Lecture 6 Dynamics of Gravity - I

Plan : 1. Complete $T^{\mu}{}_{\nu}$ a la Noether

2. Variational derivation of E Eqn. from E-H action

3. $T^{\mu\nu}$ as $\delta S_{matter}/\delta g_{\mu\nu}$

→ S. N. Gupta + Weinberg on graviton coupling to its own "$t^{\mu\nu}$".

4 Begin ADM

Q: (1) Why do we stop at small symmetry variation
(2) Are there symmetry variations non-linear in the fields ψ_a

(1) $\rightarrow$ small suffice for Noether.

(2) $\rightarrow$ don't know.

But Y-M does have "large gauge transf" whose effect is not captured in Noether's theorem.

Jackiw - Rebbi (1973?) $\rightarrow$ "θ-vacua"

$\rightarrow$ Topological charges (Non-Noether)

$\rightarrow$ Characterise soliton solutions

θ vacua of $SU(2)_L \otimes U(1)_Y$ electroweak

$\rightarrow$ relevance to B+L number violation; BAU.

SUSY (?) Secret non-linearity of transf.s in SUSY → use of auxiliary fields

S. P. Martin "SUSY: A Primer"

Returning to $T^{\mu}{}_{\nu}$ a la Noether

$$x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu \leadsto \text{not functions of } x$$

$$\tilde{\psi} - \psi \equiv \delta\psi(x) = -a^\mu \partial_\mu \psi(x)$$

$$\delta S = \int_\Omega d^4x \left\{ \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \delta(\partial_\mu \psi) + \frac{\delta \mathcal{L}}{\delta \psi} \delta\psi \right\} + \int_{\partial\Omega} \delta(d^4x)\, \mathcal{L}$$

$$= \int_{\partial\Omega} d^3\Sigma_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \cdot \delta\psi + \left\{ \begin{matrix} \text{E-L} \\ \text{eqn.s} \end{matrix} \right\}$$

$$+ \int_{\partial\Omega} \text{“} \underbrace{a^\mu d\Sigma_\mu}_{} \text{”} \, \mathcal{L}$$

$$d^4x \to \frac{a^\mu}{\delta x^\mu} \times d\Sigma_\mu$$

Subject to E–L eqn.s ~~satisfied~~ satisfied,

$$\delta_a S = -a^\nu \int d^3\Sigma_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial_\nu \psi + a^\nu \int d^3 \Sigma_\nu \mathcal{L}$$

$$= a^\nu \int d^3 \Sigma_\mu \left\{ \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial_\nu \psi - \delta^\mu_\nu \mathcal{L} \right\}$$

Recall $\int_{\partial\Omega} d^3 \Sigma_\mu j^\mu \Rightarrow \int_\Omega d^4x\, \partial_\mu j^\mu \Rightarrow Q = \int j^0 d^3x$

Thus we have for each ν,

$$\partial_\mu T^\mu{}_\nu = 0$$

with $T^\mu{}_\nu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial_\nu \psi - \delta^\mu_\nu \mathcal{L}$

Note $T^0{}_0 = \frac{\delta \mathcal{L}}{\delta(\dot\psi)} \dot\psi - \mathcal{L}$

$$= \pi \dot\psi - \mathcal{L} \equiv \underbrace{\mathcal{H}}_{\text{Hamiltonian density}}$$

Thus we have 4 conserved charges

$$P_\nu \equiv \int_{t=\text{const}} d^3x \; T^0{}_\nu$$

of which $P_0 \equiv \mathcal{H}$ canonical Hamiltonian

but we have space components of physical ~~total~~ momentum of the field

$$P_i = \int T^0{}_i \, d^3x \neq \left\{ \begin{array}{l} \text{canonical} \\ \text{momentum} \end{array} \right.$$

Generalises simple case of particle mechanics — translation invariance

$$\Rightarrow V(\vec{x}_i) \equiv V(\vec{x}_1 - \vec{x}_2, \vec{x}_1 - \vec{x}_3 \cdots)$$

difference only

$$\Rightarrow \vec{\nabla}_1 V(\vec{x}_1 - \vec{x}_2) = -\vec{\nabla}_2 V(\vec{x}_1 - \vec{x}_2) \rightarrow \text{Newton's third law}$$

$$\text{Since } \vec{F}_{tot} = \sum_i F_{ij} = 0 = \frac{d}{dt}\left(\sum_i p_i\right)$$

as for t - translation

Jacobi inv. $J = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$

same as $T^0{}_0$ same as H

Einstein - Hilbert action

$$S = \int d^4x \sqrt{-g}\, R \times \frac{1}{16\pi G}$$

the only generally covariant scalar upto $\partial\partial g$ other than a constant

$$\delta S = \int d^4x\, \delta(\sqrt{-g})\, R + \int d^4x \sqrt{-g}\, \delta R_{ab}\, g^{ab} + \int d^4x \sqrt{-g}\, R_{ab}\, \delta(g^{ab})$$

