EQUIVALENCE TESTING OF PRINCIPAL MINORS

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PRINCIPAL MINOR EQUIVALENCE TESTING

- Given an $n \times n$ matrix A and $S, T \subseteq [n]$, let $A[S, T]$ be the submatrix of A with rows indexed by S and columns indexed by T. Let $A[S] = A[S, S]$.
- Principal minor corresponding to set $S \subseteq [n]$ is $\det(A[S])$.
- Question (PMET): Given $A, B \in \mathbb{F}^{n \times n}$, check whether all principal minors of A and B are same. If yes then we say $A \equiv B$.
- Example:

$$\mathrm{A} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 2 & 1/2 & -2 \\ 6 & -1 & -2 \\ 2 & -1/2 & 0 \end{pmatrix}.$$

- $\{1\} \to 2; \{2\} \to -1; \{3\} \to 0; \{1,2\} \to -5; \{1,3\} \to -4; \{2,3\} \to 1; \{1,2,3\} \to -2.$

DETERMINANTAL POINT PROCESS

- A Determinantal Point Process (DPP) on $[n]$ is a random subset $Y \subseteq [n]$ such that there exists a matrix A that satisfies
$$\mathbb{P}[S \subseteq Y] = \det(A[S]) \quad \forall S \subseteq [n].$$
- Such a matrix A is called a kernel of the DPP.
- DPPs have applications in
  - Recommender systems
  - Image search and segmentation
  - Audio signal Processing
- Equivalence testing of Principal Minors $\Rightarrow$ Testing whether two matrices are kernels of a same DPP.

POLYNOMIAL IDENTITY TESTING

- PIT: Input is a polynomial from a class of polynomials, the goal is to check whether it is 0.
- In Blackbox setting, the input polynomial is given as an oracle.
- In Whitebox setting, the polynomial access is given via
  - Algebraic circuits
  - Algebraic branching programs
  - Symbolic matrices etc.
- Symbolic Determinant: Determinant of a matrix with linear forms

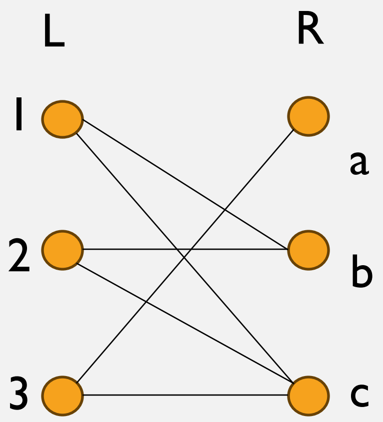
$$A = \sum_{i=1}^{m} y_i A_i$$

- [PIT Lemma]: At a random point, a non-zero polynomial evaluates to non-zero with high probability.
- Big open question: Deterministic algorithm for PIT?
  - Known for some restricted classes of polynomials.

SYMBOLIC DETERMINANT WITH RANK ONE CONSTRAINT

- $\text{DET1}_{n,m} = \{\det(\sum_{i=1}^{m} y_i A_i) \mid A_i \in \mathbb{F}^{n\times n}, \text{rank}(A_i) = 1\}$.
- DET1: $m = \text{poly}(n)$.
- Polynomial Identity Testing for this class captures
  - Bipartite Perfect Matching
  - Linear Matroid Intersection
  - Full Matrix Rank Completion.
- White box PIT [Lovász 89], Blackbox PIT [Gurjar-Thierauf 18]

EXAMPLE: BIPARTITE PERFECT MATCHING



$A_G =$

	a	b	c
1	0	x_{1b}	x_{1c}
2	0	x_{2b}	x_{2c}
3	x_{3a}	0	x_{3c}

$$\det(A_G) = x_{1b}x_{2c}x_{3a} + x_{1c}x_{2b}x_{3a}$$

- $A_\mathrm{G} = \sum_{i=1}^{m} x_i A_i$, where A_i has single non-zero entry 1.
- In $\det(A_\mathrm{G})$, each monomial correspond to a perfect matching.
  - $\det(A_G) \neq 0 \Leftrightarrow G$ has a perfect matching.

