Some Select Topics in Extremal Graph Theory

These notes arise from the notes that the students of MA 5109 (Graph theory) scribed for certain select topics of the course (this is not the entire course!) I offered in this semester. The proofs here are not mine; in fact Conlon's course material contains the first two 'chapters' of these notes. The part on the Regularity lemma are by now pretty standard material, and the applications are a minute dip into the wonderful survey paper of Komlós and Simonovits. The two topics on the van der Waerden theorem and the Hales Jewett theorem also appear in the old monograph of Graham/Rothschild/Spencer. What is possibly new here is the exposition, but I don't make that claim either. This is simply the way I taught these topics in this course.
There is also likely 'irregularity' in the way the different topics are presented here, and that would be owing to the fact that the scribes brought in their perspective in their writing. The students were given a directive to scribe the notes as closely as possible to the spirit of the lectures in class. Some of them have added some additional notes on their own to improve the quality of the notes and their enthusiasm in this regard has resulted in this net output.

I thank all the students for participating in this project wholeheartedly, and for their proactive interest in this little project. I however take responsibility for any mistakes/errors that remain amongst these notes.

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Graph Theory	Fall 2018

Turán Number for Odd Cycles: $ex(n; C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$ **for** $n \gg 0$

Scribe: S Deepak Mallya

In this lecture we will consider the extremal number for odd cycles. We use the Erdős-Stone-Simonovits theorem and the result on extremal number for paths. Both are stated below.

Theorem 1 (Erdős-Stone-Simonovits). *For any fixed graph H and any fixed $\epsilon > 0$, there is $n_0 = n_0(H, \epsilon)$ such that, for any $n > n_0$,*

$$\left(1 - \frac{1}{\chi(H) - 1}\right) \frac{n^2}{2} \le ex(n; H) \le \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right) \frac{n^2}{2}.$$

Theorem 2. *If G satisfies*

$$e(G) > \frac{k-1}{2} |G|$$

then G contains a path of length k.

Theorem 3 (Relative Density Theorem). *For $k \ge 2$ and $\epsilon > 0$, there exists $\delta = \delta(k, \epsilon) > 0$ and $n_0 = n_0(k, \epsilon)$ such that if G is C_{2k+1} - free on $n \ge n_0$ vertices with at least $(\frac{1}{4} - \delta)n^2$ edges, then G may be made bipartite by removing at most ϵn^2 edges.*

Proof. Let us prove this result for $\delta = \frac{\epsilon^2}{L}$, with $L \gg 0$ and n sufficiently large. First, let us find a subgraph G' of G with large minimum degree, $\delta(G') \ge \frac{1}{2}(1 - 4\sqrt{\delta}) |G'|$. If G has vertices which don't satisfy the minimum degree requirements, we drop those vertices one by one, iteratively, till the requirement is met on the induced subgraph of G. The number of vertices in the final subgraph G' must necessarily satisfy $|G'| > (1 - 4\sqrt{\delta})n$. Because for $|G'| = (1 - 4\sqrt{\delta})n$ we have,

$$\begin{aligned} e(G') &> e(G) - \sum_{i=|G'|+1}^{|G|} \frac{1}{2}(1 - 4\sqrt{\delta})i \\ &\ge (\frac{1}{4} - \delta)n^2 - \frac{1}{2}(1 - 4\sqrt{\delta})(\binom{|G| + 1}{2} - \binom{|G'| + 1}{2}) \\ &\ge |G'|^2 (\frac{1}{4} + \eta) \end{aligned} \tag{1}$$

for some $\eta > 0$. This will contradict the assumption that $C_{2k+1} \nsubseteq G$ (from the Erdős - Stone - Simonovits Theorem with $\chi(H) = 3$ for C_{2k+1}). Therefore, we have a subgraph $G' \in G$ with $|G'| > (1 - 4\sqrt{\delta})n$ and $\delta(G') \ge \frac{1}{2}(1 - 4\sqrt{\delta}) |G'|$.
Since $ex(n; C_{2k}) = o(n^2)$, we know that for n (and therefore for $|G'|$) sufficiently large, the graph G' must contain C_{2k}. Let $v_1 v_2 \cdots v_{2k}$ be such a cycle with vertices in that order. Note that, $N(v_1)$ and $N(v_2)$ cannot intersect, else there will be a cycle of length $2k+1$. Let A and B be two induced graphs such that, $V(A) = N(v_1) - \{v_1, ..., v_{2k}\}$ and $V(B) = N(v_2) - \{v_1, ..., v_{2k}\}$.
For $n \gg 0$,

$$|A|, |B| \ge \frac{|G'|}{2}(1 - 4\sqrt{\delta}) - 2k \ge \frac{n}{2}(1 - 8\sqrt{\delta})$$

Both A and B contains at most $k\,|G'|$ edges (by $ex(n, P_{2k})$).
Deleting all edges in A, in B and from both A and B to $V(G) \setminus (A \cup B)$, we get a bipartite graph. The number of edges deleted is at most,

$$2kn + 8\sqrt{\delta}n^2.$$

So given $\epsilon > 0$, $n \gg 0$ and $\delta = \frac{\epsilon^2}{L}$, we would have deleted at most ϵn^2 edges to get the bipartite graph. □

Using the Relative Density Theorem, now let us move on to the main theorem.

Theorem 4. *For $n \gg 0$,*

$$ex(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor$$

Proof. Let G be a C_{2k+1} - free graph on n vertices with the maximum number of edges. It will have atleast $\lfloor \frac{n^2}{4} \rfloor$ edges (from $e(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) = \lfloor \frac{n^2}{4} \rfloor$ and $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ does not contain C_{2k+1}).
From the previous theorem, we may also assume that $\delta(G) \geq \frac{1}{2}(1 - 4\sqrt{\epsilon})n$. If not, we can add more edges to it without introducing C_{2k+1}.
Also, from the previous theorem, we know that G is approximately bipartite between two sets of size roughly $\frac{n}{2}$. Consider a bipartition $V(G) = A \cup B$ such that $e(A) + e(B)$ is minimized. Then, $e(A) + e(B) < \epsilon n^2$ where $\epsilon > 0$ may be taken to be arbitrarily small provided $n >> 0$. We may assume that A and B have size $(\frac{1}{2} \pm \sqrt{\epsilon})n$. Else, $e(G) < |A|\,|B| + \epsilon n^2 < \frac{n^2}{4}$ which is a contradiction to the maximum size of G.
We claim that there are no vertices $a \in A$ such that $|N_A(a)| > 2\sqrt{\epsilon}n$. If $|N_A(a)| > 2\sqrt{\epsilon}n$, then $|N_B(a)| > 2\sqrt{\epsilon}n$. This is because, else we can find a better partition with less number of edges in $e(A) + e(B)$. Moreover, $A \cap N(a)$ and $B \cap N(a)$ span a bipartite graph with no path length of $2k - 1$ and therefore there are at most kn edges between them. For sufficiently large n, this gives the number of edges missing between A and B to be

$$4\epsilon n^2 - kn > \epsilon n^2 > e(A) + e(B).$$

In that case, $e(G) < |A|\,|B| - 3\epsilon n^2 < \frac{n^2}{4}$ again a contradiction to the number of edges in maximal G. Therefore, for $a \in A$, $|N_A(a)| < 2\sqrt{\epsilon}n$. Similarly, for $b \in B$, $|N_B(b)| < 2\sqrt{\epsilon}n$. Now suppose there is an edge in A say aa'. Then,

$$\left|N_B(a) \cap N_B(a')\right| > N(a) - 2\sqrt{\epsilon}n + N(a') - 2\sqrt{\epsilon}n - |B| > (\frac{1}{2} - 9\sqrt{\epsilon})n$$

. Let $A' = A - \{a, a'\}$ and $B' = N_B(a) \cap N_B(a')$. There is no path of length 2k - 1 of the form $b_1 a - 1 b_2 a_2 ... b_{k-1} a_{k-1} b_k$ between A' and B'. But this implies there is no path of length $2k$. This implies that $e(A', B') < kn$, which in turn says,

$$e(G) = e(A', B') + e(A - A', V(G)) + e(V(G), B - B') \geq kn + 2n + \frac{n^2}{4} - 81\epsilon n^2.$$

