The Weighted Davenport constant of a group and a related extremal problem - II

Niranjan Balachandran* and Eshita Mazumdar†

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Abstract

For a finite abelian group G with $\exp(G) = n$ and an integer $k \geq 2$, Balachandran and Mazumdar [3] introduced the extremal function $f_G^{(D)}(k)$ which is defined to be $\min\{|A| : \emptyset \neq A \subseteq [1, n-1]$ with $D_A(G) \leq k\}$ (and ∞ if there is no such A), where $D_A(G)$ denotes the A-weighted Davenport constant of the group G. Denoting $f_G^{(D)}(k)$ by $f^{(D)}(p, k)$ when $G = \mathbb{F}_p$ (for p prime), it is known ([3]) that $p^{1/k} - 1 \leq f^{(D)}(p, k) \leq O_k(p \log p)^{1/k}$ holds for each $k \geq 2$ and p sufficiently large, and that for $k = 2, 4$, we have the sharper bound $f^{(D)}(p, k) \leq O(p^{1/k})$. It was furthermore conjectured that $f^{(D)}(p, k) = \Theta(p^{1/k})$. In this short paper we prove that $f^{(D)}(p, k) \leq 4^{k^2} p^{1/k}$ for sufficiently large primes p.

Keywords: Zero-sum problems, Davenport constant of a group.

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1 Introduction

For a prime p, and $a \neq b, a, b \in \mathbb{F}_p$, we shall borrow the notation $[a, b]$ (from the usual integer case) to denote the set $\{a, a+1, \ldots, b\}$ for $a \neq b \in \mathbb{F}_p$. Throughout this paper, we follow the standard Landau asymptotic notation (see [5] for instance): For functions f, g, we write $f(n) = O(g(n))$ if there exists an absolute constant $C > 0$ and an integer n_0 such that for all $n \geq n_0, |f(n)| \leq C|g(n)|$. We write $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$. If the constant $C = C(k)$ depends on another parameter k (but not on n) then we shall denote this by writing $f = O_k(g)$. We also write $f \ll g$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

*Department of Mathematics, IIT Bombay. Email: niranj (at) math.iitb.ac.in
†Stat-Math Unit, ISI Bengaluru. Email: eshita_vs (at) isibang.ac.in

Suppose G is a finite abelian group (written additively) with $\exp(G) = n$, and suppose $A \subseteq [1, n-1]$. The *A-weighted Davenport constant* $D_A(G)$ of the group G (introduced by Adhikari *et al*, see [1]) is the least positive integer k for which the following holds: Given an arbitrary sequence $(x_1, \ldots, x_k)$, with $x_i \in G$, there exists a non-empty subsequence $(x_{i_1}, \ldots, x_{i_t})$ along with $a_j \in A$ such that $\sum_{j=1}^{t} a_j x_{i_j} = 0$, where as usual, $ax = \overbrace{x + \cdots + x}^{a \text{ times}}$. For general finite abelian group $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$ with $n_i | n_{i+1}$ and $\exp(G) = n_s$. If we consider the sequence $S = (g)$ where $g = (0, \cdots, 0, 1)$ in G then clearly $kg = (0, \cdots, 0)$ in G iff $k = 0$. So $D_A(G) \geq 2$ for any finite abelian group G. The weighted Davenport constant has been the primary object of study in several papers (see [1, 2] for instance and some of the references in [3]) and determining $D_A(G)$ for some 'natural' choices for the weight set A for various categories of groups are questions that have garnered sufficient interest.

In [3], Balachandran and Mazumdar introduced a natural extremal problem associated to the weighted Davenport constant as follows. Suppose G is a finite abelian group with $\exp(G) = n$ and $k \geq 2$. Define

$$\begin{aligned} f_G^{(D)}(k) &:= \min\{|A| : \emptyset \neq A \subseteq [1, n-1] \text{ satisfies } D_A(G) \leq k\}, \\ &:= \infty \text{ if there is no such } A. \end{aligned}$$

Given a group G, determine $f_G^{(D)}(k)$ (if it is finite).