PIT FOR SUM OF TWO DET1

- Given $A = \sum_{i=1}^{m} y_i A_i, \quad B = \sum_{i=1}^{m} y_i B_i$ such that $\text{rank}(A_i) = \text{rank}(B_i) = 1$,

  Check whether $\det(A) + \det(\mathrm{B}) = 0$ or $\det(\mathrm{A}) = \det(\mathrm{B})$.

- Example: Check if two bipartite graphs G_1, G_2 have same set of perfect matchings.
  - Check whether $\det(A_{G_1}) = \det(A_{G_2})$.
- Theorem: Let $P = \det(\sum_{i=1}^{m} y_i A_i)$ such that $A_i = u_i . v_i^T$. Let $U, V \in \mathbb{F}^{n \times m}$ with ith column as u_i and v_i respectively. Then for $|T| = n$ and $y_T = \prod_{e \in T} y_i$
  - $\text{Coeff}(y_T) = \det(U_T)\det(V_T)$ where $U_T = U[[n], T], V_B = V[[n], T]$
- Given $(U_1, V_1), (U_2, V_2)$, check $\det(U_{1,T})\det(V_{1,T}) = \det(U_{2,T})\det(V_{2,T})$ $\forall T$.
- Example: Check whether two pairs of binary matroids have same common bases.

PIT FOR SUM OF TWO $\mathrm{DET1}$

- Claim: Equivalence testing of principal minors $\Rightarrow$ PIT for Sum of two $\mathrm{DET1}$.
- $(U_1, V_1): P_1, (U_2, V_2): P_2$, check $\det(U_{1,T})\det(V_{1,T}) = \det(U_{2,T})\det(V_{2,T})\ \forall\, T$.
- Proof: Find T_0 such that $\det(U_{1,T_0})\det(V_{1,T_0}) \neq 0$.
  - If $\det(U_{1,T_0})\det(V_{1,T_0}) \neq \det(U_{2,T_0})\det(V_{2,T_0})$, then $P_1 \neq P_2$.
- Wlog $T_0 = [n], U_i = (I_n \quad \widehat{U_i})$ and $V_i = (I_n \quad \widehat{V_i})$ for $i \in \{1,2\}$.
- $$P_1 = P_2 \Leftrightarrow \begin{pmatrix} 0_n & \widehat{V_1}^T \\ -\widehat{U_1} & 0_{m-n} \end{pmatrix} \equiv \begin{pmatrix} 0_n & \widehat{V_2}^T \\ -\widehat{U_2} & 0_{m-n} \end{pmatrix}.$$

[redacted]

PRINCIPAL MINOR PRESERVING OPERATIONS

- Transpose: $\mathrm{A} \equiv A^T$.
- Diagonally Equal (DE) : $D \rightarrow$ A diagonal matrix with non-zero entries.

$$A \equiv DAD^{-1}.$$

  - For any $\sigma: [n] \rightarrow [n]$, $\prod_{i=1}^{n} A[i, \sigma(i)] = \prod_{i=1}^{n} \frac{D[i]}{D[\sigma(i)]} A[i, \sigma(i)]$.
- Example: $\mathrm{A} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 2 & 1/2 & -2 \\ 6 & -1 & -2 \\ 2 & -1/2 & 0 \end{pmatrix}$

  $\mathrm{B} = DAD^{-1}$ where $D = \mathrm{diag}\left\{1, \frac{1}{2}, -1\right\}$.
- Question: Are these two operations sufficient to describe any B with $B \equiv A$?
  - True for a certain class of irreducible matrices.