Recall $\delta \Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\delta g_{\sigma\nu;\rho} - \delta g_{\nu\rho;\sigma} + \delta g_{\sigma\rho;\nu} \right)$

Thus it can be shown

$$\delta R_{\mu\nu} = (\delta \Gamma^\sigma_{\mu\sigma})_{;\nu} - (\delta \Gamma^\sigma_{\mu\nu})_{;\sigma}$$

Then r.c.b.s.t.

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{\partial}{\partial x^\nu}\left(\sqrt{g}\, g^{\mu\nu}\delta\Gamma^\sigma_{\mu\sigma}\right)$$

$$- \frac{\partial}{\partial x^\sigma}\left(\sqrt{g}\, g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu}\right)$$

cov. divergences converted

Thus it is a total derivative

$$\int_\Omega d^4x\,(g^{\mu\nu}\delta R_{\mu\nu}) = \int_{\partial\Omega} d^3\Sigma_\mu\, J^\mu \to \text{zero}$$

for suitable choice $\delta g\big|_{\partial\Omega} = 0$

Next

$$\delta(\sqrt{-g}) = \frac{1}{2}\frac{1}{\sqrt{-g}}\,\delta(-\det g)$$

use identity for matrix variation

$\delta(\det A) = |\det A|\, \mathrm{Tr}\, A^{-1}\delta A$

$$= \frac{1}{2\sqrt{-g}}(-g)\, g^{ab}\delta g_{ab}$$

$$= +\frac{1}{2}\sqrt{-g}\, g^{ab}\delta g_{ab}$$

Note $0 = \delta(g^{ab}g_{ab}) = g^{ab}\delta g_{ab} + (\delta g^{ab})g_{ab}$

$$= -\frac{1}{2}\sqrt{-g}\, g_{ab}\,\delta g^{ab}$$

Thus we recover Einstein's eqn-s as E-L eqn-s

$$R_{ab} - \frac{1}{2} g_{ab} R = 0$$

Energy-Momentum tensor (covariant)

Need to compare

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

Since total action

$$S = S_{grav} + S_{matter}$$

We do have $\frac{1}{16\pi G}$ on S_{grav}.

by convention so that we must have

$$T_{ab} = \frac{\delta S_{matter}}{\delta g^{ab}}$$

Note $S_{matter}[\psi_m, \nabla_a \psi_m]$

Note : $T_{ab} = \dfrac{\delta S_{matter}}{\delta g_{ab}}$

① is naturally symmetric

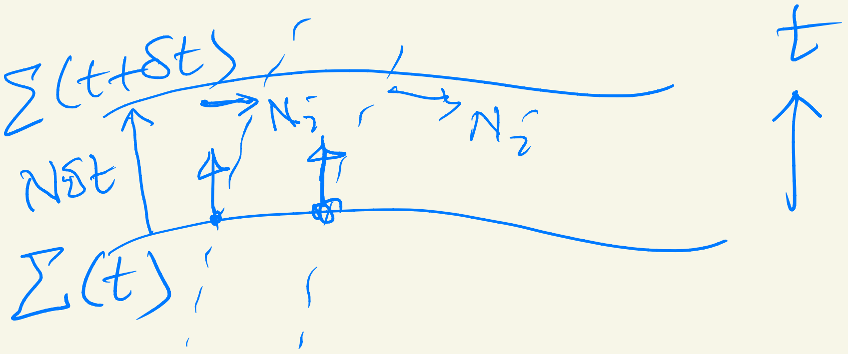
② will also come out satisfying

$$\nabla_a T^{ab} = 0$$ → needs proof which follows from $S[\psi, \nabla\psi]$

Proved in Straumann.

Properties typically not satisfied by Noether $T^{\mu}{}_{\nu}$

Quick preview of ADM



MTW Chap. 21

$$\Sigma(t+\delta t) \quad dx^i \longrightarrow N^i + dx^i$$
$$dt = N\,dt$$

Thus,

$$ds^2 = -N^2 dt^2 + {}^{(3)}g_{ij}(N^i + dx^i)(N^j + dx^j)$$

Extrinsic curvature

→ arises due to cov. deriv. restricted to Σ's

→ determines variation of vectors normal to Σ

→ is equiv to momentum canonically conjugate to ${}^{(3)}g_{ab}$