Bounded fractional intersecting families are linear in size

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Abstract

Using the sunflower method, we show that if $\theta \in (0,1) \cap \mathbb{Q}$ and $\mathcal{F}$ is a $O(n^{1/3})$-bounded θ-intersecting family over $[n]$, then $|\mathcal{F}| = O(n)$, and that if $\mathcal{F}$ is $o(n^{1/3})$-bounded, then $|\mathcal{F}| \leq (\frac{3}{2} + o(1))n$. This partially solves a conjecture raised in [5] that any θ-intersecting family over $[n]$ has size at most linear in n, in the regime where we have no very large sets.

Keywords: sunflower; fractionally intersecting family; hierarchically intersecting family
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1 Introduction

Intersecting families of set systems are well-studied in extremal combinatorics, and the most natural extremal question investigated here has the following template: how large can a family of subsets of $[n]$ be under the constraint that the sets satisfy some intersection properties? Some of the classical results of this kind include the de Bruijn–Erdős theorem [3], the Erdős–Ko–Rado theorem [8], the Ray-Chaudhuri–Wilson inequality [11], the Frankl–Wilson inequality [10], the Alon–Babai–Suzuki inequality [1] and many more (see Babai–Frankl [2] for more).

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In this note, our interest is in a fractional variant of intersecting families defined by Balachandran–Mathew–Mishra [5], that goes as follows. Let $\theta \in (0,1) \in \mathbb{Q}$. A *(fractional) θ-intersecting family* $\mathcal{F}$ over $[n]$ is a collection of subsets of $[n]$ such that for all $A, B \in \mathcal{F}$ with $A \neq B$, $|A \cap B| \in \{\theta|A|, \theta|B|\}$.[1] In [5], the following upper bound is proved for the size of any θ-intersecting family over $[n]$.

Theorem 1.1 (Balachandran–Mathew–Mishra [5], 2019)**.** *Let $\mathcal{F}$ be a θ-intersecting family over $[n]$. Then, $|\mathcal{F}| = O(n \log n)$.*

On the other hand, the best-known constructions give θ-intersecting families over $[n]$ of size only linear in n.

Example 1.1. The *sunflower family* $\mathcal{F}_s$ over $[n]$ is defined as follows:

$$\mathcal{F}_s = \begin{cases} \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-1)n\}, & n \equiv 0 \pmod 2; \\ \{12, 13, \ldots, 1n, 1234, 1256, \ldots, 12(n-2)(n-1)\}, & n \equiv 1 \pmod 2. \end{cases}$$

This is easily seen to be a $\frac{1}{2}$-intersecting family, also called a *bisection closed* family. Note that $|\mathcal{F}_s| = \lfloor 3n/2 \rfloor - 2$.

Example 1.2. The *Hadamard family* $\mathcal{F}_H$ over $[2m]$ is constructed from an $m \times m$ normalized Hadamard matrix H as follows. View the rows $A_1, \ldots, A_{3m}$ of the following block matrix as the $\{\pm 1\}$-incidence vectors of subsets of $[2m]$, where J denotes the $m \times m$ all-ones matrix:

$$\begin{bmatrix} H & H \\ H & -H \\ H & -J \end{bmatrix}.$$

Then, $\mathcal{F}_H = \{A_i : i \in [3m] \setminus \{1, m+1\}\}$. One can show using the orthogonality of the rows of H that $\mathcal{F}_H$ is a bisection closed family over $[2m]$. Writing $2m = n$, we see that $|\mathcal{F}_H| = 3n/2 - 2$.

It was conjectured in [5] that *any* θ-intersecting family over $[n]$ is at most linear in size.[2]

Conjecture 1.2 (Balachandran–Mathew–Mishra [5], 2019)**.** *For $\theta \in (0,1) \cap \mathbb{Q}$, there is a constant $c > 0$ such that for any θ-intersecting family $\mathcal{F}$ over $[n]$, $|\mathcal{F}| \leq cn$.*

Moreover, the fact that two very different constructions give rise to maximal bisection closed families over $[n]$ of the same size raises the question whether, for $\theta = 1/2$, we have $|\mathcal{F}| \leq \lfloor 3n/2 \rfloor - 2$ for any bisection closed family $\mathcal{F}$ over $[n]$. In [6], there are constructions of bisection closed families over $[n]$ for $n \leq 15$ which have size greater than $\lfloor 3n/2 \rfloor - 2$, so the constructions in Examples 1.1 and 1.2 are possibly extremal only for large n.