IRREDUCIBLE MATRICES

- A matrix A is called reducible if $\exists$ permutation matrix P such that PAP is block upper triangular.
  - A matrix that is not reducible is called irreducible matrix.
- Example: $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 2 & 0 & -2 \\ 1 & 5 & -1 & 6 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, Exchange 2 & 3 $\to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 5 & 6 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$
- For a block upper triangular matrix A with blocks $(T_1, T_2, \dots, T_k)$,

$$\det(A) = \prod_{i=1}^{k} \det(A[T_1]).$$

  - $A[T_i]$ is irreducible.
- $A \equiv B \Leftrightarrow A[\{1,3\}] \equiv B[\{1,3\}]$ & $A[\{2,4\}] \equiv B[\{2,4\}]$.
- PMET for Irreducible matrices $\Rightarrow$ PMET for general matrices [Hartfiel-Leowy, 84].

PMET FOR IRREDUCIBLE MATRICES WITH CONSTRAINTS

- Cut: For $n \times n$ matrix A, $S \subset [n]$ with $2 \leq |S| \leq n-2$ is a cut if
$$\text{rank}(A[S,S^c]) \leq 1 \text{ and } \text{rank}(A[S^c,S]) \leq 1.$$
  - Example: $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & 2 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 1 & 1 & -1 \end{pmatrix}$
  - $\{1,2\}$ is a cut as $A[\{1,2\},\{3,4\}] = \langle 1,2\rangle^T.\langle 1,2\rangle$ & $A[\{3,4\},\{1,2\}] = \langle 1,-1\rangle^T.\langle 1,-1\rangle$.
- For an irreducible symmetric matrix A,
$$A \equiv B \Leftrightarrow B = DAD^{-1} \;\; or\; B = DA^TD^{-1}.$$ [Engel-Schneider, 80]
- For an irreducible matrix A with no cuts,
$$A \equiv B \Leftrightarrow B = DAD^{-1} \;\; or\; B = DA^TD^{-1}.$$ [Hartfiel-Leowy, 84]
- Not sufficient for irreducible matrices with cuts. [Ahmadieh, 23]

DETERMINANT OF MATRIX WITH A CUT

- Let A be an irreducible matrix with cut S,

$$\mathrm{A} = \begin{array}{c} \\ S \\ S^c \end{array} \begin{array}{c} \begin{array}{cc} S & S^c \end{array} \\ \begin{pmatrix} M & p.q^T \\ u.v^T & N \end{pmatrix} \end{array}.$$

- Generalized Laplace Theorem: For an $n \times n$ matrix P and $S \subset [n]$,

$$\det(P) = \sum_{|T|=|S|} \operatorname{sgn}(S,T) \det(P[S,T])\det(P[S^c,T^c]) .$$

  - $\operatorname{sgn}(S,T) \in \{-1,1\}$.

- $\det(A) = \det(M)\det(N) + \sum_{S'=S-e+f} \pm\det(A[S,S'])\det(A[S^c,S'^c]) + 0$

$$\pm\det\begin{pmatrix} M & p \\ v^T & 0 \end{pmatrix} \det\begin{pmatrix} 0 & q^T \\ u & N \end{pmatrix}$$

"TWIST" OPERATION

- For A = $\begin{array}{c c} & \begin{array}{cc} S & S^c \end{array} \\ \begin{array}{c} S \\ S^c \end{array} & \begin{bmatrix} M & p.q^T \\ u.v^T & N \end{bmatrix} \end{array}$, $\det(A) = \det(M)\det(N) \pm \det\begin{pmatrix} M & p \\ v^T & 0 \end{pmatrix} \det\begin{pmatrix} 0 & q^T \\ u & N \end{pmatrix}$.
- Let tw(A, S) = $\begin{array}{c c} & \begin{array}{cc} S & S^c \end{array} \\ \begin{array}{c} S \\ S^c \end{array} & \begin{bmatrix} M & p.u^T \\ q.v^T & N^T \end{bmatrix} \end{array}$: Twist of matrix A w.r.t. cut S
  - $\det(\mathrm{tw}(A, S)) = \det(M)\det(N^T) \pm \det\begin{pmatrix} M & p \\ v^T & 0 \end{pmatrix} \det\begin{pmatrix} 0 & q^T \\ u & N \end{pmatrix}$.
  - $\det(N) = \det(N^T)$ and $\det\begin{pmatrix} 0 & q^T \\ u & N \end{pmatrix} = \det\begin{pmatrix} 0 & u^T \\ q & N^T \end{pmatrix}$.
- $A \equiv \mathrm{tw}(A, S)$.