As it turns out, $f_G^{(D)}(k) < \infty$ for k not 'too large' (see [3]) and the most interesting case is when $G = \mathbb{F}_p$ with p being a prime (We denote $f_G^{(D)}(k)$ by $f^{(D)}(p, k)$ for simplicity).

In addition to being an interesting problem of independent interest, the problem of determining $f^{(D)}(p, k)$ is a generalization of certain well-studied notions for abelian groups. For instance, determining $f^{(D)}(p, 2)$ is equivalent to finding a smallest difference base in the cyclic group $\mathbb{Z}_p$ (see [4] for the definition and related results) and the general case is a vast generalization of this notion.

One of the main results in [3] is the following

Theorem 1. *Let $k \in \mathbb{N}$, $k \geq 2$.*

(a) There exists an integer $p_0(k)$ and an absolute constant $C > 0$ such that for all primes $p > p_0(k)$,

$$p^{1/k} - 1 \leq f^{(D)}(p, k) \leq C(p \log p)^{1/k}.$$

(b) $f^{(D)}(p, k) \leq C p^{1/k}$ for $k = 2, 4$, for some absolute constant $C > 0$.

It was conjectured in [3] that $f^{(D)}(p, k) = O(p^{1/k})$. The main result of this short paper is the following

Theorem 2. *Suppose $k \geq 2$. Then there exists $p_0 = p_0(k)$ such that for all primes $p \geq p_0$,*

$$f^{(D)}(p, k) \leq 4^{k^2} p^{1/k}.$$

So, $f^{(D)}(p, k) = \Theta_k(p^{1/k})$.

This does not settle the aforementioned conjecture in the strong form mentioned there, but it is a substantial improvement on Theorem 1. The constants that are involved in our proof are far from optimal, and we make no attempt to optimize for them.

We prove Theorem 2 in the next section. We ignore floors and ceilings in the expressions that appear, in order to increase clarity of expression and facilitate ease of comprehension. The last section contains a few remarks and questions for further inquiry.

2 Proof of Theorem 2

We first consider the case $f^{(D)}(p, 2k)$. We generalize the proof of the case $k = 4$ in [3] along with some other ideas. The basic scheme of proof is somewhat similar, so we recall it first, for the readers' convenience.

In order to show $f^{(D)}(p, 2k) \leq O_k(p^{1/2k})$, it suffices to show the existence of $A \subset \mathbb{F}_p^*$ of size $27^{k^2} p^{1/2k}$ (which is stronger than the result stated) such that for any $\alpha_1, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_{k-1} \in \mathbb{F}_p^*$,

$$\mathbb{F}_p^* \subseteq \frac{A + \alpha_1 A + \cdots + \alpha_{k-1} A}{A + \beta_1 A + \cdots + \beta_{k-1} A}. \tag{1}$$

To see why this will suffice, note that $D_A(\mathbb{F}_p) \leq 2k$ implies that for any sequence $(x_1, \ldots, x_{2k}) \in (\mathbb{F}_p^*)^{2k}$, we have $0 \in Ax_1 + \cdots + Ax_{2k}$. This is equivalent to requiring that

$$-\frac{x_1}{x_{k+1}} = \frac{a_{k+1} + a_{k+2}(x_{k+2}/x_{k+1}) + \cdots + a_{2k}(x_{2k}/x_{k+1})}{a_1 + a_2(x_2/x_1) + \cdots + a_k(x_k/x_1)}$$

holds for some $a_i \in A$, and if (1) holds, then this is indeed satisfied.

The following observation, which is also the starting point in [3], is again the key to our scheme of proof.

Observation 2.1. *If $A, B \subseteq \mathbb{F}_p$ satisfy $|A||B| > p$, then $\mathbb{F}_p = \frac{A-A}{B-B}$.*

To see why this holds, consider for an arbitrary $x \in \mathbb{F}_p^*$, the map $\phi_x : A \times B \to \mathbb{F}_p^*$ defined by $\phi_x(a, b) := ax + b$. Since $|A| \cdot |B| > p$ it follows that this map is not injective, so there exist pairs $(a, b) \neq (a', b')$ such that $\phi_x(a, b) = \phi_x(a', b')$. In particular, this implies that $a \neq a'$, which further implies that $x = \frac{b'-b}{a-a'} \in \frac{B-B}{(A-A)_*}$, and since x was arbitrary, the proof of the observation is complete.