This is also a contradiction for large n. Hence this shows, there can be no aa' in A. Hence, graph G with maximum number of edges is a bipartite graph with $\lfloor \frac{n^2}{4} \rfloor$ edges. □

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Cycles of Even Lengths in Graphs

Scribe: Kunal Mittal

In the last lecture, we saw the following theorem:

Theorem 5. *For every $k \geq 1$, $ex(n, C_{2k+1}) = \lfloor n^2/4 \rfloor$ for sufficiently large n (depending on k), where C_{2k+1} means a simple cycle on $2k+1$ vertices.*

This leads us to the next question, that of graphs without even cycles. The following result was first conjectured by Erdős (1964) and then later proved by Bondy and Simonovits (1974):

Theorem 6. *(Bondy-Simonovits) For all natural numbers $k \geq 2$, there exists a constant $c_0 = c_0(k)$, such that $ex(n, C_{2k}) \leq c_0 n^{1+1/k}$, for sufficiently large n (depending on k).*

We see that this gives a better upper bound than that obtained by excluding $K_{k,k}$, best known to be $O(n^{2-1/k})$ from the Zarankiewicz problem. It is conjectured that this result by Bondy-Simonovits is tight. For the special case of $k = 2$, that is C_4, we have already seen tightness through the idea of projective planes: $ex(n, C_4) = (1/2 + o(1))n^{3/2}$.

As for lower bounds, we'll prove later in the course that $ex(n, C_{2k}) \geq \Omega(n^{1+1/(2k-1)})$ by using a simple probabilistic method by Erdős. Explicit constructions are also known, as by Lazebnik, Ustimenko and Woldar (1994) which give lower bounds of $\Omega(n^{1+2/(3k-3)})$ for odd values of k and $\Omega(n^{1+2/(3k-4)})$ for even values of k, but these are outside the scope of this course.

The rest of this lecture will be devoted to the proof of Theorem 6.

Proof of Theorem 6

We first state a lemma (without proof):

Lemma 7. *Every graph H has a subgraph whose minimum degree is at least half the average degree of H.*

Suppose that H is a C_{2k} free graph on n vertices (n sufficiently large) with $e(H) > c_0 n^{1+1/k}$. Then $\bar{d}(H) = 2e(H)/n > 2c_0 n^{1/k}$, where $\bar{d}(H)$ denotes the average degree H. By Lemma 7, H must contain a subgraph G of minimum degree $\delta(G) \geq \bar{d}/2 > c_0 n^{1/k}$.

Pick a vertex $x \in V(G)$ and perform breadth-first search (BFS) starting from x. Define $V_0 = \{x\}$ and $V_i = \{y \in V(G) \mid d(x, y) = i\}$ for $1 \leq i \leq k$, where $d(x, y)$ denotes the distance between x and y in the BFS tree.

For relevant values of i, denote by $G[V_i]$ the induced subgraph on the vertex set V_i and by $G[V_i, V_{i+1}]$ the bipartite subgraph induced by partitions V_i and V_{i+1}. For clarity, we'll denote their number of edges by $e(V_i)$ and $e(V_i, V_{i+1})$, and their average degrees by $\bar{d}(V_i, V_{i+1})$ and $\bar{d}(V_i)$ respectively.
The main proof relies on the fact that both $G[V_i]$ and $G[V_i, V_{i+1}]$ are sparse, for all valid i. This is formalized in the following lemma, which we will prove in the next section.

Lemma 8. *There exist constants $c_1 = c_1(k)$ and $c_2 = c_2(k)$ such that for $1 \le i \le k-1$ the following hold:*

- $\bar{d}(V_i) \le c_1 k$.
- $\bar{d}(V_i, V_{i+1}) \le c_2 k$.

Using this Lemma, we next show that number of vertices in V_i blows up as we increase i. Let $n_i = |V_i|$. The following lemma captures this:

Lemma 9. *For $0 \le i \le k-1$, $\frac{n_{i+1}}{n_i} \ge \frac{c_0}{2c_2 k} n^{1/k}$.*

Proof. We Induct on i. In the base case i=0, $n_1/n_0 = d(x)/1 \ge \delta(G) > c_0 n^{1/k}$, where $d(x)$ denotes the degree of x. Thus if $c_2 \ge 1/2$, we are through.
For $1 \le i \le k-1$, we have that

$$\begin{aligned} e(V_i, V_{i+1}) &= \sum_{y \in V_i} \big(d(y) - d_i(y) - d_{i-1,i}(y)\big) = \Big(\big(\sum_{y \in V_i} d(y)\big) - 2e(V_i) - e(V_{i-1}, V_i)\Big) \\ &\ge \Big(\delta(G) n_i - \bar{d}(V_i) n_i - \frac{1}{2}\bar{d}(V_{i-1}, V_i)(n_{i-1} + n_i)\Big) \ge \Big(c_0 n^{1/k} n_i - c_1 k n_i - c_2 k n_i\Big) \\ &= (c_0 n^{1/k} - c_1 k - c_2 k) n_i \ge \frac{c_0}{2} n^{1/k} n_i \text{ (for } n \gg 0). \end{aligned}$$

where the second inequality uses Lemma 8 and fact that $n_i \ge n_{i-1}$ which comes by induction and that n is large. Here $d_i(y)$ denotes the degree of y to vertices in V_i and $d_{i-1,i}(y)$ denotes its degree to vertices in V_{i-1}.

Also, by Lemma 8, we have that

$$e(V_i, V_{i+1}) = \frac{1}{2}\bar{d}(V_i, V_{i+1})(n_i + n_{i+1}) \le \frac{c_2 k}{2}(n_i + n_{i+1})$$

Combining both equations, we get $\frac{c_0}{2} n^{1/k} n_i \le \frac{c_2 k}{2}(n_i + n_{i+1})$, giving $\frac{n_{i+1}}{n_i} \ge \frac{c_0}{c_2 k} n^{1/k} - 1 \ge \frac{c_0}{2c_2 k} n^{1/k}$ (for $n \gg 0$). □

From Lemma 9, it is easy to prove the main theorem, i.e., Theorem 6.
Applying the lemma, we get that $n_k \ge \left(\frac{c_0}{2c_2 k} n^{1/k}\right)^k \ge \left(\frac{c_0}{2c_2}\right)^k n$ which gives a contradiction in the case when $c_0 \ge 2c_2 k$.

A note on the constant factors involved:
In the following sections, we'll show that Lemma 8 holds for $c_1 = 4$ and $c_2 = 2$. This c_2 works in the base case of Lemma 9, and shows that $c_0 \ge 4k$ works.
Hence, we have proved $ex(n, C_{2k}) \le (4k) n^{1+1/k}$.

In fact, a careful analysis of this proof by Pikhurko (2012) shows that $ex(n, C_{2k}) \le (k - 1 + o(1)) n^{1+1/k}$. In our proof, at two places we say that $n \gg 0$ and relax by a factor of 2. This gives the improvement of a factor of 4 on better analysis. Also, in Lemma 8, we choose $c_1 k$ and $c_2 k$ as $2k$ and $4k$, whereas $2(k-1)$ and $4(k-1)$ suffice.

Proof of Lemma 8

Before we go on to prove this lemma, we make note of some useful lemmas.

Lemma 10. *Every graph G has a bipartite subgraph in which each vertex has degree at least half of its degree in the original graph.*

Proof. Done earlier. □

Lemma 11. *Given a bipartite graph G, with $\delta(G) \geq d$ ($d \geq 3$), then G has a cycle of length at least $2k$ with a chord.*

Proof. Consider a longest path in the graph. Observe that the first vertex must have all its neighbours in the path itself (since the path is longest). The rest comes from the fact that the degree of the first vertex is at least d and that G is bipartite. □

The proof of the next lemma is quite non-trivial and will be dealt in a separate section.

