# Almost full rank matrices arising from transitive tournaments

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**ABSTRACT**
Let $\mathbb{F}$ be a field and suppose $\mathbf{a} \coloneqq (a_1, a_2, \dots)$ is a sequence of non-zero elements in $\mathbb{F}$. For a tournament $T$ on $[n]$, associate the $n \times n$ symmetric matrix $M_T(\mathbf{a})$ (resp. skew-symmetric matrix $M_{T,\text{skew}}(\mathbf{a})$) with zero diagonal as follows: for $i < j$, if the edge $ij$ is directed as $i \to j$ in $T$, then set $M_T(\mathbf{a}) = a_i$ (resp. $M_{T,\text{skew}}(\mathbf{a}) = a_i$), else set $M_T(\mathbf{a}) = a_j$ (resp. $M_{T,\text{skew}}(\mathbf{a}) = a_j$). Let $\mathcal{M}_n(\mathbf{a})$ (resp. $\mathcal{M}_{n,\text{skew}}(\mathbf{a})$) be the family consisting of all the $n \times n$ symmetric matrices $M_T(\mathbf{a})$ (resp. skew-symmetric matrices $M_{T,\text{skew}}(\mathbf{a})$) as $T$ varies over all tournaments on $[n]$. We show that any matrix in $\mathcal{M}_n(\mathbf{a})$ or $\mathcal{M}_{n,\text{skew}}(\mathbf{a})$ corresponding to a transitive tournament has rank at least $n-1$, and this is best possible. This settles in a strong form a conjecture posed in [Balachandran N, Bhattacharya S, Sankarnarayanan B. An ensemble of high-rank matrices arising from tournaments. 2021. 9 p. Located at: https://arxiv.org/abs/2108.10871v1]. As a corollary, any matrix in these families has rank at least $\lfloor \log_2(n) \rfloor$.

**KEYWORDS**
Minimum rank; symmetric matrix; skew-skymmetric matrix; determinant; transitive tournament

**AMS CLASSIFICATION**
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## 1. Introduction

By $[n]$ we shall mean the set $\{1, \dots, n\}$. Let $\mathbb{F}$ be a field and suppose $\mathbf{a} \coloneqq (a_1, a_2, \dots)$ is a sequence of non-zero elements in $\mathbb{F}$. Write $\mathbf{a}_n \coloneqq (a_1, \dots, a_n)$. Let $\mathcal{M}_n(\mathbf{a})$ consist of the family of all symmetric $n \times n$ matrices over $\mathbb{F}$ with all diagonal entries being zero and such that for $1 \le i < j \le n$ the $(i, j)$th entry is either $a_i$ or $a_j$. Note that to each $M \in \mathcal{M}_n(\mathbf{a})$ there corresponds a tournament on the vertex set $[n]$ in the following natural manner: for $i < j$ we direct the edge $ij$ as $i \to j$ if $M(i, j) = a_i$, and the edge is directed in the reverse direction if $M(i, j) = a_j$. Conversely, for a tournament $T$ on $[n]$, we can associate the matrix $M_T(\mathbf{a}) \in \mathcal{M}_n(\mathbf{a})$ in exactly the same way, namely, for $i < j$, set $M_T(i, j) = a_i$ if $i \to j$, and $M_T(i, j) = a_j$ otherwise. Note that this correspondence is not necessarily one-to-one, since the $a_i$ need not be distinct.

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In [1], the first author together with Mathew and Mishra raised the following problem for $\mathbb{F} = \mathbb{R}$:

**Problem 1.1.** *Is there a constant $c > 0$ such that* $\operatorname{rank}(M) \geq cn$ *for all* $M \in \mathcal{M}_n(\mathbf{a})$*?*

The ranks of general matrices associated to graphs has emerged as a powerful and useful tool in several combinatorial problems in recent times, for instance see [2–5]. The above question, in particular, arises from a problem in extremal combinatorics concerning so-called *bisection-closed families*: a family $\mathcal{F}$ of subsets of $[n]$ is bisection-closed if, for any two distinct members $A, B \in \mathcal{F}$, we have either $|A \cap B| = \frac{|A|}{2}$ or $|A \cap B| = \frac{|B|}{2}$. One is then interested in finding the maximum size of a bisection-closed family. In [1], the authors showed that any bisection-closed family in $[n]$ has size at most $O(n \log_2 n)$. Furthermore, the authors presented two distinct constructions of bisection-closed families in $[n]$ of size $\frac{3n}{2} - 2$.