PMET FOR IRREDUCIBLE MATRICES

- Question: Are these three operations sufficient for PME for irreducible matrices?
  - This work: YES!
- Theorem: Let A & B be irreducible matrices s.t. $A \equiv B$. Then, $\exists$ $(A = A_0, A_1, \ldots, A_k)$
  - $A_i = \mathrm{tw}(A_{i-1}, X_i)$ where X_i is a cut of A_{i-1} for each $i \in [k]$.
  - A_k is diagonally equal to B or B^T.
- Theorem: For $n \times n$ matrices, there exists polynomial time algorithm that
  - Outputs $(A = A_0, A_1, \ldots, A_k)$ such that $\mathrm{k} \leq 2n$ iff $A \equiv B$.
  - Otherwise output "No".

MINIMAL CUT

- For $n \times n$ matrix $A, S \subseteq [n]$ is cut if $\operatorname{rank}(A[S, S^c]) = \operatorname{rank}(A[S^c, S]) = 1.$
- S is a minimal cut if there is no cut $T \subset S$.
- Lemma: S is a minimal cut of A, $|S| \geq 3 \Rightarrow$ For $\mathrm{t} \in S^c$, $\mathrm{A}[\mathrm{S} + \mathrm{t}]$ has no cuts.
- Proof: Suppose not true. Let X be a cut of $A[S+t]$ with $t \in X$.

A=

	$S \setminus X$	$X - t$	t	$S^c - t$
$S \setminus X$	*		u_1	
$X - t$		*	u_2	
t			*	*
$S^c - t$			*	*

$A[S \setminus X, (S \setminus X)^c]$
$= u_1.\langle v_1 | 1 | v_2 \rangle^T$

$A[(S \setminus X)^c, \setminus X]$
$= \langle p_1 | 1 | p_2 \rangle. q_2^T$

Contradicts the minimality of S.

MINIMAL CUT

- **Lemma**: $A \equiv B, S$ is a minimal cut of A, $|S| \geq 3 \Rightarrow$ S is also a cut of B.
- **Proof Idea**: Claim $\Rightarrow$ For $\forall t \in S^c$, $A[S+t]$ has no cuts.
- $A[S+t] \equiv B[S+t] \Rightarrow B[S+t] = DA[S+t]D^{-1} or\ DA[S+t]D^{-1}$ [Hartfiel-Leowy, 84]
- Fix $t_0 \in S^c$. Wlog, assume $A[S+t_0] = B[S+t_0]$, then

$$A[S+\{t_0,t\}] = \begin{array}{c} \\ S \\ t_0 \\ t \end{array} \begin{array}{c} \begin{matrix} S & t_0 & t \end{matrix} \\ \begin{pmatrix} M & p & \alpha p \\ q^T & n & * \\ \beta q^T & * & * \end{pmatrix} \end{array} \ \&\ \mathrm{B}[S+\{t_0,t\}] = \begin{array}{c} \\ S \\ t_0 \\ t \end{array} \begin{array}{c} \begin{matrix} S & t_0 & t \end{matrix} \\ \begin{pmatrix} M & p & x \\ q^T & n & * \\ y^T & * & * \end{pmatrix} \end{array}$$

- $B[S+t] = DA[S+t]D^{-1} \Rightarrow x = \alpha_1 p$ & $y = \beta_1 q$.
  - $B[S+t] = DA[S+t]^T D^{-1}$ can't happen.