Lemma 12. *Suppose G is a cycle of size t with a chord, and suppose (A, B) is a non-trivial partition of its vertex set. Then for any $1 \leq \ell < t$, there is a path of length ℓ starting in A and terminating in B, unless of course G is bipartite with respect to the partition (A, B).*

Now we go on to prove Lemma 8 (in two parts).

Lemma 13. *For $1 \leq i \leq k-1$, $\bar{d}(V_i, V_{i+1}) \leq c_2 k$*

Proof. We prove the lemma for $c_2 = 2$. Let $\bar{d}(V_i, V_{i+1}) > 2k$. Then by Lemma 7, $G[V_i, V_{i+1}]$ has a subgraph F with minimum degree as k. Since $G[V_i, V_{i+1}]$ is bipartite, F is also bipartite and by Lemma 11, the graph F and hence $G[V_i, V_{i+1}]$ has a cycle of length at least $2k$ with a chord. Call this cycle + chord as the subgraph C. Let $X \cup Y$ be the partition of $V(C)$ with $X \in V_i$ and $Y \in V_{i+1}$.

Consider the BFS tree starting at x and let $y \in V_j$ ($j < i$) be a closest vertex to V_i (j is max possible) such that all vertices of X are descendants of y. Since y is closest, no child of y admits all vertices of X as its descendants. Let z be a child of y which admits a non-trivial and non-entire subset of X as its descendants. Let $A =$ descendants of z in X, and $B = (X \cup Y) \setminus A$. Clearly (A, B) is not a valid bipartition of C, since B has vertices in both X and Y and has edges of C among its vertices.
Then by Lemma 12, there is some $a_0 \in A$ and $b_0 \in B$ and a path of length $2k - 2(i-j)$ starting from a_0 and ending in b_0 that lies in $G[V_i, V_{i+1}]$. Also, since $a_0 \in A \subset X$ and the path is of even length, we have that $b_0 \in X$. Since $b_0 \notin A$, the earliest common ancestor of a_0 and b_0 in the BFS tree is y. But then $y \rightsquigarrow b_0 \rightsquigarrow a_0 \rightsquigarrow z \to y$ is a cycle of length $2k$ (where $b_0 \rightsquigarrow a_0$ is in C), giving a contradiction. □

Lemma 14. *For $1 \leq i \leq k-1$, $\bar{d}(V_i) \leq c_1 k$*

Proof. We prove the lemma for $c_1 = 4$. Let $\bar{d}(V_i) > 4k$. Then by Lemma 10, $G[V_i]$ has a bipartite subgraph T with $\bar{d}(T) > \bar{d}(V_i)/2 > 2k$ (since each vertex in T has degree at least half that in $G[V_i]$ the average degree of T is also atleast half of that of $G[V_i]$).
The rest of the proof follows exactly as in Lemma 13, by replacing V_i and V_{i+1} by the two bipartitions of T. □

Proof of Lemma 12

This Lemma is independent of our main theorem. We state it again and then give a proof.

Lemma 15. *Suppose G is a cycle of size t with a chord, and suppose (A, B) is a non-trivial partition of its vertex set. Then for any $1 \le \ell < t$, there is a path of length ℓ starting in A and terminating in B, unless of course G is bipartite with respect to the partition (A, B).*

Proof. Let the vertex set of G be $V = \{0, 1, \dots, t-1\}$ and let the chord be between vertices 0 and r. We can assume that $r \le t/2$ by looking at the cycle clockwise or anti-clockwise. Also, $r \ge 2$ since $r = 1$ means that it is an edge and not a chord.

By a *simple* path, we mean a path starting in A, terminating in B and not using the chord. Let $BAD = \{i \mid G \text{ has no } simple \text{ path of length } i\}$. If $BAD = \emptyset$, we are through. Suppose $BAD \ne \emptyset$. Let s be the minimum element of BAD. We must have that $s \le t/2$, since by symmetry $s \in BAD$ if and only if $t - s \in BAD$. Also, $s \ge 2$, since *simple* paths of length 1 (edges) from A to B must exist.

Let $\chi = \chi_A$ be the characteristic function of A, that is $\chi : V \to \{0, 1\}$ is given by $\chi(i) = 1$ if and only if $i \in A$. Let $x \in V$. We must have

$$\chi(x + \lambda s) = \chi(x) \text{ for all } \lambda \in \mathbb{Z} \tag{2}$$

where all additions are modulo t. This also gives that $\chi(x) = \chi(x + \lambda s + \mu t)$ for all $\lambda, \mu \in \mathbb{Z}$, giving that $\chi(x) = \chi(x + d)$ where d is the *gcd* of s and t. This uses the fact that $gcd(s, t)$ can be represented as $\lambda s + \mu t$ for appropriate $\lambda, \mu \in \mathbb{Z}$. Since this is true for all $x \in V$, we have that $d \in BAD$. But s is the minimum element of BAD giving that $s = d$ and $s|t$.
The same argument shows that for any $s_1 \in BAD$, $s|s_1$.

What is remaining is to find paths of length is, for $1 \le i \le t/s - 1$, starting and ending in different partitions. Consider the following cases:

1. $s \ge r$: We have $is < is + r - 1 < (i+1)s$ and hence there must exist a *simple* path of length $is + r - 1$ (s divides the length of all bad paths). We can assume by shifting (due to equation 2) that this path starts at $-s < j \le 0$. This *simple* path ends at $j' = j + is + r - 1 \ge r$ giving that $\chi(j) \ne \chi(j')$. Since $1 \le i \le t/s - 1$, the path doesn't wrap around and $j' < t + j$. Consider the path $j \rightsquigarrow 0 \to r \rightsquigarrow j'$. This is a path of length is beginning at j and ending at j' with $\chi(j) \ne \chi(j')$.

2. $s < r < t - s$: Let $-s < j \le 0$ and consider the following paths (of length s):

   (a) $j \rightsquigarrow 0 \rightsquigarrow r - j - s + 1$

   (b) if $j \ne 0$ then $s + j \rightsquigarrow 0 \to r \rightsquigarrow r - j - 1$
   else if $j = 0$ then $0 \to r \rightsquigarrow r + s - 1$

   If either of these is a path starting in A and ending in B (or vice-versa) we have found the required path of length s. Now we can extend it in both directions (using equation 2) to get longer paths (multiples of s) until the number of vertices not in the path is less than s (and non zero) on both sides of the chord. Then at this point we have $t - 2(s-1) \le is + 1$, giving

$t/s - 1 \leq i$ since $s|t$. Thus we have found all required paths.

In the remaining case, we must have that for all $-s < j \leq 0$ both the paths start and end in the same partition. That is, for $-s < j \leq 0$, we have

$$\chi(r-j-1) = \chi(s+j) = \chi(j) = \chi(r-j-s+1) = \chi(r-j+1)$$

using shifting as per equation 2 (works for $j = 0$ as well by appropriate shifts). The above equation gives that $\chi(i) = \chi(i+2)$ for $r-1 \leq i < r+s-1$. Since we have this for an interval of length s, we can obtain it for all i by shifting by s.
Hence we have that for all $i \in V$, $\chi(i) = \chi(i+2)$. This gives that $2 \in BAD$ and hence $s = 2$. In this case we have an even cycle (s divides t) and vertices of the cycles alternate between A and B. Now, if the chord is between the same partition, we have the required paths of all lengths less than t. Otherwise, the graph is bipartite with respect to the partition (A, B), completing the proof.

□

Random Graphs

Suppose $p \in (0,1)$ is a fixed real. $G(n,p)$, the Erdos-Renyl Random Graph is given as follows: $G(n,p)$: Suppose for each edge $e \in \binom{[n]}{2}$, X_e is an i.i.d. sequence of $Ber(p)$ random variables, $X_e \sim \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix}$ and X_e is independent i.e. for any $\epsilon_e \in \{0,1\}$, $\mathbb{P}(\wedge_e(X_e = \epsilon_e)) = \prod_e \mathbb{P}(X_e = \epsilon_e)$.