A positive answer to Problem 1.1 will improve this upper bound to $O(n)$, which would be asymptotically tight, and for the sake of completeness, we include the straightforward argument below: Given a bisection-closed family $\mathcal{F}$ of $[n]$ of size $m$, define the $m \times n$ matrix $X$ by $X(A, x) := 1$ if $x \in A$, and $X(A, x) := -1$ otherwise. For two sets $A, B \in \mathcal{F}$, define $\operatorname{Tor}(A, B) := A$ if $|A \cap B| = \frac{1}{2}|B|$, and $\operatorname{Tor}(A, B) := B$ otherwise. Then, the matrix $XX^T$ whose rows and columns are indexed by the members of $\mathcal{F}$ is described by

$$\begin{aligned} XX^T(A, A) &= n, \\ XX^T(A, B) &= n - 2(|A| + |B|) + 4|A \cap B| \\ &= n - 2|\operatorname{Tor}(A, B)|. \end{aligned}$$

Thus, if $\mathcal{F} = \{A_1, \ldots, A_m\}$, and if $J$ denotes the $m \times m$ all-ones matrix, then $\frac{1}{2}(nJ - XX^T) \in \mathcal{M}_n(\mathbf{a})$ for the sequence $\mathbf{a}_n = (|A_1|, \ldots, |A_m|)$. Hence, if the conjecture holds, then $\operatorname{rank}(XX^T) \geq cm$. But, since $\operatorname{rank}(XX^T) \leq \operatorname{rank}(X) \leq n$, we have $m \leq (n+1)/c$. In particular, by the above argument, the existence of bisection-closed families of size $\frac{3n}{2} - 2$ implies that a constant $c$ as in Problem 1.1 cannot be greater than $2/3$, if it exists.

The notion of bisection-closed families in $[n]$ (more generally, of *fractional* $L$-*intersecting families*) has been generalized in other directions. For instance, in [6] the authors consider a fractional variant of $l$-cross-intersecting pairs of families in $[n]$. They characterize the maximal $\frac{c}{d}$-cross-intersecting pairs, and in particular the maximal cross-bisecting pairs (i.e., when $\frac{c}{d} = \frac{1}{2}$). In [7], the authors consider fractional $L$-intersecting families of *subspaces* of an $n$-dimensional vector space over a finite field, instead of *subsets* of $[n]$. In particular, they show that the maximum size of a bisection-closed family of subspaces is at most $O([n]_q \log_2 n)$, where $[n]_q$ is the $q$-analog of the integer $n$. Furthermore, they exhibit examples of bisection-closed families of size at least $\Omega([n]_q)$, so the logarithmic factor that separates the upper and lower bounds persists even in the $q$-analog of the set version of bisection-closed families. One is strongly led to believe that removing the logarithmic factor in the set case, i.e. resolving Problem 1.1, can lead to corresponding improvements in the bounds in the $q$-analog case, too.

We have made some partial progress in the direction of Problem 1.1 in a previous work (see [8]). Here, by the phrase "with high probability" (*whp*) we mean that the probability that the said event occurs asymptotically tends to 1 as $n \to \infty$.

**Theorem 1.2.** *Let* $\mathrm{char}(\mathbb{F}) \neq 2$ *and* $\mathbf{a}$ *be a sequence of non-zero elements in* $\mathbb{F}$. *Suppose that* $T$ *is a uniformly random tournament on* $[n]$, *that is, one whose edges are directed in either direction with probability* $1/2$ *each and independently. Then* whp $\mathrm{rank}(M_T(\mathbf{a})) \geq \frac{n}{2} - 21\sqrt{n\log(n)}$.

*Furthermore, over a field of arbitrary characteristic, if* $T$ *is a transitive tournament on* $[n]$, *then* $\mathrm{rank}(M_T(\mathbf{a})) \geq \lfloor \frac{2n}{3} \rfloor - 1$.

We further conjectured that for transitive tournaments the bound should be much better:

**Conjecture 1.3.** *If* $T$ *is a transitive tournament on* $[n]$, *then* $\mathrm{rank}(M_T(\mathbf{a})) \geq n - o(n)$.