Hence to show (1), it suffices to show the existence of a set A of size at most the bound mentioned earlier, that satisfies the following:

$$\text{For any } \alpha_1 \ldots, \alpha_{k-1} \in \mathbb{F}_p^*, \quad |A + \alpha_1 A + \cdots + \alpha_{k-1} A| > \sqrt{p}.$$

Let $L = Cp^{1/2k}$ where we shall determine C later. For a positive integer t, let $X_t := t[-L, L] = \{-Lt, \ldots, -t, 0, t, \ldots, Lt\}$. It immediately follows that $\alpha X_t = X_{\alpha t}$. The following lemma is a special case of Lemma 3.10 in [5] for generalized arithmetic progressions in finite abelian groups (see the definition of generalized arithmetic progression in [5]).

Lemma 3. *For distinct t_i, $X_{t_1}+\cdots+X_{t_k}$ contains a subset Y of size at least $|X_{t_1}+X_{t_2}+\cdots+X_{t_k}|/2^k$ such that $Y-Y\subseteq X_{t_1}+X_{t_2}+\cdots+X_{t_k}$.*

Proof. For $t\in\mathbb{F}_p^*$, set $Y_t:=a[0,L]$. Then it is trivial to see that the set $Y:=Y_{t_1}+\cdots+Y_{t_k}$ satisfies $Y-Y=X_{t_1}+\cdots+X_{t_k}$. Furthermore, Y is a generalized arithmetic progression of rank k. Lemma 3.10 in [5] states that for any generalized arithmetic progression $P=a+[0,L]\cdot\mathbf{v}$ of rank k, we have $|P-P|\leq 2^k|P|$. The lemma now follows immediately. □

We shall now introduce some notation. We shall write $I:=X_1=\{-L,\ldots,-1,0,1,\ldots,L\}$ for convenience. For sets $A_1,\ldots,A_r\subset\mathbb{F}_p$ we shall denote the sum set $A_1+\cdots+A_r$ by $\sum_{i=1}^r A_i$. For $t\in\mathbb{F}_p^*$, we shall denote the set $\{ta:a\in A\}$ by tA. We write $A_*:=A\setminus\{0\}$, and finally, for $\mathbf{a}:=(\alpha_1,\ldots,\alpha_k),\mathbf{b}:=(\beta_1,\ldots,\beta_k)$ we shall write $\mathbf{a}\cdot\mathbf{b}:=(\alpha_1\beta_1,\ldots,\alpha_k\beta_k)$.

For $A_1,\ldots,A_{k-1},B\subseteq\mathbb{F}_p$ define

$$S(A_1,\ldots,A_{k-1};B):=\left\{(t_1,\ldots,t_{k-1})\in(\mathbb{F}_p^*)^{k-1}:(\sum_{i=1}^{k-1}t_iA_i)\cap B\neq\emptyset\right\}.$$

Definition 4. *For a given $\mathbf{t}:=(t_1,\ldots,t_{k-1})\in(\mathbb{F}_p^*)^{k-1}$, we say that $\mathbf{a}:=(\alpha_1,\ldots,\alpha_{k-1})\in(\mathbb{F}_p^*)^{k-1}$ is good for $\mathbf{t}$ if*

$$|(I+\alpha_1X_{t_1}+\cdots+\alpha_{k-1}X_{k-1})_*|\geq\frac{L^k}{4^k}.$$

Suppose $\mathbf{t}\in(\mathbb{F}_p^*)^{k-1}$. The following lemma tells us that if $\mathbf{a}$ is not good for $\mathbf{t}$ then, in a sense (made precise by the lemma), $\mathbf{a}$ is restricted to a very small subset of $(\mathbb{F}_p^*)^{k-1}$.