Suppose $G = G(n,p)$, and for $x \in [n]$ define $d_G(x) := \{y \in [n] : xy \in E(G)\}$ = Sum of $n-1$ independent $Ber(p) \sim Bin(n-1,p)$. Then

$$\begin{aligned} \mathbb{E}[d_G(x)] &= (n-1)p, \\ \mathbb{V}(d_G(x)) &= (n-1)p(1-p), \text{ and} \\ e(X,Y) &= Bin(|X||Y|,p) \end{aligned}$$

for $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$.

ϵ-Regularity

Definition 16. *Given $0 < \epsilon < 0.1$, Suppose X, Y are disjoint subsets of $V(G)$. We say that (X,Y) is ϵ-regular if for any $X' \subseteq X, Y' \subseteq Y$ with $\frac{|X'|}{|X|}, \frac{|Y'|}{|Y|} \geq \epsilon$, we have $|d(X',Y) - d(X,Y)| \leq \epsilon$, where $d(X,Y) = \frac{e(X,Y)}{|X||Y|}$ denotes the density of pair (X,Y).*

Regularity Lemma

Theorem 17. *(The Regularity Lemma of Szemerédi): Given $0 < \epsilon < 0.1$, there exists $M = M(\epsilon)$ such that any graph G on n vertices admits a partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ such that the following hold:*

(i) $|V_0| \leq \epsilon n$.

(ii) $|V_i| = |V_j|$ *for all* $1 \leq i, j \leq k$

(iii) $k \leq M$.

*(iv) At most ϵk^2 of the pairs (V_i, V_j) are **not** ϵ regular.*

Note that

- The regularity lemma of Szemerédi gives non trivial information only when $e(G) = \Omega(n^2)$.
- $M(\epsilon)$ turns out to be humongously large.

Suppose X, Y are both partitioned further by a partition P'. Pick $x \in X, y \in Y$ uniformly at random. Consider the random variable $z = d(X', Y')$ where X' and Y' are parts in P' and $x \in X', y \in Y'$. If (X, Y) is a disjoint pair, define $q(X, Y) := \frac{|X||Y|}{n^2} d^2(X, Y)$. And if P' is a partition of $X \cup Y$, define

$$q(P') = \sum_{\substack{X' \subseteq X \\ Y' \subseteq Y \\ X', Y' \in P'}} q(X', Y').$$

$q(P)$ is called the **mean square density** of the partition P.

Lemma 18. *Suppose (X, Y) is a disjoint pair, and suppose P' is a partition of $X \cup Y$. Then $q(P') \geq q(P)$ where $P = X, Y$.*

Proof. We claim that

$$q(P') - q(P) = \sum_{\substack{X' \in X \\ Y' \in Y \\ X', Y' \in P'}} (d(X, Y) - d(X', Y'))^2 \frac{|X'||Y'|}{n^2}.$$

Indeed, the RHS can be expressed as the sum of the following three terms:

- $$\frac{1}{n^2} \sum_{\substack{X' \in X \\ Y' \in Y \\ X', Y' \in P'}} d^2(X, Y)|X'||Y'| = d^2(X, Y)\frac{1}{n^2} = q(P).$$

- $$\frac{1}{n^2} \sum_{\substack{X' \in X \\ Y' \in Y \\ X', Y' \in P'}} d^2(X', Y')|X'||Y'| = q(P').$$

- $$\frac{-2}{n^2} \sum_{\substack{X' \in X \\ Y' \in Y \\ X', Y' \in P'}} d(X', Y')|X'||Y'| = \frac{-2}{n^2} d^2(X, Y)|X||Y| = -2q(P).$$

Thus, the total summation, which is clearly positive as it is a sum of squares is equal to $q(P') - q(P)$, proves that $q(P') \geq q(P)$. $\square$

Remark: By the Cauchy Schwartz inequality it can also be seen that $q(P) \leq 1$.

Lemma 19. *Suppose (X, Y) is not ϵ regular, and consider $P' = \{X^*, X/X^*\} \cup \{Y^*, Y/Y^*\}$ where (X^*, Y^*) witnesses the ϵ irregularity of the pair (X, Y). Then*

$$q(P') - q(P) > \epsilon^4 \frac{|X||Y|}{n^2}.$$

Proof. By the definition of ϵ-irregularity, $\frac{|X^*|}{|X|}, \frac{|Y^*|}{|Y|} \geq \epsilon$ and $|d(X^*, Y^*) - d(X,Y)| > \epsilon$. Then,

$$\begin{aligned} q(P') - q(P) &= \sum_{\substack{X' \in \{X^*, X/X^*\} \\ Y' \in \{Y^*, Y/Y^*\}}} (d(X', Y') - d(X,Y))^2 \frac{|X'||Y'|}{n^2} \\ &\geq \epsilon^2 |X^*||Y^*| \frac{1}{n} \\ &> \epsilon^4 \frac{|X||Y|}{n^2} \end{aligned}$$

and that completes the proof. □

Now, we are in a position to complete the proof of the Regularity lemma.

Proof. Suppose we have a parition $V_0 \cup V_1 \cup V_2 \ldots \cup V_k$ which is equitable, with $|V_0| < \epsilon n$ and suppose $P = \{V_0, \ldots, V_k\}$ is not ϵ- regular i.e. there are more than ϵk^2 pairs (V_i, V_j) such that these pairs are not ϵ-regular. For each irregular (V_i, V_j), get, as indicated in lemma 19 a partition $(V_i^*, Vi/V_i^*)$ and $(V_j^*, Vj/V_j^*)$ and consider the common refinement induced by all these irregular pairs. Call this partition P_1. By lemma 19 we have

$$q(P1) - q(P) > \epsilon^4 \epsilon \frac{k^2}{n^2} \geq \epsilon^5 (1-\epsilon)^2 \geq \frac{\epsilon^5}{2}.$$

Note that the V_i is partitioned into at most 2^{k-1} parts in P. Cut each of these so that the V_i is now partitioned into 4^k parts and put the residual vertices into V_0. Call this resulting Partition P'. Then

(i) P' is equitable of size $b = \frac{c}{4^k}$.

(ii) $q(P') \geq q(P_1) > q(P) + \frac{\epsilon^5}{2}$.

(iii) $|V_0| \leq |V_0| + k2^{k-1}\frac{c}{4^k} \leq |V_0| + \frac{n}{2^k}$.

Note that

$$q(P) = \sum_{1 \leq i < j \leq k} \frac{|V_i||V_j|}{n^2} d^2(V_i, V_j) < \frac{1}{2}.$$

Thus this suggests: Start with an arbitrary partition of V into k_0 parts with $\frac{1}{2^{k_0}} < \frac{\epsilon^6}{2}$ and $k_0 > \frac{2}{\epsilon}$, so that initially, $|V_0| < \frac{n}{k_0} < \frac{\epsilon n}{2}$. Furthermore, if P is not ϵ-regular, then as stated above, get a refinement of P. Note that the refinement iterations must happen at most $\leq \frac{1}{\epsilon^5}$ times as $q(P) \leq \frac{1}{2}$. Also, eventually,

$$|V_0^{fin}| \leq |V_0^{ini}| + \frac{n}{2^k}\frac{1}{\epsilon^5} \leq \epsilon n$$

by the choice of the parameters. This completes the proof. □

Remarks

1. One may wonder if the size of $M(\epsilon)$ may be reduced by a 'better' proof. But as was first proved by Gowers (1996) and improved by several others since then, there exists graphs for which a tower type bound for $M(\epsilon)$ is in fact necessary, i.e. there exist graphs with $M(\epsilon) = \Omega(Tower(\epsilon^{-c}))$ for some constant $c > 0$.

2. The definition of ϵ regularity of a partition only requires that the number of irregular pairs at most ϵk^2. As it turns out, there are graphs for which the irregular pairs one will nonetheless get is of the order $\frac{k^2}{log^* k}$ where $log^* n$ is defined to be the least integer m such that $\underbrace{\log(\log \cdots \log) \cdots)}_{\log \text{ appears } m \text{ times}} n \leq 1$.