In this short paper, we show the truth of the above conjecture by proving the following result. The transitive tournament on $[n]$ with edges oriented as $1 \to 2 \to \cdots \to n$ is said to be in the *natural orientation*, and is denoted $T_n$.

**Theorem 1.4.** *If* $T_n$ *is the transitive tournament on* $[n]$ *in the natural orientation, then* $\mathrm{rank}(M_{T_n}(\mathbf{a})) \geq n - 1$. *Moreover, for all* $n \geq 1$, $M_{T_n}(\mathbf{a})$ *and* $M_{T_{n+1}}(\mathbf{a})$ *cannot both be singular.*

Since any transitive tournament $T$ is isomorphic to $T_n$, the matrix $M_T(\mathbf{a})$ is similar to $M_{T_n}(\mathbf{a})$, and thus has the same rank. Hence, Theorem 1.4 resolves Conjecture 1.3. Also note that Theorem 1.4 holds over arbitrary fields.

The natural correspondence between tournaments on $[n]$ and the family $\mathcal{M}_n(\mathbf{a})$ motivates the following analogous problem. Define $\mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$ to be the family of all skew-symmetric $n \times n$ matrices over $\mathbb{F}$ such that the diagonal entries are all zero, and for $1 \leq i < j \leq n$ the $(i, j)$th entry is either $a_i$ or $a_j$. Given a tournament $T$ on $[n]$ and a sequence $\mathbf{a}$ of non-zero elements in $\mathbb{F}$, we denote by $M_{T,\mathrm{skew}}(\mathbf{a})$ the corresponding skew-symmetric matrix in $\mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$, and vice-versa. One may view the skew-symmetry as a consequence of incorporating the direction of the edges of the tournament $T$ into the data that defines the matrix $M_{T,\mathrm{skew}}(\mathbf{a})$, *à la* (generalized) skew-adjacency matrices of tournaments. For instance, questions about the rank ([9]) and spectra ([10]) of the skew-adjacency matrices of directed graphs, the existence of skew-Hadamard matrices in constructing orthogonal designs ([11]), and unimodular tournaments ([12]) are widely studied in the literature. It is also worth mentioning a remarkable result in [13] which generalizes the classical matrix-tree theorem of Kirchhoff and Tutte to 3-graphs by using a specially concocted skew-symmetric matrix. All in all, the analogous problem in our case over skew-symmetric matrices merits examination, independent of the original combinatorial motivation. We now state the problem for the sake of completeness, over an arbitrary field $\mathbb{F}$:

**Problem 1.5.** *Is there a constant* $c > 0$ *such that* $\mathrm{rank}(M) \geq cn$ *for all* $M \in \mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$?

While it is not immediately clear how to prove an analogue of the random result in Theorem 1.2 for the family $\mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$ (it is not clear if such a result even holds), we are able to show in this paper that for transitive tournaments, it is true that the corresponding skew-symmetric matrices have high rank.

**Theorem 1.6.** *If* $T_n$ *is the transitive tournament on* $[n]$ *in the natural orientation, then* $\mathrm{rank}(M_{T_n,\mathrm{skew}}(\mathbf{a})) \geq n-1$. *Moreover, for all* $n \geq 1$, $M_{T_n,\mathrm{skew}}(\mathbf{a})$ *and* $M_{T_{n+1},\mathrm{skew}}(\mathbf{a})$ *cannot both be singular.*

As a corollary, we then have the following result:

**Corollary 1.7.** *If* $M \in \mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$, *then* $\mathrm{rank}(M) \geq \lfloor \log_2(n) \rfloor$.

Note that Theorem 1.6 and Corollary 1.7 hold over arbitrary fields, whereas the random result in Theorem 1.2 requires $\mathrm{char}(\mathbb{F}) \neq 2$.

