which $S_{nor} = \{2, 4\}$. Then, there is a permutation σ of $[n]$ such that $\sigma(\mathcal{F}) = \mathcal{F}_{\max}$. In particular, if $\mathcal{F}$ is an extremal family for which $|\mathcal{F}(2)|$ is maximum, then $\sigma(\mathcal{F}) = \mathcal{F}_{\max}$ for some permutation σ of $[n]$.*

Note that $|\mathcal{F}(2)| \leq n - 1$ for any $\mathcal{F}$, and equality holds for the extremal family $\mathcal{F}_{\max}$. Now, we reformulate Claim 36 to avoid any extra assumptions on the size of $\mathcal{F}(2)$:

Claim 38. *Let $B \in \mathcal{F}(4)$ and $\mathsf{Pet}(B) = \{a, b\}$. Then:*

1. $|\{a, b\} \cap \mathsf{Pet}(\mathcal{F}(2))| \in \{0, 2\}$, *or*

2. $|\{a,b\}\cap \mathsf{Pet}(\mathcal{F}(2))| = 1$, *and if* $b \in \mathsf{Pet}(\mathcal{F}(2))$, *then there is a unique set* $A \in \mathcal{F}(\geq 6)$ *such that* $a \in A$. *Moreover,* $A \in \mathcal{F}(6)$.

Proof. Suppose that $b \in \mathsf{Pet}(\mathcal{F}(2))$ and $a \notin \mathsf{Pet}(\mathcal{F}(2))$. If $a \notin B'$ for any $B' \in \mathcal{F}$ distinct from B, then we contradict the extremality of $\mathcal{F}$ as follows: the family $\mathcal{F}' := \mathcal{F} \cup \{A'\}$, where $A' := \mathsf{Cor}(\mathcal{F}(2)) \cup \{a\}$, is r-bisection closed and satisfies $|\mathcal{F}'| > |\mathcal{F}|$.

So, there is a set $A \in \mathcal{F}$ distinct from B for which $a \in A$. In particular, $A \in \mathcal{F}(\geq 6)$. Note that $\mathsf{Cor}(B) \cup \{a\} \subseteq A$, so $|A \cap B| \geq 3 > \frac{1}{2}|B|$. Thus, $|A \cap B| = \frac{1}{2}|A|$. So, if $A \in \mathcal{F}(\geq 8)$, then in fact $A \in \mathcal{F}(8)$ and $B \subseteq A$. But this implies that $a \in A$, which contradicts Claim 34. Thus, $A \in \mathcal{F}(6)$.

Lastly, if $\mathcal{F}(6)$ is a singleton, then A is clearly unique, and if there are at least two sets in $\mathcal{F}(6)$, then $a \notin A'$ for any $A' \in \mathcal{F}(6)$ distinct from A because $\mathcal{F}(6)$ is a sunflower and $a \in \mathsf{Pet}(A)$. □

Now, in terms of Claim 38 we have (without any change in the proof):

Corollary 39. *If Claim 38(1) holds for all* $B \in \mathcal{F}(4)$, *then* $S_{nor} = \{2,4\}$.

We are now ready to prove:

Proposition 40. *There is no extremal family* $\mathcal{F}$ *over* $[n]$ *for which* $|\mathcal{F}(2)| < n-1$.

Proof. Suppose for the sake of contradiction that $\mathcal{F}$ is an extremal family over $[n]$ for which $|\mathcal{F}(2)| < n-1$. Then, $S_{\text{nor}} \supsetneq \{2,4\}$ by Lemma 37 and Claim 33. Also, Claim 38(2) holds for some $B \in \mathcal{F}(4)$ by Corollary 39 and Lemma 37.

Now, let $\mathcal{F}_0 := \mathcal{F}$. For $n \in \mathbb{N}$, if the extremal r-bisection closed family $\mathcal{F}_n$ has been defined, and there is a set $B_n \in \mathcal{F}_n(4)$ for which Claim 38(2) holds, then we define $\mathcal{F}_{n+1}$ as follows. Let $\mathsf{Pet}(B_n) = \{a_n, b_n\}$ with $b_n \in \mathsf{Pet}(\mathcal{F}_n(2))$. Let $A_n \in \mathcal{F}_n(6)$ be the unique set in $\mathcal{F}_n(\geq 6)$ such that $a_n \in A_n$. Then, define $\mathcal{F}_{n+1} := (\mathcal{F}_n \setminus \{A_n\}) \cup \{A'_n\}$, where $A'_n := \mathsf{Cor}(\mathcal{F}_n(2)) \cup \{a_n\}$. Note that $\mathcal{F}_{n+1}$ is also an r-bisection closed family that is extremal, since $|\mathcal{F}_n| = |\mathcal{F}_{n+1}|$.

Applying this procedure inductively by starting with $\mathcal{F}_0 := \mathcal{F}$, for some $N \in \mathbb{N}$ we get an extremal family $\mathcal{F}' = \mathcal{F}_N$ such that Claim 38(1) holds for all $B' \in \mathcal{F}'(4)$. Hence, by Corollary 39, $\mathcal{F}'$ has only two normal sunflowers, namely $\mathcal{F}'(2)$ and $\mathcal{F}'(4)$. Since the only sets from $\mathcal{F}$ that were thrown out in the construction of $\mathcal{F}'$ were those of size 6, $\mathcal{F}$ has only three normal sunflowers, namely $\mathcal{F}(2)$, $\mathcal{F}(4)$, and $\mathcal{F}(6)$. Now, let $B \in \mathcal{F}(6)$, and let $\mathsf{Pet}(B) = \{a,b,c\}$. Define $\mathcal{G} = (\mathcal{F}^* \setminus \{B\}) \cup \{D_a, D_b, D_c\}$, where $D_i := \mathsf{Cor}(\mathcal{F}(2)) \cup \{i\}$, for $i \in \{a,b,c\}$. Then, $\mathcal{G}$ is an r-bisection closed family for which $|\mathcal{G}| \geq |\mathcal{F}| + 1$, contradicting the extremality of $\mathcal{F}$. □

This completes the proof of [1, Theorem 4(2)] that the family $\mathcal{F}_{\max}$ over $[n]$ of Example 1 is the unique extremal r-bisection closed family (up to permutations of $[n]$).

References

[1] N. Balachandran, S. Bhattacharya, K. V. Kher, R. Mathew, B. Sankarnarayanan. *On hierarchically closed fractional intersecting families*, Electron. J. Comb. **30** (2023), no. 4, #P4.37, doi:10.37236/11651. MR4672585