SRL Applications

Suppose $e(G_n) \geq \epsilon n^2$ for some fixed $\epsilon > 0$. Let $\{V_0, \ldots, V_k\}$ be an $\frac{\epsilon}{8}$ regular partition. Also, let $k \geq \frac{4}{\epsilon}$. Now we begin a 'purification' process by deleting the following edges in the graph:

(i) Edges incident to V_0. This removes at most $\frac{\epsilon}{8} n|V_0| < \frac{\epsilon}{8} n^2$ edges from G.

(ii) Edges within V_i for each i. This removes at most $\frac{n^2}{2k} < \frac{\epsilon}{8} n^2$ edges from G.

(iii) Edges between irregular pairs. This removes at most $\epsilon k^2 c^2 < \frac{\epsilon}{8} n^2$ edges from G.

(iv) Edges between pairs having density less than $\frac{\epsilon}{4}$. This removes at most $\frac{\epsilon}{8} n^2$ edges of G.

The 'purified' graph still has more than $\frac{\epsilon}{2} n^2$ edges and is an $(\frac{\epsilon}{8}, \delta)$ regular graph with density between pairs at least $\delta = \frac{\epsilon}{4}$. This essentially means that each pair (V_i, V_j) is $\frac{\epsilon}{8}$ regular with density $\geq \delta = \frac{\epsilon}{4}$.
This leads us to define the 'reduced' graph $R_\delta(\Pi)$ whose vertex set corresponds to the parts obtained after the aforementioned purification process, with (V_i, V_j) being adjacent in $R(\Pi)$ iff the pair is ϵ-regular with density at least δ for some prefixed $\delta > 0$. This is directly used in the counting lemma (that we shall prove later) which says that if the reduced graph has a copy of a graph H, then the original graph G has $\geq \eta n^{|H|}$ copies of H for some $\eta = \eta(\epsilon, H)$.

Lemma 20. *(Counting Lemma): Let H be a fixed graph. Suppose $R_\delta(\Pi)$ is the reduced graph as described above (from an ϵ-regular partition of a graph G), and suppose H appears as a subgraph of $R_\delta(\Pi)$. Then there exists $\eta := \eta(\delta, \epsilon, H)$ such that G contains at least $\eta n^{|H|}$ copies of H.*

Regularity Lemma: Applications

Scribe: Anasuya Acharya

The Regularity Lemma for dense graphs basically states that there exists a partitioning of the vertex set of the graph into almost equal parts s.t. between all pairs of vertex sets, for large enough subsets the *density* of edges between them is close to *density* of edges between the vertex sets.

Definition 21. *A* ***random graph*** $G_{n,p}$ *is a graph on* n *nodes and each edge is present with the probability* p.

For a graph $G_{n,p}$, let $X, Y \subseteq V(G)$, then the number of edges between X and Y,

$$\begin{aligned}
P(e \in E(G)) &= p \\
\mathbb{E}(X+Y) &= \mathbb{E}X + \mathbb{E}Y \\
\mathbb{E}(e(X,Y)) &= \mathbb{E}(\sum_{x\in X, y\in Y} \mathbb{I}(x,y)) \\
&= \sum_{x\in X, y\in Y} P(e(x,y)) \\
&= p|X||Y|
\end{aligned}$$

$\therefore$ the number of edges between X and Y is very close to $p|X||Y|$ with high probability.

Szemerédi's Regularity Lemma

Some preliminary definitions:

Definition 22. *For a graph* G*, let* $X, Y \subset V$ *be two disjoint non-empty parts of the vertex set. The* ***density*** *of* (X,Y) *is*

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|} = \frac{\#\ \textit{edges between } X \textit{ and } Y}{|X||Y|}$$

Definition 23. *For a graph* G *and* $\epsilon > 0$*, let* $A, B \subset V$ *be two disjoint non-empty parts of the vertex set. The pair* (A,B) *is* ***ϵ-regular*** *if* $\forall X \subseteq A, |X| \geq \epsilon|A|$ *and* $\forall Y \subseteq B, |Y| \geq \epsilon|B|$,

$$|d(X,Y) - d(A,B)| \leq \epsilon$$

Definition 24. *A partition of the vertex set* $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ *is said to be an* ***equipartition*** *if* $|V_1| = \cdots = |V_k|$*, with* V_0 *being an exceptional set.*

Definition 25. *For a graph* $G(V,E)$ *and* $\epsilon > 0$*, an equipartition* $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ *with* $|V_0| \leq \epsilon|V|$ *is an* ***ϵ-regular partition*** *if all but at most* ϵk^2 *pairs* (V_i, V_j)*,* $1 \leq i \leq j \leq k$*, are* ϵ*-regular.*

Theorem 26. *(Regularity Lemma) $\forall \epsilon > 0, t \in \mathbb{N}, \exists$ integer $T = T(t, \epsilon)$, s.t. every graph $G(V, E)$ with $|V| = T$ has an ϵ-regular partition $V(G) = V_1 \cup \cdots \cup V_k$, $t \leq K \leq T$*

That is, $\forall \epsilon, t \exists T$ s.t. for every graph G on at least T vertices there is a **Szemerédi Partition** of G, i.e. a partition with the properties:

- $V(G) = V_1 \cup \cdots \cup V_k$, $t \leq k \leq T$
- $|V_1| = \cdots = |V_k| \pm 1$
- all but ϵk^2 pairs are ϵ-regular

Lemma 27. *(Counting/Embedding Lemma) Let H be a graph and $\delta > 0$, then there exists an $\epsilon = \epsilon(H, \delta) > 0$ sufficiently small and $n_0 = n_0(\epsilon) \in \mathbb{N}$ sufficiently large such that:*
If $r = \chi(H)$, and G is a graph with $V(G) = A_1 \cup \cdots \cup A_r$ s.t.

- $|A_1| = \cdots = |A_r| = n \geq n_0$
- *(A_i, A_j) is ϵ-regular and δ-dense for all $1 \leq i \leq j \leq r$*

then graph G contains $\geq \frac{1}{2}\delta^{e(H)} n^{v(H)}$ copies of H.

Proof Outline using Regularity Lemma:

1. Apply the **Regularity Lemma** on graph G with ϵ sufficiently small. This gives us an equipartition of V into k sets.

2. Derive a **Reduced Graph** R with
   $V(R) = \{V_1, \cdots, V_k\}$
   $E(R) = \{(V_i, V_j) : (V_i, V_j)$ is ϵ-regular and has density $\geq \delta\}$

3. **Clean up** the graph, removing edges from less dense sections. The edges removed are as follows:

   - edges meeting V_0: $\leq \epsilon n.n$
   - non ϵ-regular pairs: $\leq \epsilon k^2.(n/k)^2 = \epsilon n^2$
   - non δ-dense pairs: $\leq \delta n^2$
   - edges within single part: $\leq k.\binom{n/k}{2} \leq k.\frac{n}{k}^2 = \frac{n^2}{k} \leq \frac{n^2}{t} = \epsilon n^2$

   Therefore, total edges removed is at most $(\delta + 3\epsilon)n^2 < 2\delta n^2$ edges
   Find the number of edges R has

4. Apply the appropriate **Standard Theorem** in Graph Theory (e.g. Turán, Hall's, etc) to R

5. Apply the **Embedding Lemma** to find the subgraph H with desired property.

Application 1: Roth's Theorem and Arithmetic Progressions

Theorem 28. *(Roth's Theorem) For every $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, $A \subset \{1, \cdots, n\}$ with $|A| \geq \delta n$, then A contains a 3-AP: $\{a, a+d, a+2d\}$*

The proof of Roth's Theorem makes use of another lemma stated as follows:

Lemma 29. *(Triangle-Removal Lemma) For every $\alpha > 0$ there exists an $\beta > 0$ such that if graph G on n vertices contains $\leq \beta n^3$ triangles, then all of them can be destroyed by removing αn^2 edges.*

Proof. Using Regularity Lemma Proof Outline:

1. Let $\alpha > 0$ be small, suppose G has $\leq \beta n^3$ triangles.
   Let $\delta = \delta(\alpha) > 0$, and $\epsilon = \epsilon(\delta) > 0$ be small.
   Apply Regularity Lemma to G with $t = \frac{1}{\epsilon}$.
   We obtain a Szemerédi Partition $V(G) = V_1 \cup \cdots \cup V_k$ where $\frac{1}{\epsilon} \leq k \leq T(\epsilon, t)$

2. Form reduced graph R with $V(R) = \{V_1, \cdots, V_k\}$ and
   $E(R) = \{(V_i, V_j) : (V_i, V_j)$ is ϵ-regular and has density $\geq \delta\}$.