## 2. Proof of main results

**Lemma 2.1.** *Let* $\mathbf{a}$ *be a sequence of non-zero elements in the field* $\mathbb{F}$. *For each* $n \geq 1$, *let* $T_n$ *be the transitive tournament on* $[n]$ *with the natural orientation.*

*(1) For all* $n \geq 2$, $\det(M_{T_n}(\mathbf{a}))$ *satisfies the recurrence relation*

$$\det(M_{T_n}(\mathbf{a})) = -a_{n-1}^2 \det(M_{T_{n-2}}(\mathbf{a})) - 2a_{n-1} \det(M_{T_{n-1}}(\mathbf{a})), \tag{1}$$

*where* $M_{T_1}(\mathbf{a})$ *is the* $1 \times 1$ *zero matrix, and* $M_{T_0}(\mathbf{a})$ *is taken to be the empty matrix, so that* $\det(M_{T_1}(\mathbf{a})) = 0$ *and* $\det(M_{T_0}(\mathbf{a})) = 1$.

*(2) For all* $n \geq 2$, $\det(M_{T_n,\mathrm{skew}}(\mathbf{a}))$ *satisfies the recurrence relation*

$$\det(M_{T_n,\mathrm{skew}}(\mathbf{a})) = a_{n-1}^2 \det(M_{T_{n-2},\mathrm{skew}}(\mathbf{a})), \tag{2}$$

*where* $M_{T_1,\mathrm{skew}}(\mathbf{a})$ *is the* $1 \times 1$ *zero matrix, and* $M_{T_0,\mathrm{skew}}(\mathbf{a})$ *is taken to be the empty matrix, so that* $\det(M_{T_1,\mathrm{skew}}(\mathbf{a})) = 0$ *and* $\det(M_{T_0,\mathrm{skew}}(\mathbf{a})) = 1$.

***Proof.*** We shall prove part 1 below; the proof of part 2 will proceed along similar lines.

Since $\det(M_{T_2}(\mathbf{a})) = -a_1^2 = -a_1^2 \det(M_{T_0}(\mathbf{a})) - 2a_1 \det(M_{T_1}(\mathbf{a}))$, recurrence (1) is verified for $n = 2$. Fix $n \geq 3$, and view $a_{n-1}$ as a variable. Then, $\det(M_{T_n}(\mathbf{a}))$ is a formal polynomial in $a_{n-1}$ of degree at most 2, since $a_{n-1}$ occurs only in the $(n-1, n)$ and $(n, n-1)$ positions in $M_{T_n}(\mathbf{a})$. Observe that for a field $\mathbb{F}$ of any characteristic, the constant term of a polynomial over $\mathbb{F}$ can be found by applying the evaluation map that sends the indeterminate to $0 \in \mathbb{F}$; similarly, the coefficient of the linear term of a polynomial over $\mathbb{F}$ can be found by first computing its formal derivative and then applying the same evaluation map.

In our case, the constant term of the polynomial $\det(M_{T_n}(\mathbf{a}))$ must be zero, since by plugging in $a_{n-1} = 0$ the last two columns of $M_{T_n}(\mathbf{a})$ become identical. Write $\det(M_{T_n}(\mathbf{a})) = \alpha a_{n-1}^2 + \beta a_{n-1}$, and expand the determinant as

$$\det(M_{T_n}(\mathbf{a})) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} [M_{T_n}(\mathbf{a})]_{i,\sigma(i)}, \tag{3}$$

where $\mathfrak{S}_n$ is the set of permutations of $[n]$. The terms containing $a_{n-1}^2$ in the RHS are only obtained from those $\sigma \in \mathfrak{S}_n$ such that $\sigma(n-1) = n$, $\sigma(n) = n-1$. Viewing $\mathfrak{S}_{n-2}$ as the set of those permuations in $\mathfrak{S}_n$ that fix $n-1$ and $n$, and denoting the transposition that swaps $n-1$ and $n$ by $(n-1,n)$, we see that

$$\alpha = \sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \sigma(n)=n-1,\\ \sigma(n-1)=n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n-2} [M_{T_n}(\mathbf{a})]_{i,\sigma(i)} = \sum_{\tau\in\mathfrak{S}_{n-2}} \operatorname{sgn}(\tau\circ(n-1,n)) \prod_{i=1}^{n-2} [M_{T_n}(\mathbf{a})]_{i,\tau(i)} = -\det(M_{T_{n-2}}(\mathbf{a})).$$