Lemma 5. *Let $L=Cp^{1/2k}$, $B:=[-2L,2L]$, and for each $1\leq i,j\leq k-1$, define $A_i(j):=[\frac{-L}{2},\frac{L}{2}]$ if $i\neq j$, and $A_i(i)=[1,L/2]$. Let $\mathcal{S}_j:=S(A_1(j),\ldots,A_{k-1}(j);B)$ and $\mathcal{S}:=\cup_{j=1}^{k-1}\mathcal{S}_j$. Suppose $\mathbf{a}:=(\alpha_1,\ldots,\alpha_{k-1})\in(\mathbb{F}_p^*)^{k-1}$ is not good for $\mathbf{t}:=(t_1,\ldots,t_{k-1})\in(\mathbb{F}_p^*)^{k-1}$, then $\mathbf{a}\cdot\mathbf{t}\in\mathcal{S}$. Moreover,*

$$|\mathcal{S}|<3kL^k(p-1)^{k-2}<3kC^k(p-1)^{k-\frac{3}{2}}\text{ for sufficiently large }p.$$

Proof. Since $\alpha X_t=X_{\alpha t}$, it suffices to show that

$$|(I+X_{t_1}+\cdots+X_{t_{k-1}})_*|<\frac{L^k}{4^k}\ \Rightarrow\ \mathbf{t}\in\mathcal{S}.$$

Set $X_t^+:=\{0,t,\ldots,Lt\}$. If $\mathfrak{j}$ denotes the $(k-1)$-tuple $(j_1,\ldots,j_{k-1})$, then observe that

$$I+X_{t_1}^++\cdots+X_{t_{k-1}}^+=\bigcup_{\mathfrak{j}\in[0,L]^{k-1}}[j_1t_1+\cdots+j_{k-1}t_{k-1}-L,j_1t_1+\cdots+j_{k-1}t_{k-1}+L].$$

Put an arbitrary linear order $\leq$ on $[0,L]^{k-1}$ with least element $\mathbf{0}:=(0,\ldots,0)$ and define

$$\begin{aligned}\mathcal{X}(\mathbf{0})&:=[-L,L],\\ \mathcal{X}(\mathfrak{i})&:=\bigcup_{\mathfrak{j}\leq\mathfrak{i}}[\mathfrak{j}\cdot\mathbf{t}-L,\mathfrak{j}\cdot\mathbf{t}+L]=\bigcup_{\mathfrak{j}\leq\mathfrak{i}}[j_1t_1+\cdots+j_{k-1}t_{k-1}-L,j_1t_1+\cdots+j_{k-1}t_{k-1}+L].\end{aligned}$$

Call the set $\mathcal{X}(\mathfrak{i})$ *valid* if it is the union of pairwise disjoint intervals each of length $2L$, centred around an element of $\sum_{j=1}^{k-1} X_{t_j}^{+}$. Clearly, $\mathcal{X}(0,\ldots,0)$ is valid.

We now claim that there exists $\mathfrak{i}$ with some $i_j \le L/2$ such that $\mathcal{X}(\mathfrak{i})$ is not valid. Indeed, suppose $\mathcal{X}(\mathfrak{i})$ is valid for all such $\mathfrak{i}$. In particular, for $M_j = \lceil L/4 \rceil$, we have

$$|\mathcal{X}(M_1,\ldots,M_{k-1})| = 2L\prod_{j=1}^{k-1} M_j \ge (2L)\left(\frac{L}{4}\right)^{k-1} \ge 2\frac{L^k}{4^{k-1}}$$

contradicting the hypothesis.