   Claim: R is triangle-free.

   Proof. By contradiction following from the Embedding Lemma:
   If R has a triangle, then $\exists V_{i_1}, V_{i_2}, V_{i_3}$ of size $\frac{n}{T}$ each, with edges between them being ϵ-regular and δ-dense. By choosing $\beta = \beta(\alpha)$ sufficiently small, we ensure $\frac{n}{T}$ is sufficiently large. By Embedding Lemma, G contains $\geq \frac{1}{2}\delta^3 n^3$ triangles, which contradicts the fact that $\beta < \frac{1}{2}\delta^3$ □

3. Count the edges not in ϵ-regular and δ-dense pairs:

   - non ϵ-regular pairs: $\leq \epsilon n^2$
   - non δ-dense pairs: $\leq \delta n^2$
   - edges within single part: $\leq \epsilon n^2$

   Clean up graph: Removes $(\delta + 2\epsilon)n^2$ edges. So, setting $\alpha = \delta + 2\epsilon$ destroys all edges not in R.

4. Only edges left in G correspond to edges in R. As R is triangle-free, there are no triangles in G.

□

Proof. of Roth's Theorem: Let $\delta > 0$, r be sufficiently large. Let $A \subset \{1, \cdots, n\}$ with $|A| = \delta n$.
Claim: A contains a 3-AP.
Consider the tripartite graph G with $V(G) = X \cup Y \cup Z$, $X \equiv [n]$, $Y \equiv [2n]$ and $Z \equiv [3n]$. The edge set $E(G) = \{xy : y = x + a$ for some $a \in A\} \cup \{yz : z = y + a$ for some $a \in A\} \cup \{xz : z = x + 2a$ for some $a \in A\}$.

A triangle in G is $\{x, y, z\}$ s.t. $y = x + a$, $z = y + b$, and $z = x + 2c$, with $a, b, c \in A$. So, $a + b = 2c$ and (a, c, b) are a 3-AP.
Suppose A contains no 3-AP. This implies that G contains $\leq n|A|$ triangles (only trivial ones with $a = b = c$). $|A| = \delta n$. Therefore, G contains $\leq \delta n^2 \leq \beta n^3$ triangles.
But all the triangles of the form $\{x, x + a, x + 2a\}$ are edge-disjoint. This gives $\geq \delta n^2$ triangles.
Let $\alpha = \alpha(\delta) > 0$ be small. The triangle Removal Lemma implies that $\exists \beta = \beta(\alpha)$ s.t. if there are $\leq \beta n^3$ triangles in G then they can be all destroyed by removing $\leq \alpha n^2$ edges. We take n to be large and so $\beta n^3 >> \delta n^2$, which implies that all triangles can be destroyed by removing αn^2 edges. This is a contradiction since G has at least δn^2 edge disjoint triangles. □

Application 2: No-Corners Theorem (Ajtai-Szemerédi)

A lattice point is an element of $\mathbb{N}^2 \subseteq \mathbb{R}^2$.

Definition 30. *A **corner** is a triple of lattice points of the form (x, y), $(x, y + d)$, $(x + d, y)$.*

Theorem 31. *Given $\delta > 0$, there exists $N_0 = N_0(\delta)$ s.t. for all $N \geq N_0$, if $A \subseteq [N]^2 = \{1, \cdots, n\} \times \{1, \cdots, n\}$ s.t. A has no corners, then $|A| < \delta N^2$.*

Proof. Consider the graph G with vertex set partitions: $H \equiv [N], V \equiv [N], S_{-1} \equiv [2n - 1]$ representing lines that are horizontal, vertical, and having slope -1 respectively. An edge in G between two vertices (representing lines) corresponds to an intersection between the lines in A.
Let $A \subseteq [N]^2$ be corner free and suppose $|A| \geq \delta N^2$. A triangle in G would correspond to intersections in a horizontal (H), a vertical (V), and a slope = -1 (S_{-1}) line, i.e. either a corner in R^2 or all three lines intersecting at a point: a 'trivial' triangle.
Note that every point of A corresponds to a 'trivial' triangle in G. But these trivial triangles are edge disjoint. Then $\exists \geq \delta N^2$ edges that need to be deleted to make G triangle free. This, along with the triangle removal lemma, further implies that G has $\geq \beta N^3$ triangles. Therefore, there exists a non-trivial triangle and further, a corner in A.
Hence we have a contradiction. □

Application 3: Induced Matching Theorem

Definition 32. *A matching M in graph G is **induced** if the only edges of G connecting vertices of M are those of M, i.e. the subgraph of G induced by the vertices of M is exactly M.*

Theorem 33. *(Induced Matching) If G_n is the union of n induced matchings, then $e(G_n) = o(n^2)$.*

Let $\epsilon > 0$ be arbitrary and $n \geq 2M(\epsilon)/\epsilon^2$. If G_n is the union of k induced matchings, then $e(G_n) < 2\epsilon n^2 + k\epsilon n$ for all large enough n

Proof. Applying the Regularity Lemma with density $\delta = 2\epsilon$, let $G'' = G' - V_0$.
Claim: any induced matching in G'' has at most ϵn edges.
Let I_M be an induced matching in G'' with $U = V(I_M)$ as its vertex set and $U_i = U \cap V_i$. Define $I = \{i : |U_i| > \epsilon |V_i|\}$, and set $L = \bigcup_{i \in I} U_i$ and $S = \frac{U}{L}$. $|S| \leq \epsilon n$. Hence, if $|U| > 2\epsilon n$, then $|L| > |U|/2$, and $\exists u, v \in L$ that are vertices adjacent in I_M. Let $u \in V_i$ and $v \in V_j$. We then have an edge between V_i and V_j in the reduced graph R of G'', and hence density $> 2\epsilon$ between them.

The sets U_i and U_j, being of size larger than ϵm each, would have density $> \epsilon$ between them. This means more than $\epsilon|U_i||U_j| \geq min\{|U_i|, |U_j|\}$ edges, which is a contradiction with I_M being induced. $\square$

The theorem of van der Waerden

Scribe: Shubhansu

In this part, we prove the van der Waerden Theorem which implies the existence of arbitrarily long monochromatic arithmetic progressions in any coloring of natural numbers with a finite number of colors.

Theorem 34. *Given $k, r \in \mathbb{N}$, there exists $W = W(k,r) \in \mathbb{N}$ such that every r-coloring $\chi : [1, W] \to [r]$ admits a monochromatic A.P. of length k, i.e., there exist $a, d \in \mathbb{N}$ and $c \in [r]$ such that $\chi(a + id) = c$ for all $i \in \{0, \ldots, k-1\}$.*

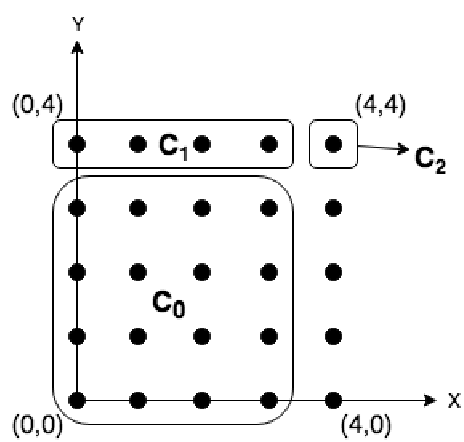
Example 35. *Note that $W(1,r) = 1$ and $W(k,1) = k$ trivially and $W(2,r) = r+1$ by pigeon-hole principle.*

Proof Strategy

For $l, m \in \mathbb{N}$, we define $m+1$ disjoint subsets of $[0,l]^m$ (called l-equivalence classes) as follows.