In this note, we make some progress towards resolving the conjecture by proving the following result. We say that a family of sets is *w-bounded*, for a positive real w, if every set in the family has size at most w.

Theorem 1.3. *Let $\theta \in (0,1) \cap \mathbb{Q}$ and $w = O(n^{1/3})$ be a positive real. There is a constant $C > 0$ such that the following holds: for all sufficiently large n, if $\mathcal{F}$ is a w-bounded θ-intersecting family over $[n]$, then $|\mathcal{F}| \leq Cn$.*

[1] More generally, given a set L of proper fractions, a *(fractional) L-intersecting family* $\mathcal{F}$ over $[n]$ is a collection of subsets of $[n]$ such that for all $A, B \in \mathcal{F}$ with $A \neq B$, $|A \cap B| \in \{\theta|A|, \theta|B|\}$ for some $\theta \in L$.

[2] The conjecture is implicit in [5], and explicitly stated for the case when $\theta = 1/2$.

Under the slightly stronger assumption that the family is $o(n^{1/3})$-bounded, we give an explicit constant that is often tight.

Theorem 1.4. *Let $a, b \in \mathbb{N}$ such that $1 \leq a < b$ and $\gcd(a, b) = 1$. Let $\theta = a/b$, and let $\mathcal{F}$ be a $o(n^{1/3})$-bounded θ-intersecting family over $[n]$. Then $|\mathcal{F}| \leq (C_\theta + o(1))n$, where $C_\theta = \frac{1}{b-a}\sum_{i=1}^{\lfloor b/a \rfloor} \frac{1}{i}$, and this constant is best possible for $\theta \in \{1/3\} \cup [1/2, 1)$.*

Our results also have implications in the setting of *hierarchically r-closed* θ-intersecting families, as defined in [4]. Given $r \geq 2$, we say $\mathcal{F}$ is hierarchically r-closed θ-intersecting if, for any $2 \leq t \leq r$ and any t-subset $\{A_1, \dots A_t\}$ of $\mathcal{F}$, we have $|\bigcap_{i=1}^{t} A_i| \in \{\theta|A_i| : i \in [t]\}$. From our previous examples, note that $\mathcal{F}_s$ is hierarchically r-closed for all r, while $\mathcal{F}_H$ is not hierarchically r-closed for any $r \geq 3$. Thus, in this sense, the two families are at opposite ends of a spectrum, despite having the same size.[3]

In [4], it was shown that Conjecture 1.2 holds for hierarchically closed fractional intersecting families, with a sharp bound for $\theta = 1/2$.

Theorem 1.5 (Balachandran–Bhattacharya–Kher–Mathew–Sankarnarayanan [4], 2023)**.** *Let $r \geq 3$ and $\theta \in (0, 1) \cap \mathbb{Q}$. There is a positive constant $c_\theta \leq 3/2$ such that, if $\mathcal{F}$ is a hierarchically r-closed θ-intersecting family over $[n]$, then $|\mathcal{F}| \leq c_\theta n$. Moreover, when $\theta = 1/2$, we have $|\mathcal{F}| \leq \lfloor 3n/2 \rfloor - 2$, and $\mathcal{F}_s$ is the unique family (up to permutations of $[n]$) that attains this bound.*

The authors of [4] posed the problem of determining the optimal constant for other values of θ. Our proofs of Theorems 1.3 and 1.4 follow their framework, but with a couple of new ideas. Essentially, we replace their main idea of using a dyadic grouping of sunflowers with a double-counting argument that provides sharper estimates. This allows us to improve the constant c_θ in Theorem 1.5 to C_θ from Theorem 1.4, and the tightness results carry over as well.