Let $\mathfrak{i} = (i_1,\ldots,i_{k-1})$ be the first $(k-1)$-tuple with respect to the linear order such that $\mathcal{X}(i_1,\ldots,i_{k-1})$ is not valid. In particular, there exists $\mathfrak{j} = (j_1\ldots,j_{k-1}) \ne \mathfrak{i}$ and $1 \le r \le k-1$ such that $i_r < j_r$ and

$$\sum_{l=1}^{k-1} i_l t_l + \xi_1 = \sum_{l=1}^{k-1} j_l t_l + \xi_2$$

for some $\xi_1, \xi_2 \in [-L, L]$. Consequently,

$$\mathbf{t}\cdot(\mathfrak{j}-\mathfrak{i}) = \xi_1 - \xi_2 \;\Rightarrow\; \mathbf{t} \in \mathcal{S}_r.$$

To complete the proof, we need to show the bound on $|\mathcal{S}|$. We shall first show that $|\mathcal{S}_1| < 3L^k(p-1)^{k-2}$ for sufficiently large p.

First observe that $(t_1\ldots,t_{k-1}) \in \mathcal{S}_1$ implies that there exists $a_i \in A_i(1)$ for $1 \le i \le k-1$ and $b \in B$ such that $t_1a_1 + \cdots t_{k-1}a_{k-1} = b$. For fixed choices of $a_i \in A_i(1)$ for $1 \le i \le k-1$ and $b \in B$, along with choices of $t_2,\ldots,t_{k-1} \in \mathbb{F}_p^*$, the equation $t_1a_1 + \cdots t_{k-1}a_{k-1} = b$ admits a unique solution for $t \in \mathbb{F}_p$. In particular, it follows that

$$|\mathcal{S}_1| \le (L/2)\cdot(L+1)^{k-2}\cdot(4L+1)\cdot(p-1)^{k-2} < 3L^k p^{k-2}$$

for p sufficiently large as claimed.

Now the bound on $|\mathcal{S}|$ follows by a similar bound for each $|\mathcal{S}_j|$. □

In the rest of the paper, $\mathcal{S}$ shall denote the set described in the statement of Lemma 5.

Lemma 6. *Suppose that there exist* $\mathbf{y}_i = (y_1^{(i)},\ldots,y_{k-1}^{(i)}) \in (\mathbb{F}_p^*)^{k-1}$ *for* $1 \le i \le N$ *such that*

$$\bigcap_{i=1}^{N} \mathbf{y}_i \cdot \mathcal{S} = \emptyset.$$

Write $x_r^{(i)} = (y_r^{(i)})^{-1}$ *for* $1 \le i \le N, 1 \le r \le k-1$, *and set* $\mathbf{x}_i = (x_1^{(i)},\ldots,x_{k-1}^{(i)})$. *Then the set*

$$A = \left(I \cup \bigcup_{\substack{1\le j\le N\\ 1\le r\le k-1}} X_{x_j^{(r)}} \right)_* \subset [1, p-1]$$

satisfies $D_A(\mathbb{F}_p) \le 2k$. *In particular,* $f^{(D)}(p, 2k) \le 2kNL$.

Proof. Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_{k-1}) \in (\mathbb{F}_p^*)^{k-1}$. We shall show that

$$|I + \alpha_1 X_{x_1^{(i)}} + \cdots + \alpha_{k-1} X_{x_{k-1}^{(i)}}| \ge \frac{L^k}{4^k}$$

for some $1 \le i \le N$. Then by Observation 3, there exists $Y_{\mathbf{a}} \subset I + \alpha_1 X_{x_1^{(i)}} + \ldots + \alpha_{k-1} X_{x_{k-1}^{(i)}}$ with $|Y_{\mathbf{a}}| > p^{1/2}$ (if $C > 8$) such that $Y_{\mathbf{a}} - Y_{\mathbf{a}} \subseteq I + \alpha_1 X_{x_1^{(i)}} + \ldots + \alpha_{k-1} X_{x_{k-1}^{(i)}}$. Since this holds for all $\mathbf{a} \in (\mathbb{F}_p^*)^{k-1}$ the proof of Lemma 6 is complete by Observation 2.1.

Since $\bigcap_0^N \mathbf{y}_i \cdot \mathcal{S} = \emptyset$, there exists $1 \le i \le N$ such that $\mathbf{a} \notin \mathbf{y}_i \cdot \mathcal{S}$. But then, by Lemma 5, this implies that $\mathbf{a}$ is good for $\mathbf{y}_i^{-1} = \mathbf{x}_i$, or equivalently,

$$|I + \alpha_1 X_{x_1^{(i)}} + \ldots + \alpha_{k-1} X_{x_{k-1}^{(i)}}| \ge \frac{L^k}{4^k}$$

as required. □

We are now in a position to prove Theorem 2.