The coefficient of $a_{n-1}$ can be found by computing $\left.\frac{d(\det M_{T_n}(\mathbf{a}))}{da_{n-1}}\right|_{a_{n-1}=0}$, where $\frac{d}{da_{n-1}}$ is the formal derivative operator, applied on the polynomial $\det(M_{T_n}(\mathbf{a}))$. Writing $M_{T_n}(\mathbf{a})$ as

$$M_{T_n}(\mathbf{a}) = \begin{pmatrix} M_{T_{n-2}}(\mathbf{a}) & \mathbf{a}_{n-2}^T & \mathbf{a}_{n-2}^T \\ \mathbf{a}_{n-2} & 0 & a_{n-1} \\ \mathbf{a}_{n-2} & a_{n-1} & 0 \end{pmatrix},$$

we get

$$\frac{d(\det M_{T_n}(\mathbf{a}))}{da_{n-1}} = \begin{vmatrix} M_{T_{n-2}}(\mathbf{a}) & \mathbf{a}_{n-2}^T & \mathbf{a}_{n-2}^T \\ \mathbf{0}_{n-2} & 0 & 1 \\ \mathbf{a}_{n-2} & a_{n-1} & 0 \end{vmatrix} + \begin{vmatrix} M_{T_{n-2}}(\mathbf{a}) & \mathbf{a}_{n-2}^T & \mathbf{a}_{n-2}^T \\ \mathbf{a}_{n-2} & 0 & a_{n-1} \\ \mathbf{0}_{n-2} & 1 & 0 \end{vmatrix},$$

where $\mathbf{0}_{n-2} = (0,\dots,0) \in \mathbb{F}^{n-2}$. Substitute $a_{n-1} = 0$ and expand the first determinant along the $(n-1)$th row and the second determinant along the $n$th row to get that $\beta = -2\det(M_{T_{n-1}}(\mathbf{a}))$. This proves the recurrence (1) for all $n \geq 3$.

The proof of part 2 proceeds analogously. Since $M_{T_n,\text{skew}}(\mathbf{a})$ is skew-symmetric, the expression that was found for $\alpha$ in the proof of part 1 is precisely the coefficient of $-a_{n-1}^2$, rather than of $a_{n-1}^2$. After taking the formal derivative of $\det(M_{T_n,\text{skew}}(\mathbf{a}))$ with respect to $a_{n-1}$ and then setting $a_{n-1} = 0$, the two determinants on the RHS cancel each other: expanding along the same rows chosen in the proof of part 1 easily shows that the two expressions only differ by a sign. This suffices to prove the recurrence (2). □

**Theorem 2.2.** *Let $\mathbb{F}$ be a field and $\mathbf{a}$ be a sequence of non-zero elements in $\mathbb{F}$. Let $T_n$ be the transitive tournament on $[n]$ in the natural orientation. Then, for all $n \geq 1$,*

*(1)* $\operatorname{rank}(M_{T_n}(\mathbf{a})), \operatorname{rank}(M_{T_n,\text{skew}}(\mathbf{a})) \geq n-1$*;*
*(2)* $M_{T_n}(\mathbf{a})$ *(resp.* $M_{T_n,\text{skew}}(\mathbf{a})$*) and* $M_{T_{n+1}}(\mathbf{a})$ *(resp.* $M_{T_{n+1},\text{skew}}(\mathbf{a})$*) cannot both be singular.*

***Proof.*** We shall prove the above theorem for the symmetric matrices $M_{T_n}(\mathbf{a})$. The proof is by induction. Note that $\det(M_{T_1}(\mathbf{a})) = 0$ and $\det(M_{T_2}(\mathbf{a})) = -a_1^2 \neq 0$. Hence, $\operatorname{rank}(M_{T_1}(\mathbf{a})) \geq 0$ and $\operatorname{rank}(M_{T_2}(\mathbf{a})) \geq 1$; moreover, $M_{T_1}(\mathbf{a})$ is singular and $M_{T_2}(\mathbf{a})$ is non-singular. This verifies the base case.