Definition 36. *The i^{th} equivalence class for $0 \le i \le m$ C_i consists of all those elements of $[0,l]^m$ where last i coordinates are all l and there is no l among the first $m-i$ coordinates.*

Example 37. *For $m = 2$, $l = 4$, the equivalence classes C_0, C_1 and C_2 can be represented as*



We shall prove the following:

Proposition 38. *Given $l, m, r \in \mathbb{N}$, there exists $N = N(l,m,r)$ such that for any $\chi : [1, N] \to [r]$, $\exists\ a, d_1, ..., d_m \in \mathbb{N}$ such that $a + \sum_{i=1}^{m} x_i d_i$ all have the same color on each $(x_1, ..., x_m) \in C_j$ for each $j \in \{0, 1, ..., m\}$, i.e each $m+1$ of the l-equivalence classes of $[0,l]^m$ are monochromatic (but not necessarily the same color).*

Note that in particular $N = N(k, 1, r) < \infty$ implies there exists a monochromatic A.P. of length k in any r-coloring of $[N]$. Hence, $W(k,r) \le N(k,1,r) < \infty$ which proves Theorem 34.

Proof of Proposition

We use double induction on l and m. Let $S(l,m)$ denote the statement: $N(l,m,r) < \infty \ \forall \ r$. Observe that $N(1,1,r) = 2 < \infty \ \forall \ r \in \mathbb{N}$. Hence $S(1,1)$ is true. We break the induction argument into two parts:

1. $S(l,m) \implies S(l,m+1)$

2. $S(l,m) \ \forall \ m \in \mathbb{N} \implies S(l+1,1)$

Proof of 1

We need to prove that $N(l,m+1,r) < \infty \ \forall \ r \in \mathbb{N}$.
Given an r, let $N = N(l,m,r)$ and $M = N(l,1,r^N)$ and take $N' = N \times M$. For any r-coloring $\chi : [N'] \to [r]$, divide the set $\{1,...,N'\}$ into M blocks each of size N as $\{B_1,...,B_M\}$. Consider an induced coloring on $\{B_1,...,B_M\}$ where the color of each block is the coloring sequence of its elements. Each block can be colored in one of r^N ways. Since $M = N(l,1,r^N) < \infty$, there exists a one-dimensional l-length A.P., i.e., $\exists \ a_0, d$ such that the equivalence classes $\{B_{a_0}, B_{a_0+d}, B_{a_0+2d}..., B_{a_0+(l-1)d}\}$ and $\{B_{a_0+ld}\}$ are monochromatic.

Since B_{a_0} has $N = N(l,m,r)$ elements, $\exists \ a \in B_{a_0}$, $d_1,...,d_m \in \mathbb{N}$ and $c_0,c_1,...,c_m \in [r]$ such that $\chi(a + \sum_{i=1}^m x_i d_i) = c_j$ for all $(x_1,...,x_r) \in C_j$ and $j \in \{0,..,m\}$, where $C_0, C_1,...,C_m$ are l-equivalence classes of $[0,l]^m$. Define $a' = a$, $d'_1 = d \times N$, $d'_2 = d_1$, $d'_3 = d_2,...,$ $d'_{m+1} = d_m$, $c'_0 = c_0$, $c'_1 = c_1,...,$ $c'_m = c_m$ and $c'_{m+1} = \chi(a' + \sum_{i=1}^{m+1} l d'_i)$. Now, it can be seen that $\chi(a' + \sum_{i=1}^{m+1} x'_i d'_i) = c'_j$ holds for all $(x'_1,...,x'_{m+1}) \in C'_j$ and $j \in \{0,..,m+1\}$, where $C'_0, C'_1,...,C'_m$ are l-equivalence classes of $[0,l]^{m+1}$.

This implies $N(l,m+1,r) < N' < \infty$ since $N = N(l,m,r) < \infty$ and $M = N(l,1,r^N) < \infty$ follow from $S(l,m)$ and $S(l,1)$, both of which are true by induction hypothesis. Hence, $S(l,m+1)$ is true.

Proof of 2

We need to prove $N(l+1,1,r) < \infty \ \forall \ r \in \mathbb{N}$.

Given an r, let $N = N(l,r,r)$. $N < \infty$ follows from $S(l,r)$ which is true for all $r \in \mathbb{N}$ by induction hypothesis. Consider any r-coloring of $[N]$, i.e., $\chi : [N] \to [r]$. By definition of N, $\exists \ a_0, d_1, ..., d_r \in \mathbb{N}$ and $c_0,...,c_r \in [r]$ such that $\chi(a_0 + \sum_{i=1}^r x_i d_i) = c_j$ holds $\forall \ (x_1,...,x_r) \in C_j$ and $j \in \{0,...,r\}$, i.e., each of the $r+1$ l-equivalence classes $C_0,...,C_r$ are monochromatic. By pigeon-hole principle, $\exists \ 0 \le p < q \le r$ such that the classes C_p and C_q are colored the same (say $c \in [r]$), i.e., $\chi(a+\sum_{i=1}^r x_i d_i) = c$ $\forall \ (x_1,...,x_r) \in C_p \cup C_q$. Then, $\chi(a_0 + z\sum_{k=r-j+1}^{r-i} d_k + l\sum_{k=r-i+1}^{r} d_k) = c \ \forall \ z \in \{0,1,...,l\}$, i.e., we get a monochromatic A.P. of length $l+1$.

Thus, in any r-coloring of $[N]$, we get a monochromatic A.P. of length $l+1$. Let this be $\{a+kd\}, k \in \{0,1,...,l\}$. Then $d < N/l$ and hence $a + (l+1)d < N + N/l$. Hence, we get monochromatic $(l+1)$-equivalence classes, C_0 and C_1 of $[0,l+1]$ in any r-coloring of $[\lfloor N(1+\frac{1}{l})\rfloor]$ i.e. $N(l+1,1,r) \le \lfloor N(1+\frac{1}{l})\rfloor < \infty$ as $N < \infty$ by induction hypothesis. Hence, $S(l+1,1)$ is true.

The proof of Proposition 38 is now complete by double induction.

Observation 39. *(Hilbert Cube Lemma): Substituting $l = 2$ and considering only the equivalence class C_0 in Proposition 38, we get the following result due to Hilbert -*
Given $n, r \in \mathbb{N}$, there exists $H = H(n, r)$ such that for any r-coloring $\chi : [1, H] \to [r]$, there exist $a, d_1, \ldots d_n \in \mathbb{N}$ and $c \in [r]$ such that $\chi(a + \sum_{i=1}^{n} \epsilon_i d_i) = c$ for all $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$.

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The Hales-Jewett Theorem

Scribe: Yash Karnik

We present a proof of the Hales-Jewett theorem due to Saharon Shelah (1988). This proof gives better bounds than the original proof by Hales -Jewett, whose bounds were of the Ackermann type

Definition

A hypercube of size t in n dimensions, $[t]^n$ is defined as:

$$[t]^n = \{(x_1, x_2, ...x_n) : x_i \in \{1, 2, ..t\}\}$$

Definition

A combinatorial line in $[t]^n$ is defined as:

$$L = (\mathbf{x_1}, \mathbf{x_2}, ...\mathbf{x_t})$$

where each $\mathbf{x_i} \in [t]^n$ and satisfies

$$\mathbf{x_i} = (x_{i1}, x_{i2}, ...x_{in})$$

where, for at least one $1 \leqslant j \leqslant n$

$$x_{ij} = i \; \forall \; 1 \leqslant i \leqslant t$$

and for the rest $j's$

$$x_{1j} = x_{2j} = ... = x_{tj}$$

Statement of the theorem

Given $r, t \in \mathbb{N}$, there exists $H = HJ(r, t)$ such that if $n \geqslant H$, the r-coloring

$$\chi : [t]^n \to [r]$$

admits a monochromatic combinatorial line.