2 Main results

For a family $\mathcal{F}$ over $[n]$, we denote by $\mathcal{F}(i)$ the maximal i-uniform subfamily of $\mathcal{F}$. We say $\mathcal{F}(i)$ is a *sunflower* if, for all distinct $F, F' \in \mathcal{F}(i)$, we have $F \cap F' = \bigcap_{F'' \in \mathcal{F}(i)} F''$. The common intersection $C = \bigcap_{F'' \in \mathcal{F}(i)} F''$ is called the *core*, while the (pairwise disjoint) remainders of the sets $F \setminus C$ are called *petals*.

Note that the family $\mathcal{F}_s$ is the union of 2- and 4-uniform sunflowers, whose cores are nested. We generalize this notion of neatly-arranged sunflowers in the following definition.

Definition. Let $\mathcal{F}$ be a family over $[n]$, and let $i_1 < \cdots < i_t$ be the sizes of sets in $\mathcal{F}$. We say that $\mathcal{F}$ is a *bouquet* if

1. each $\mathcal{F}(i_j)$ is a sunflower with at least two petals;
2. $C_{i_1} \subsetneq C_{i_2} \subsetneq \cdots \subsetneq C_{i_t}$, where C_{i_j} denotes the core of $\mathcal{F}(i_j)$;
3. for any $F \in \mathcal{F}$ we have $F \cap C_{i_t} = C_{|F|}$.

Note that the third property implies that in a bouquet, every petal is disjoint from all cores.

[3] Note that a hierarchically 2-closed family is just a θ-intersecting family as defined in Section 1. So, when we say that a θ-intersecting family $\mathcal{F}$ is *hierarchically closed*, we mean that it is hierarchically r-closed for some $r \geq 3$.

Our proofs then split into two parts. First, we show that bounded θ-intersecting families contain large bouquets.

Proposition 2.1. *Let $\mathcal{F}$ be a w-bounded θ-intersecting family over $[n]$. Then $\mathcal{F}$ contains a bouquet $\mathcal{F}^*$ with $|\mathcal{F} \setminus \mathcal{F}^*| \leq w^3$.*

We then utilize the structure of bouquets to bound their size.

Proposition 2.2. *Let $a, b \in \mathbb{N}$ such that $1 \leq a < b$ and $\gcd(a, b) = 1$. Let $\theta = a/b$, and let $\mathcal{F}^*$ be a θ-intersecting bouquet over $[n]$. Then $|\mathcal{F}^*| \leq C_\theta n$, where $C_\theta = \frac{1}{b-a} \sum_{i=1}^{\lfloor b/a \rfloor} \frac{1}{i}$.*

We note that the proof of Theorem 1.5 in [4] starts by essentially showing that by removing very few sets from a hierarchically closed θ-intersecting family, one obtains a bouquet. Thus, Proposition 2.2 improves the constant in Theorem 1.5 as well.

Proof of Proposition 2.1

We will require the following result of Deza [7] that implies that a large uniform θ-intersecting family must be a sunflower.

Theorem 2.3 (Deza [7], 1974)**.** *Let $\mathcal{F}$ be a w-bounded family of subsets of $[n]$ such that all pairwise intersections have the same cardinality. If $|\mathcal{F}| \geq w^2 - w + 2$, then $\mathcal{F}$ is a sunflower.*

Call a level $\mathcal{F}(i)$ *small* if $|\mathcal{F}(i)| \leq w^2$. We can bound the number of sets in small levels by

$$\sum_{i\,:\,|\mathcal{F}(i)| \leq w^2} |\mathcal{F}(i)| = |\mathcal{F}(1)| + \sum_{i>1\,:\,|\mathcal{F}(i)| \leq w^2} |\mathcal{F}(i)| \leq 1 + (w-1)w^2 < w^3.$$

We remove these sets from $\mathcal{F}$, and shall show that what remains must be a bouquet (after removing at most one more set, if needed). Let $i_1 < i_2 < \cdots < i_t$ be the remaining levels.

