Advanced topics in GR

Notation and some basic formulae [Wald Ch. 3]

"Abstract index notation"

We write $T^{\mu\nu} \xrightarrow{\text{abstract}} \mathbb{T} \xrightarrow{\text{Wald}} T^{ab}$

coordinate basis indices

abstract

Covariant derivative ∇_a ← abstract

Properties:

1. Linearity $\nabla(\alpha T^{ab} + \beta Q^{ab})$
$$= \alpha \nabla_c T^{ab} + \beta \nabla_c Q^{ab}$$

2. Leibnitz Rule $\nabla(AB) = (\nabla A)B$
$A(\nabla B)$

3. Commutes with contraction

$$\nabla_d (A^{a_1 \ldots c \ldots a_k}{}_{b_1 \ldots c \ldots b_l}) = (\nabla_d A)^{a_1 \ldots c \ldots a_k}{}_{b_1 \ldots c \ldots b_l}$$

4 Reproduce usual derivative when acting on a scalar

$$X^a \nabla_a f = X(f)$$

5 [Special to cov. deriv. in Gauss-Riemann geometry]

"Compatibility with the metric."

In general we might have

$$\nabla_a \nabla_b f - \nabla_b \nabla_a f = -\underbrace{T^c{}_{ab}}_{\text{Torsion}} \nabla_c f + B_{ab} f$$

Set torsion to zero

Comment on contraction

Recap Diff. Geom. philosophy:

Topological space → differential structure

(point set topology ... metric space ... Cauchy sequences)

Existence of smooth functions – coordinates from Euclidean n-space for each nbhd.

→ Tangent space ≈ local Euclidean approx.

→ Basis for vectors of the tangent space vect. space V^a basis $\{e_\alpha\}$

Dual vector space V^{a*} on the tangent space.

$$W(V) \longrightarrow \text{scalar}$$

$$W \in V^{a*} \qquad V \in V^a$$

Wald notation $W_a(V^b) \longrightarrow$ scalar

We also have metric g_{ab}

Thus for every $W_a(V^b) = r$ we introduce

$g_{ab}\tilde{W}^a V^b = r \longrightarrow$ defines $\tilde{W}^a \in V^a$ corresp. to $W_a \in V^{a*}$

Contractions are carried using g_{ab}

$\rightarrow$ Hicks $\longrightarrow g(W, V)$

Diff. Geom.

$g_{\mu\nu}\tilde{T}^{\mu\nu} = T^{\nu}_{\ \nu}$