For the induction hypothesis, suppose that $\operatorname{rank}(M_{T_n}(\mathbf{a})) \geq n-1$, and moreover that if $M_{T_{n-1}}(\mathbf{a})$ is singular then $M_{T_n}(\mathbf{a})$ is non-singular, for some $n \geq 2$. Consider $M_{T_{n+1}}(\mathbf{a})$. Since $M_{T_n}(\mathbf{a})$ is a submatrix of $M_{T_{n+1}}(\mathbf{a})$, if $M_{T_n}(\mathbf{a})$ is non-singular, then $\operatorname{rank}(M_{T_{n+1}}(\mathbf{a})) \geq n$, as required. So, suppose that $M_{T_n}(\mathbf{a})$ is singular. By the recurrence (1),

$$\det(M_{T_{n+1}}(\mathbf{a})) = -a_n^2 \det(M_{T_{n-1}}(\mathbf{a})) - 2a_n \det(M_{T_n}(\mathbf{a})) = -a_n^2 \det(M_{T_{n-1}}(\mathbf{a})).$$

Now, by the induction hypothesis, $\det(M_{T_{n-1}}(\mathbf{a})) \neq 0$, so $\det(M_{T_{n+1}}(\mathbf{a})) \neq 0$. Hence, $\operatorname{rank}(M_{T_{n+1}}(\mathbf{a})) = n+1 \geq n$, and $M_{T_{n+1}}(\mathbf{a})$ is non-singular, as required.

This completes the proof for the symmetric matrices $M_{T_n}(\mathbf{a})$. The proof for the skew-symmetric matrices $M_{T_n,\text{skew}}(\mathbf{a})$ follows analogously by applying the recurrence (2) in place of the recurrence (1). □

We conclude this section with a few remarks concerning the case $\mathbb{F} = \mathbb{C}$. Here, one can define the families $\mathcal{M}_n^{\mathbb{C}}(\mathbf{a})$ and $\mathcal{M}_{n,\text{skew}}^{\mathbb{C}}(\mathbf{a})$ of Hermitian and skew-Hermitian matrices, respectively, corresponding to tournaments on $[n]$ in the analogous fashion. This naturally raises questions analogous to Problems 1.1 and 1.5, respectively. We note that one can suitably modify the proof of Lemma 2.1 to a more combinatorial flavour to show the following:

**Theorem 2.3.** *Let* $\mathbf{a}$ *be a sequence of non-zero elements in the field* $\mathbb{F}$*. For each* $n \geq 1$*, let* $T_n$ *be the transitive tournament on* $[n]$ *with the natural orientation. For all* $n \geq 2$*, we have the following recurrences:*

$$\det(M_{T_n}^{\mathbb{C}}(\mathbf{a})) = -|a_{n-1}|^2 \det(M_{T_{n-2}}^{\mathbb{C}}(\mathbf{a})) - 2\Re(a_{n-1} \det(M_{T_{n-1}}^{\mathbb{C}}(\mathbf{a}))),$$
$$\det(M_{T_n,\text{skew}}^{\mathbb{C}}(\mathbf{a})) = |a_{n-1}|^2 \det(M_{T_{n-2},\text{skew}}^{\mathbb{C}}(\mathbf{a})) + 2i\Im(a_{n-1} \det(M_{T_{n-1},\text{skew}}^{\mathbb{C}}(\mathbf{a}))),$$

*where* $M_{T_n}^{\mathbb{C}}(\mathbf{a}) \in \mathcal{M}_n^{\mathbb{C}}(\mathbf{a})$ *and* $M_{T_n,\text{skew}}^{\mathbb{C}}(\mathbf{a}) \in \mathcal{M}_{n,\text{skew}}^{\mathbb{C}}(\mathbf{a})$ *for all* $n$*.*

As before, $M_{T_1}^{\mathbb{C}}(\mathbf{a})$ (and $M_{T_1,\text{skew}}^{\mathbb{C}}(\mathbf{a})$) is the $1 \times 1$ zero matrix, and $M_{T_0}^{\mathbb{C}}(\mathbf{a})$ (and $M_{T_0,\text{skew}}^{\mathbb{C}}(\mathbf{a})$) is taken to be the empty matrix. Hence, $\det(M_{T_1}^{\mathbb{C}}(\mathbf{a})) = 0 = \det(M_{T_1,\text{skew}}^{\mathbb{C}}(\mathbf{a}))$ and $\det(M_{T_0}^{\mathbb{C}}(\mathbf{a})) = 1 = \det(M_{T_0,\text{skew}}^{\mathbb{C}}(\mathbf{a}))$.