1. By Thereom 2.3, each $\mathcal{F}(i_j)$ is a sunflower with at least two sets.
2. Let C_{i_j} be the core of $\mathcal{F}(i_j)$. Since $\mathcal{F}$ is θ-intersecting, $|C_{i_j}| = \theta i_j$ for all $1 \leq j \leq t$. Now, let $1 \leq j < j' \leq t$, and suppose $F' \in \mathcal{F}(i_{j'})$. Then $|F' \cap F| \geq \theta i_j = |C_{i_j}|$ for every $F \in \mathcal{F}(i_j)$, since $\mathcal{F}$ is θ-intersecting. If $C_{i_j} \nsubseteq F'$, then F' must intersect every petal in $\mathcal{F}(i_j)$. But then $|F'| \geq |\mathcal{F}(i_j)| > w^2$, which is not possible since $\mathcal{F}$ is w-bounded. Thus, $C_{i_j} \subseteq F'$ for every $F' \in \mathcal{F}(i_{j'})$, which implies that $C_{i_j} \subseteq C_{i_{j'}}$.
3. Let $j < t$ and $F \in \mathcal{F}(i_j)$. If $F \cap (C_{i_t} \setminus C_{i_j}) \neq \emptyset$, then for any $G \in \mathcal{F}(i_t)$ we have $|F \cap G| > |C_{i_j}| = \theta i_j$. Thus, necessarily, $|F \cap G| = \theta i_t$. Again, F is not large enough to meet every petal of $\mathcal{F}(i_t)$, and so we have $C_{i_t} \subseteq F$. Now, if there was another such set $F' \in \mathcal{F}(i_{j'})$, then we have $|F \cap F'| \geq |C_{i_t}| = \theta i_t \notin \{\theta|F|, \theta|F'|\}$, contradicting that $\mathcal{F}$ is θ-intersecting. Hence, there is at most one such set; if so, we remove it, and the remaining family satisfies $F \cap C_{i_t} = C_{|F|}$.

Then, having removed at most w^3 sets, we are left with a bouquet $\mathcal{F}^*$. □

Proof of Proposition 2.2

Let $i_1 < \cdots < i_t$ be the nonempty levels in the bouquet $\mathcal{F}^*$ over $[n]$, and set $Y = [n] \setminus C_{i_t}$. Note that for each $F \in \mathcal{F}^*$ we have $|F \cap Y| = (1-\theta)|F|$. Moreover, for each j, the sets in $\mathcal{F}^*(i_j)$ are

pairwise disjoint over Y. Thus, we have

$$|\mathcal{F}^*| = \sum_{F\in\mathcal{F}^*} 1 = \sum_{F\in\mathcal{F}^*}\sum_{y\in F\cap Y}\frac{1}{(1-\theta)|F|} = \sum_{y\in Y}\sum_{\substack{F\in\mathcal{F}^*:\\ y\in F}}\frac{1}{(1-\theta)|F|}.$$

For each $y\in Y$, let $S_y = \{|F| : F\in\mathcal{F}^*, y\in F\}$. Then we have

$$|\mathcal{F}^*| = \frac{1}{1-\theta}\sum_{y\in Y}\sum_{s\in S_y}\frac{1}{s}.$$

Now observe that if $F, F'\in\mathcal{F}^*$ and $|F| < \theta|F'|$, then we must have $|F\cap F'| = \theta|F|$. However, $F\cap F'\cap C_{i_t} = C_{|F|}$, which is of size $\theta|F|$, and so $F\cap F'\cap Y = \emptyset$. This means that for every $y\in Y$, we have $\max S_y \le \frac{1}{\theta}\min S_y$. Moreover, since $\mathcal{F}^*$ is θ-intersecting, b must divide $|F|$ for every $F\in\mathcal{F}^*$. Thus, for every $y\in Y$, we have some $m_y\in\mathbb{N}$ such that $S_y\subseteq\{bm_y, b(m_y+1),\dots,b\lfloor m_y/\theta\rfloor\}$, and

$$\sum_{s\in S_y}\frac{1}{s} \le \sum_{i=m_y}^{\lfloor m_y/\theta\rfloor}\frac{1}{bi} = \frac{1}{b}\sum_{i=m_y}^{\lfloor m_y/\theta\rfloor}\frac{1}{i}.$$