Idea of Proof

We use induction on t.
$HJ(r, 1) = 1$ trivially
Let $s = HJ(r, t-1)$, that is, $\chi : [t-1]^s \to [r]$ ensures a monochromatic combinatorial line.
To make the induction work, we impose an extra criterion.

Definition

A coloring $\chi : [t]^n \to [r]$ is called fliptop if, whenever $\mathbf{x}, \mathbf{y}$ differ in exactly one coordinate, and that coordinate has values $t-1$ and t, $\chi(\mathbf{x}) = \chi(\mathbf{y})$
So if coloring on $[t]^n$ is fliptop, then any monochromatic line on $[t-1]^n$ of length $t-1$ extends to length t.

Definition

By a Shelah line in $[t]^n$ we mean a combinatorial line whose first i coordinates are constant at $t-1$, the next $j-i$ coordinates are moving coordinates and the final $n-j$ coordinates are constant at t, with $0 \leqslant i < j \leqslant n$
A point on any Shelah line is caled a Shelah point.

We can see that the number of Shelah lines is $\binom{n+1}{2}$ because i and j, which are distinct, with $i < j$ can be chosen independently from $\{0, 1, 2, ...n\}$

Let $N(n)$ denote the number of Shelah points in a hypercube $[t]^n$ of dimension n.
Clearly, N(n) $\leqslant \binom{n+1}{2}t$

Definition

A Shelah s-space is defined by

$$L_1 \times L_2 \times ...L_s$$

where L_i is a Shelah line.
Through the moving coordinates of each of the s Shelah lines, which move from 1 to t, the Shelah s-space admits a canonical isomorphism with $[t]^s$.

Question 1

How large an n do we need such that $\chi : [t]^n \to [r]$ admits a fliptop Shelah line, that is a Shelah line with last two points colored the same?

Answer

$n = r$ works. This can be seen by considering the $r + 1$ points:

$$(t-1, t-1, ...t-1), (t-1, t-1, ...t-1, t), (t-1, t-1, ...t-1, t, t)...(t, t, ...t)$$

each point having r coordinates with the ith point having the last $i-1$ coordinates equal to t and rest equal to $t-1$. By pigeonhole principle, we have at least 2 of the $r+1$ points colored the same, as there are r colors. Thus, a fliptop Shelah line corresponding to those two coordinates is ensured.

Question 2

How large an n do we need such that there exists a fliptop Shelah 2-space in the coloring $\chi : [t]^n \to [r]$, that is, the canonical isomorphism of the Shelah 2-space has a fliptop coloring?

Answer

We will try to get $L_1 \times L_2$ as $L_1 \subset [t]^{n_1}$, $L_2 \subset [t]^{n_2}$, with $n_1 + n_2 = n$
The Shelah 2-space looks like $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x}$ is a Shelah point of dimension n_1 and $\mathbf{y}$ is a Shelah point of dimension n_2.

We will try to get a deriverd coloring $\hat{\chi}$ on $[t]^{n_2}$

$$\hat{\chi} : [t]^{n_2} \to [r]^{N(n_1)}$$

that is,

$$\mathbf{y} \longmapsto \Big(\chi(\mathbf{x_1}, \mathbf{y}), \chi(\mathbf{x_2}, \mathbf{y}), ...\chi(\mathbf{x_{N(n_1)}}, \mathbf{y})\Big)$$

By answer to question 1, if $n_2 \geqslant r^{N(n_1)}$ then $\hat{\chi}$ admits a fliptop Shelah line L_2
As shown before, $N(n_1) \leqslant \binom{n_1+1}{2}\, t$. So, $n_2 \geqslant r^{\binom{n_1+1}{2}t}$ suffices.
For any Shelah point $\mathbf{x}$, in $[t]^{n_1}$ and if $\mathbf{y_1}, \mathbf{y_2}$, are last two points of L_2, $\chi(\mathbf{x}, \mathbf{y_1}) = \chi(\mathbf{x}, \mathbf{y_2})$.

Consider another derived coloring

$$\hat{\hat{\chi}} : [t]^{n_1} \to [r]^t$$

$$\mathbf{x} \longmapsto \Big(\chi(\mathbf{x}, \mathbf{y_1}), \chi(\mathbf{x}, \mathbf{y_2}), ...\chi(\mathbf{x}, \mathbf{y_t})\Big)$$

If $n_1 \geqslant r^t$ from before, this coloring contains a fliptop Shelah line L_1.
Thus $L_1 \times L_2$ is the required fliptop Shelah 2-space.

Extending the Argument to Shelah 3-space

We will try to find three Shelah lines L_1, L_2, L_3 of dimensions n_1, n_2, n_3 respectively.

Define a derived coloring

$$\hat{\chi} : [t]^{n_3} \to [r]^{N(n_1)\times N(n_2)}$$

mapping all Shelah points $\mathbf{z}$ in $[t]^{n_3}$ to a vector of colors $[\chi(\mathbf{x}, \mathbf{y}, \mathbf{z})]$ of length $N(n_1)N(n_2)$, where $\mathbf{x}$ and $\mathbf{y}$ are Shelah points in $[t]^{n_1}$ and $[t]^{n_2}$ respectively.
So if $n_3 \geqslant r^{\binom{n_1+1}{2}\binom{n_2+1}{2}t^2}$ there exists a fliptop Shelah line L_3.

Similarly define another derived coloring

$$\hat{\hat{\chi}} : [t]^{n_2} \to [r]^{N(n_1)t}$$

mapping all Shelah points $\mathbf{y}$ in $[t]^{n_2}$ to a vector of colors $[\chi(\mathbf{x}, \mathbf{y}, \mathbf{z_i})]$ of length $N(n_1)t$, where $\mathbf{x}$ is a Shelah point in $[t]^{n_1}$ and $\mathbf{z_i}$ is a point on the Shelah line L_3.
If $n_2 \geqslant \binom{n_1+1}{2}t^2$, we can get a fliptop Shelah line L_2.

Finally define

$$\hat{\hat{\hat{\chi}}} : [t]^{n_1} \to [r]^{t^2}$$

mapping all Shelah points $\mathbf{x}$ in $[t]^{n_1}$ to a vector of colors $[\chi(\mathbf{x}, \mathbf{y_j}, \mathbf{z_i})]$ of length t^2, where $\mathbf{y_j}$ and $\mathbf{z_i}$ are points on the Shelah lines L_2 and L_3 respectively. If $n_1 \geqslant r^{t^2}$, we can get a fliptop Shelah line L_1.
Thus $L_1 \times L_2 \times L_3$ is the required fliptop Shelah 3-space.

Extending the algorithm to the general case

$$n_1 = r^{t^{s-1}}$$
$$n_{i+1} = r^{A_i} \text{ for } 1 \leqslant i \leqslant s-1$$
$$A_i = \left[\prod_{j \leqslant i} \binom{n_j + 1}{2}\right] t^{s-1}$$

$n = n_1 + n_2 + ...n_s$ guarantees a fliptop Shelah s-space.

Proof of the Theorem

We fix r and induct on t.
The base case where $t = 1$ is trivial with $HJ(r, 1) = 1$ because the combinatorial line contains only 1 element.
Assume $s = HJ(r, t-1)$ exists and we will prove that $HJ(r, t)$ exists too.
Consider n given by the recursive formula in the general case above which ensures the existence of a fliptop Shelah s-space. We know that the Shelah s-space has a canonical isomorphism with $[t]^s$. Further, consider $[t-1]^s$, a subset of the hypercube $[t]^s$, which is isomorphic with the Shelah s-space. Any r-coloring of $[t-1]^s$, admits a monochromatic combinatorial line of length $t-1$, by the definition of s.
But, because the Shelah s-space is fliptop, the monochromatic combinatorial line in $[t-1]^s$ of length $t-1$ extends to length t. Therefore, $HJ(r, t) \leqslant n$.