Proof. (of Theorem 2) We shall denote by $\mathbf{y}$, a typical element in $(\mathbb{F}_p^*)^{k-1}$. For $\mathcal{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_{2k-3}) \in ((\mathbb{F}_p^*)^{k-1})^{2k-3}$, we say that $\mathfrak{A} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{2k-3}) \in \mathcal{S}^{2k-2}$ is *binding* for $\mathcal{Y}$ if

$$\mathbf{a}_0 = \mathbf{y}_1 \cdot \mathbf{a}_1 = \cdots = \mathbf{y}_{2k-3} \cdot \mathbf{a}_{2k-3}.$$

We shall call $\mathbf{a}_0$ the leading element of $\mathfrak{A}$. Clearly, each $\mathfrak{A}$ determines a unique $\mathcal{Y} \in ((\mathbb{F}_p^*)^{k-1})^{2k-3}$ such that $\mathfrak{A}$ is binding for $\mathcal{Y}$. For $\mathcal{Y} \in ((\mathbb{F}_p^*)^{k-1})^{2k-3}$ define

$$\begin{aligned} \mathfrak{A}(\mathcal{Y}) &:= \{\mathfrak{A} : \mathfrak{A} \text{ is binding for } \mathcal{Y}\}, \\ N(\mathcal{Y}) &:= |\mathfrak{A}(\mathcal{Y})|, \\ \text{NORMAL} &:= \{\mathcal{Y} \in ((\mathbb{F}_p^*)^{k-1})^{2k-3} : N(\mathcal{Y}) \le 2(3kC^k)^{2k-2}\}. \end{aligned}$$

Suppose $\mathcal{Y}$ is chosen uniformly at random from $((\mathbb{F}_p^*)^{k-1})^{2k-3}$. Then

$$\mathbb{E}(N(\mathcal{Y})) = \sum_{\mathfrak{A} \in \mathcal{S}^{2k-2}} \mathbb{P}\left(\mathfrak{A} \text{ is binding for } \mathcal{Y}\right) = \frac{|\mathcal{S}|^{2k-2}}{(p-1)^{(k-1)(2k-3)}} < (3kC^k)^{2k-2} := C^*.$$

Hence by Markov's Inequality, it follows that

$$|\text{NORMAL}| \ge \frac{1}{2}(p-1)^{(k-1)(2k-3)}. \tag{2}$$

Pick $\mathcal{Y}_1 = (\mathbf{y}_1^{(1)}, \ldots, \mathbf{y}_{2k-3}^{(1)}) \in$ NORMAL arbitrarily. Write $\mathfrak{A}(\mathcal{Y}_1) = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_\ell\}$ with $\ell \leq C^*$. Let $\mathfrak{A}_i = (\mathbf{a}_0[i], \mathbf{a}_1[i], \ldots, \mathbf{a}_{2k-3}[i])$. Since $\mathcal{Y}_1 \in$ NORMAL,

$$\mathbf{a}_0[i] = \mathbf{y}_1^{(1)} \cdot \mathbf{a}_1[i] = \cdots = \mathbf{y}_{2k-3}^{(1)} \cdot \mathbf{a}_{2k-3}[i] \text{ for all } 1 \leq i \leq \ell.$$