Hence we have

$$|\mathcal{F}^*| = \frac{1}{1-\theta}\sum_{y\in Y}\sum_{s\in S_y}\frac{1}{s} \le \frac{1}{(1-\theta)b}\sum_{y}\sum_{i=m_y}^{\lfloor m_y/\theta\rfloor}\frac{1}{i}.$$

Now write $b = ak + r$, where $k\in\mathbb{N}$ and $0\le r\le a-1$. Then,

$$\lfloor m_y/\theta\rfloor = \lfloor bm_y/a\rfloor = km_y + \lfloor rm_y/a\rfloor \le km_y + m_y - 1 = (k+1)m_y - 1.$$

Thus,

$$\begin{aligned}
\sum_{i=m_y}^{\lfloor m_y/\theta\rfloor}\frac{1}{i} &= \sum_{j=1}^{k-1}\sum_{i=jm_y}^{(j+1)m_y-1}\frac{1}{i} + \sum_{i=km_y}^{km_y+\lfloor rm_y/a\rfloor}\frac{1}{i}\\
&\le \sum_{j=1}^{k-1}\sum_{i=jm_y}^{(j+1)m_y-1}\frac{1}{jm_y} + \sum_{i=km_y}^{(k+1)m_y-1}\frac{1}{km_y}\\
&= \sum_{j=1}^{k}\frac{1}{j}\\
&= \sum_{j=1}^{\lfloor b/a\rfloor}\frac{1}{j}.
\end{aligned}$$

Noting that $\frac{1}{(1-\theta)b} = \frac{1}{b-a}$, and that there are at most n choices for $y\in Y$, we obtain the desired bound. □

This establishes Theorem 1.3. We now prove Theorem 1.4.

Proof of Theorem 1.4

The upper bound $|\mathcal{F}| \leq (C_\theta + o(1))n$ for any $o(n^{1/3})$-bounded θ-intersecting family follows from Propositions 1.3 and 1.4. To show that the constant C_θ cannot be improved for $\theta \in \{1/3\} \cup [1/2, 1)$, consider the following constructions of θ-intersecting $o(n^{1/3})$-bounded families over $[n]$.

- For $\theta = a/b \in (1/2, 1)$, let $\mathcal{F}$ be a maximal b-uniform sunflower over $[n]$ with core of size a. Then $\mathcal{F}$ is $\frac{a}{b}$-intersecting and $|\mathcal{F}| = \lfloor \frac{n-a}{b-a} \rfloor = \lfloor C_\theta n - \frac{a}{b-a} \rfloor$.
- For $\theta = 1/2$, the family $\mathcal{F}_s$ has size $\lfloor 3n/2 \rfloor - 2$ over $[n]$, and $C_{1/2} = 3/2$.
- For $\theta = 1/3$, assume $n \equiv 3 \pmod{24}$ for convenience, and consider the family $\mathcal{F} = \mathcal{F}(3) \cup \mathcal{F}(6) \cup \mathcal{F}(9)$, where:
  - $\mathcal{F}(3)$ is a sunflower with core $\{1\}$ and petals $\{\{2i, 2i+1\} : 2 \leq i \leq (n-1)/2\}$,
  - $\mathcal{F}(6)$ is a sunflower with core $\{1, 2\}$ and petals $\{\{24i+j, 24i+j+6, 24i+j+12, 24i+j+18\} : 0 \leq i \leq (n-27)/24,\ 4 \leq j \leq 9\}$, and
  - $\mathcal{F}(9)$ is a sunflower with core $\{1, 2, 3\}$ and petals $\{\{6i-2, 6i-1, 6i, \ldots, 6i+3\} : 1 \leq i \leq (n-3)/6\}$.