***Proof.*** Since our method of proof for Lemma 2.1 requires us to view $\det(M_{T_n}(\mathbf{a}))$ as a polynomial with respect to $a_{n-1}$, the same line of reasoning cannot be applied *mutatis mutandis* to Theorem 2.3 since a priori $\det(M_{T_n}^{\mathbb{C}}(\mathbf{a}))$ has expressions involving both $a_{n-1}$ and $\overline{a_{n-1}}$. Instead, we offer a suitably modified combinatorial proof, which will also apply equally well to Lemma 2.1.

In the formula (3) for the expansion of the determinant applied to $M_{T_n}^{\mathbb{C}}(\mathbf{a})$, we split the sum in the RHS over the sets $A_1$, $A_2$, $A_3$, and $A_4$, where

$$A_1 := \{\sigma \in \mathfrak{S}_n : \sigma(n-1) = n, \sigma(n) = n-1\},$$
$$A_2 := \{\sigma \in \mathfrak{S}_n : \sigma(n-1) = n, \sigma(n) \neq n-1\},$$
$$A_3 := \{\sigma \in \mathfrak{S}_n : \sigma(n-1) \neq n, \sigma(n) = n-1\},$$
$$A_4 := \{\sigma \in \mathfrak{S}_n : \sigma(n-1) \neq n, \sigma(n) \neq n-1\}.$$

Note that $\mathfrak{S}_n$ is the disjoint union of the $A_i$'s. Furthermore, we can define sets $B_i \subset A_i$ consisting of those permutations that are also fixed-point free. Since all the diagonal entries of $M_{T_n}^{\mathbb{C}}(\mathbf{a})$ are zero, the sum in the RHS over the set $A_i$ equals the sum over the set $B_i$ for each $1 \leq i \leq 4$.

Now, the sum over $A_1$ is easily seen to equal $-\det(M_{T_{n-2}}^{\mathbb{C}}(\mathbf{a})) \cdot a_{n-1} \cdot \overline{a_{n-1}}$, by following the same line of argument as in the proof of Lemma 2.1. The sum over $B_4$ (and hence over $A_4$) equals zero for the following reason: the map $\sigma \mapsto \sigma \circ (n-1, n)$, where $(n-1, n)$ denotes the transposition that swaps $n-1$ and $n$, is a well-defined sign-reversing involution on $B_4$ that is invariant on the expression $\prod_{i=1}^{n} [M_{T_n}^{\mathbb{C}}(\mathbf{a})]_{i,\sigma(i)}$. Lastly, the map $\sigma \mapsto \sigma^{-1}$ is a well-defined sign-preserving bijection from $A_2$ to $A_3$

such that $\prod_{i=1}^n [M^{\mathbb{C}}_{T_n}(\mathbf{a})]_{i,\sigma(i)} = \overline{\prod_{i=1}^n [M^{\mathbb{C}}_{T_n}(\mathbf{a})]_{i,\sigma^{-1}(i)}}$, so the sum over $A_2$ and $A_3$ is together equal to twice the real part of the sum over $A_2$. The latter is easily seen to equal $-a_{n-1} \cdot \det(M^{\mathbb{C}}_{T_{n-1}}(\mathbf{a}))$, by first swapping the last two rows and then comparing the expression for $a_{n-1} \cdot \det(M^{\mathbb{C}}_{T_{n-1}}(\mathbf{a}))$ with the sum over $A_2$.

This completes the proof of the first recurrence. The proof of the second recurrence goes similarly, taking into account the appropriate sign changes. This is reflected in the term $|a_{n-1}|^2$ appearing as the coefficient attached to $\det(M^{\mathbb{C}}_{T_{n-2},\mathrm{skew}}(\mathbf{a}))$ instead of $-|a_{n-1}|^2$, as well as in the imaginary part of the sum over $A_2$ appearing in place of the real part. □

## 3. Concluding remarks

- The recurrence (1) in Lemma 2.1 allows one to deduce slightly more about the (non)-singularity of the matrices $M_{T_n}(\mathbf{a})$. Indeed, if $\mathrm{char}(\mathbb{F}) \neq 2$ and, for some $n \geq 1$, $M_{T_n}(\mathbf{a})$ is singular, then both $M_{T_{n+1}}(\mathbf{a})$ and $M_{T_{n+2}}(\mathbf{a})$ must be non-singular, since the recurrence

$$\det(M_{T_{n+2}}(\mathbf{a})) = -a_{n+1}^2 \det(M_{T_n}(\mathbf{a})) - 2a_{n+1} \det(M_{T_{n+1}}(\mathbf{a}))$$

  implies that if $M_{T_{n+2}}(\mathbf{a})$ is also singular, then $a_{n+1} = 0$, a contradiction.