The number of $\mathcal{Y} \in$ NORMAL such that $(\mathbf{a}_0[1], \mathbf{b}_1, \ldots, \mathbf{b}_{2k-3}) \in \mathfrak{A}(\mathcal{Y})$ is at most $|S|^{2k-3} \ll$ |NORMAL|, so there exists $\mathcal{Y}_2 = (\mathbf{y}_1^{(2)}, \ldots, \mathbf{y}_{2k-3}^{(2)}) \in$ NORMAL such that $\mathbf{a}_0[1]$ is not a leading element of any $\mathfrak{A} \in \mathfrak{A}(\mathcal{Y}_2)$. Thus, having chosen $\mathcal{Y}_1, \ldots, \mathcal{Y}_i \in$ NORMAL, we inductively pick $\mathcal{Y}_{i+1} \in$ NORMAL such that $\mathbf{a}_0[i]$ is not a leading element of any $\mathfrak{A} \in \mathfrak{A}(\mathcal{Y}_{i+1})$, and these choices are possible by the same argument described above, and note that this procedure terminates in at most $T \leq \ell$ steps. As a consequence of these choices, it follows immediately that

$$\mathcal{S} \cap \left(\bigcap_{\substack{1 \leq i \leq 2k-3 \\ 1 \leq j \leq T}} \mathbf{y}_i^{(j)} \cdot \mathcal{S} \right) = \emptyset$$

so we can take $N = 1 + (2k-3)T \leq 2k(3kC^k)^{2k-2}$.

Putting all the ingredients together, we have

$$f^{(D)}(p, 2k) \leq 2kNL \leq (4k^2)(3kC^k)^{2k-2}(Cp^{1/2k}) \leq (27)^{k^2} p^{1/2k}$$

for all $k \geq 2$.

For the odd case, the proof moves along exactly the same lines. To bound $f^{(D)}(p, 2k+1)$, we need to describe a set $A \subset \mathbb{F}_p^*$ such that for any $\alpha_1, \ldots, \alpha_k, \beta_, \ldots, \beta_{k-1} \in \mathbb{F}_p^*$

$$\mathbb{F}_p^* \subseteq \frac{A + \alpha_1 A + \cdots + \alpha_k A}{A + \beta_1 A + \cdots + \beta_{k-1} A}$$

holds. Keeping with our scheme of proof, it will suffice to show that for $r = k-1, k$, and any $\alpha_1 \ldots, \alpha_r \in \mathbb{F}_p^*$ there exists $A \subset \mathbb{F}_p^*$ such that $|A + \alpha_1 A + \cdots + \alpha_r A| > p^{\frac{r+1}{2k+1}}$. We now imitate the same argument to obtain $A_r \subset \mathbb{F}_p^*$ that works for r, and finally $A = A_{k-1} \cup A_k$ does the job. It is somewhat routine to check that the bound on $|A|$ as stated in Theorem 2 indeed holds, so we omit those details. □

3 Concluding remarks

- It should be quite clear that the dependence on k in the constant in Theorem 2 is far from best possible. We still believe that $f^{(D)}(p, k) \leq Cp^{1/k}$ for an absolute constant C, which is still out of our reach.
- A closer inspection of the proof of Theorem 1 in [3] actually reveals that the proof of the first part of Theorem 1 holds for all $k \ll \frac{\log p}{\log \log p}$, which makes that theorem quite robust because the problem of determining $f^{(D)}(p, k)$ is relevant only for $k \leq \log_2 p + 1$ (see Proposition 4.1 in [3]). In contrast, the bound in Theorem 2 is suboptimal to the bound in Theorem 1 unless $k \ll (\log \log p)^{\frac{1}{3}}$.

- As pointed out in [3] the problem of determining $f_G^{(D)}(k)$ for arbitrary abelian groups G reduces to the case(s) $G = \mathbb{F}_p$ (resp. $G = \mathbb{F}_p^r$) as the most relevant one because one can choose weights that project all the weighted elements in to some small subgroup of G. This raises a more interesting *irreducible* variant of the same extremal problem: Given a finite abelian group G with $\exp(G) = n$ and $\mathbb{Z}_n$ being a cyclic group of order n, define

$$\begin{aligned} If_G^{(D)}(k) &:= \min\{|A| : \emptyset \neq A \subseteq \mathbb{Z}_n^* \text{ satisfies } D_A(G) \leq k\}, \\ &:= \infty \text{ if there is no such } A. \end{aligned}$$

  Given a group G, determine $If_G^{(D)}(k)$ (if it is finite).

  This latter extremal function does not permit us the trick of projecting into a smaller subgroup, and henceforth poses more interesting possibilities.

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