  $\mathcal{F}$ is then $\frac{1}{3}$-intersecting, and $|F| = (n-3)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right) = \frac{11}{12}(n-3) = C_{1/3}n - \frac{33}{12}$. □

We remark that these constructions are also hierarchically r-closed for any $r \geq 3$, showing these values of C_θ to be the best possible constant for Theorem 1.5 as well.

3 Concluding remarks

- Our constant C_θ in Theorem 1.4 is strictly smaller than $3/2$ when $\theta \neq 1/2$, and the proof of Proposition 2.2 shows that even when $\theta = 1/2$, we obtain a smaller constant unless almost all elements of $[n]$ are contained in sets of size 2. However, the existence of the Hadamard families $\mathcal{F}_H$ of Example 1.2 precludes any simple extension of the argument given in this note to try and establish an upper bound of $(\frac{3}{2} + o(1))n$ for the size of an arbitrary bisection closed family, since these are bisection closed families of size $3n/2 - 2$ that do not contain any sets of size 2 (in fact, the set sizes in $\mathcal{F}_H$ are all either $n/2$ or $n/4$).
- While Theorem 1.4 establishes the correct constant for certain values of θ, further arguments can be made to sharpen the constant for other fractions. We briefly illustrate this with the example of $\theta = 1/4$: any $o(n^{1/3})$-bounded $\frac{1}{4}$-intersecting family has size at most $(\frac{7}{12} + o(1))n$.

  For the lower bound, we construct such a family using sets of size $4, 8$ and 16. The sunflowers have nested cores of size $1, 2$ and 4 respectively, and for the petals, we divide the remaining elements into blocks of size 36, arranged in 3×12 rectangles. Each row (of size 12) is the petal of a 16-set, and is partitioned into four petals of size 3 each (for the 4-sets). The 12 columns are paired up to form the petals of the 8-sets, in such a way that they intersect each small petal at most once. For the upper bound, the double-counting argument shows that in an extremal family, almost all elements must be contained in sets of size 4 and sets of size 8. The only other permissible sizes are then 12 and 16, but one can argue (we omit the details) that the former is not possible. This is then sufficient to establish a constant of $7/12$.

It is however a more difficult task to see what the correct constant is, even for $\theta = 1/b$, for larger b, and it would be interesting to obtain further results in this direction.

- The best results known so far in the "large" regime are given in [5]: if all the sets in $\mathcal{F}$ have size at least $\frac{1}{4(1-\theta)}n - \Theta(\sqrt{n})$, then $|\mathcal{F}| = O(n)$.

- The $o(n)$ error in Theorem 1.4 is necessary, because of the existence of bisection closed families of size greater than $3n/2$ for $n \leq 15$. These are constructed in [6] using the Fano plane. Define the family $\mathcal{F}_{\text{Fano}}$ over [8] as follows:

$$\begin{aligned}\mathcal{F}_{\text{Fano}} &= \mathcal{F}_s \cup \{1357, 1368, 1458, 1467\} \\ &= \{12, 13, 14, 15, 16, 17, 18, 1234, 1256, 1278, 1357, 1368, 1458, 1467\}.\end{aligned}$$

  It is easy to check that $\mathcal{F}_{\text{Fano}}$ is a bisection closed family of size 14 over [8], and it arises from the symmetric 2-$(7, 4, 2)$ design. We can similarly modify $\mathcal{F}_s$ using the sets 1357, 1368, 1458, and 1467 to get bisection closed families over $[n]$ of size more than $\lfloor 3n/2 \rfloor - 2$ for $n \leq 15$.

- In [6], the authors consider a related problem of finding bounds on the ranks of certain symmetric matrices. Specifically, large θ-intersecting families induce such matrices of low rank. There the authors construct low rank matrices using bipartite graphs and ask whether any of them arise from θ-intersecting families. Theorem 1.3 shows that it is not possible for *bounded* bisection closed families to induce such matrices. This explains in a sense why the Fano construction does not seem to extend beyond small values of n to produce larger bisection closed families from $\mathcal{F}_s$.

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