PROOF OF THEOREM

- Theorem: Let A & B be irreducible matrices s.t. $A \equiv B$. Then, $\exists$ $(A = A_0, A_1, \dots, A_k)$
  - $A_i = \mathrm{tw}(A_{i-1}, X_i)$ where X_i is a cut of A_{i-1} for each $i \in [k]$.
  - A_k is diagonally equal to B or B^T.
- Proof: If A has no cut, $A \equiv B \Leftrightarrow A \text{ DE } B$ or B^T. [Hartfiel-Leowy, 84].
- Lemma: Let S be a cut of A, $t \in S$ & X be a cut of $A[S^c + t]$. Then, $\exists$ a cut T of A with $B = \mathrm{tw}(A, T)$ such that
  - $B[S^c + t] = \mathrm{tw}(A[S^c + t], X)$.
  - S is a minimal cut of $A \Rightarrow S$ is a minimal cut of B.

PROOF OF THEOREM

- Let S be a minimal cut of A.
- $A \equiv B \Rightarrow$ By induction hypothesis for $A[S^c + t]$, there exists $(A[S^c + t] = A'_0, A'_1, \ldots, A'_k)$ such that A'_k DE $B[S^c + t]$ or $B[S^c + t]^T$
- Lemma: Let S be a cut of A, $t \in S$ & X be a cut of $A[S^c + t]$. Then, $\exists$ a cut T of A with $B = \mathrm{tw}(A, T)$ such that
  - $B[S^c + t] = \mathrm{tw}(A[S^c + t], X)$.
  - S is a minimal cut of $A \Rightarrow S$ is a minimal cut of B.
- Lemma on repeat $\Rightarrow \exists\ (A = A_0, A_1, \ldots, A_k)$ such that $A_i = \mathrm{tw}(A_{i-1}, T_i)$
  - $A_i[S^c + t] = A'_i\ \forall \in [k]$.
  - S is a minimal cut of A_i.

$\Rightarrow A_k\ [S^c + t]$ is DE to $B[S^c + t]$ or $B[S^c + t]^T$
$\Rightarrow S$ is a minimal cut of A_k.

To complete proof of Theorem, find sequence from A_k to B.

PROOF OF THEOREM

- Given $\mathrm{A} \equiv B$ and
  - S is a minimal cut of A common to B.
  - for a $\mathrm{t}_0 \in S$, $A[S^c + t_0] = B[S^c + t_0]$.
- Lemma: B is diagonally equal to A or $\mathrm{tw}(A, S^c)$.
- Proof Idea: S is a minimal cut of $A \Rightarrow A[S + t]$ is irreducible $\forall t \in S^c$.
- $A[S + t]$ is irreducible $\Rightarrow A[S + t]$ is diagonally equal to
  - $B[S + t]$: B is diagonally equal to A.
  - $B[S + t]^{\mathrm{T}}$: B is diagonally equal to $\mathrm{tw}(A, S^c)$.

ALGORITHM OVERVIEW

- If A has no cut, check if A is DE to B or B^T.
- Else, Find a minimal cut S.
- Fix $t_0 \in S$, recursively check $A[S^c + t_0] \equiv B[S^c + t_0]$.
- If $A[S^c + t_0] \equiv B[S^c + t_0]$, get the sequence for $A[S^c + t_0]$ to $B[S^c + t_0]$.
- Get a sequence from A to A' such that $A'[S^c + t_0]$ is DE to $B[S^c + t_0]$.
- Check if B is DE to A' or $\mathrm{tw}(A', S^c)$.

OPEN PROBLEMS

- Principal Minor Assignment Problem: For an unknown $\mathrm{n} \times n$ matrix A,
  - Input: An oracle that outputs $\det(A[S])$ on input $S \subseteq [n]$
  - Goal: Find any $B \equiv A$ in $\mathrm{poly}(n)$ time.
  - Results known for special classes of matrices.
- Blackbox PIT for sum of two $\mathrm{DET1}$.
- PIT for sum of k $\mathrm{DET1}$.

  Check $\sum_{j=1}^{j=k} \det\left(\sum_{i=1}^{n} y_i\, A_i^j\right) = 0$

THANK YOU!