  Furthermore, Theorem 1.4 is best possible: if $\mathrm{char}(\mathbb{F}) \neq 2$, then the sequence $\mathbf{a} = (a_1, a_2, a_3, \dots)$ whose entries are defined recursively by

$$a_n := \begin{cases} 1, & n \equiv 1, 2 \pmod 3; \\ -\dfrac{2\det(M_{T_{n-1}}(\mathbf{a}))}{\det(M_{T_{n-2}}(\mathbf{a}))}, & n \equiv 0 \pmod 3, \end{cases}$$

  satisfies the condition $\mathrm{rank}(M_{T_n}(\mathbf{a})) = n - 1$ iff $n \equiv 0 \pmod 3$.
- Similarly, the recurrence (2) shows that $M_{T_{2n},\mathrm{skew}}(\mathbf{a})$ is always non-singular, and $M_{T_{2n+1},\mathrm{skew}}(\mathbf{a})$ is always singular; in particular,

$$\det(M_{T_{2n},\mathrm{skew}}(\mathbf{a})) = \prod_{i=1}^n a_{2i-1}^2$$

  for all $n$. Thus, Theorem 1.6 is also best possible.

  Of course, it is well-known that the determinant of an $n \times n$ skew-symmetric matrix vanishes when $n$ is odd, and is the square of a polynomial in the entries of the matrix (called the Pfaffian) when $n$ is even, and the above formula verifies this. However, we note that the recurrence (1) does not appear to be amenable to a simple closed-form formula in a similar fashion. Also observe that when $\mathrm{char}(\mathbb{F}) = 2$ the two families $\mathcal{M}_n(\mathbf{a})$ and $\mathcal{M}_{n,\mathrm{skew}}(\mathbf{a})$ are identical. So, if $\mathrm{char}(\mathbb{F}) = 2$, then $\mathrm{rank}(M_{T_n}) = n - 1$ iff $n \equiv 0 \pmod 2$.
- For a sequence $\mathbf{a}$ of non-zero elements in an ordered field $\mathbb{F}$, such as $\mathbb{F} = \mathbb{R}$, consider the symmetric $n \times n$ matrices $M(\overline{\mathbf{a}})$ and $M(\underline{\mathbf{a}})$ defined by

$$M(\overline{\mathbf{a}})_{i,j} = \max\{a_i, a_j\} \quad \text{and} \quad M(\underline{\mathbf{a}})_{i,j} = \min\{a_i, a_j\}$$

for all $i < j$. We also consider, similarly, the skew-symmetric versions of these $n \times n$ matrices, denoted $M(\overline{\mathbf{a}}, \text{skew})$ and $M(\underline{\mathbf{a}}, \text{skew})$.

Theorem 2.2 implies that these matrices all have rank at least $n-1$. Such max- and min-type matrices are natural to consider in various contexts (for instance, see [14]), so it is interesting to note that they are all of full rank or nearly so.

- Furthermore, since any tournament on $[n]$ contains a transitive subtournament of size at least $\lfloor \log_2(n) \rfloor + 1$ (see [15]), any skew-symmetric matrix $M \in \mathcal{M}_{n,\text{skew}}(\mathbf{a})$ has rank at least $\lfloor \log_2(n) \rfloor$. This proves Corollary 1.7.
- Additionally, one may consider such matrices having constant (non-zero) diagonal as well. If the diagonal entry, say $d$, differs from all the off-diagonal entries, then the rank is at least $n-2$, since we get a matrix of the above form with zero diagonal by subtracting $dJ$, where $J$ is the all-ones matrix, which has rank 1.
- The analogues of Theorem 2.2 and Corollary 1.7 hold for the families $\mathcal{M}_n^{\mathbb{C}}(\mathbf{a})$ and $\mathcal{M}_{n,\text{skew}}^{\mathbb{C}}(\mathbf{a})$, with their proofs going through in a similar fashion. Furthermore, the second remark in this section applies equally well to these matrices, too.

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The authors report there are no competing interests to declare.

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