Fractional L-intersecting families

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Abstract

Let $L = \{\frac{a_1}{b_1}, \dots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let $\mathcal{F} = \{A_1, \dots, A_m\}$ be a family of subsets of $[n]$. We say $\mathcal{F}$ is a *fractional L-intersecting family* if for every distinct $i, j \in [m]$, there exists an $\frac{a}{b} \in L$ such that $|A_i \cap A_j| \in \{\frac{a}{b}|A_i|, \frac{a}{b}|A_j|\}$. In this paper, we introduce and study the notion of fractional L-intersecting families.

Mathematics Subject Classifications: 05D05, 05C50, 05C65

1 Introduction

Let $[n]$ denote $\{1, \dots, n\}$ and let $L = \{l_1, \dots, l_s\}$ be a set of s non-negative integers. A family $\mathcal{F} = \{A_1, \dots, A_m\}$ of subsets of $[n]$ is *L-intersecting* if for every $A_i, A_j \in \mathcal{F}$

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with $A_i \neq A_j$, we have $|A_i \cap A_j| \in L$. In 1975, it was shown by Ray-Chaudhuri and Wilson in [13] that if $\mathcal{F}$ is t-uniform, then $|\mathcal{F}| \leqslant \binom{n}{s}$. Setting $L = \{0, \ldots, s-1\}$, the family $\mathcal{F} = \binom{[n]}{s}$ is a tight example to the above bound, where $\binom{[n]}{s}$ denotes the set of all s-sized subsets of $[n]$. In the non-uniform case, it was shown by Frankl and Wilson in 1981 (see [7]) that if we don't put any restrictions on the cardinalities of the sets in $\mathcal{F}$, then $|F| \leqslant \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$. This bound is tight as demonstrated by the set of all subsets of $[n]$ of size at most s with $L = \{0, \ldots s-1\}$. The proof of this bound was using the method of higher incidence matrices. Later, in 1991, Alon, Babai, and Suzuki in [2] gave an elegant linear algebraic proof of this bound. They showed that if the cardinalities of the sets in $\mathcal{F}$ belong to the set of integers $K = \{k_1, \ldots, k_r\}$ with every $k_i > s - r$, then $|\mathcal{F}|$ is at most $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$. The collection of all the subsets of $[n]$ of size at least $s - r + 1$ and at most s with $K = \{s - r + 1, \ldots, s\}$ and $L = \{0, \ldots, s-1\}$ forms a tight example to this bound. In 2002, this result was extended by Grolmusz and Sudakov [8] to k-wise L-intersecting families. In 2003, Snevily showed in [14] that if L is a collection of s positive integers then $|\mathcal{F}| \leqslant \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$. See [11] for a survey on L-intersecting families and their variants.

In this paper, we introduce a new variant of L-intersecting families called the fractional L-intersecting families. Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$. We say $\mathcal{F}$ is a *fractional L-intersecting family* if for every distinct $i, j \in [m]$, there exists an $\frac{a}{b} \in L$ such that $|A_i \cap A_j| \in \{\frac{a}{b}|A_i|, \frac{a}{b}|A_j|\}$. When $\mathcal{F}$ is t-uniform, it is an L'-intersecting family where $L' = \{\lfloor \frac{a_1 t}{b_1} \rfloor, \ldots, \lfloor \frac{a_s t}{b_s} \rfloor\}$ and therefore (using the result in [13]), $|\mathcal{F}| \leqslant \binom{n}{s}$. A tight example to this bound is given by the family $\mathcal{F} = \binom{[n]}{t}$ where $L = \{\frac{0}{t}, \ldots, \frac{t-1}{t}\}$. So what is interesting is finding a good upper bound for $|\mathcal{F}|$ in the non-uniform case. Unlike in the case of the classical L-intersecting families, it is clear from the above definition that if A and B are two sets in a fractional L-intersecting family, then the cardinality of their intersection is a function of $|A|$ or $|B|$ (or both).

In Section 2.1, we prove the following theorem which gives an upper bound for the cardinality of a fractional L-intersecting family in the general case. We follow the convention that $\binom{a}{b}$ is 0, when $b > a$.

Theorem 1. *Let n be a positive integer. Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let $\mathcal{F}$ be a fractional L-intersecting family of subsets of $[n]$. Then, $|\mathcal{F}| \leqslant 2\binom{n}{s} g^2(t, n) \ln(g(t, n)) + \left(\sum_{i=1}^{s-1} \binom{n}{i}\right) g(t, n)$, where $g(t, n) = \frac{2(2t + \ln n)}{\ln(2t + \ln n)}$ and $t = \max(s, \max(b_i : i \in [s])\ )$. Further,*

(a) if $s \leqslant n + 1 - 2g(t, n) \ln(g(t, n))$, then $|\mathcal{F}| \leqslant 2\binom{n}{s} g^2(t, n) \ln(g(t, n))$, and

(b) if $t > n - c_1$, where c_1 is a positive integer constant, then
$|\mathcal{F}| \leqslant 2c_1 \binom{n}{s} g(t, n) \ln(g(t, n)) + c_1 \sum_{i=1}^{s-1} \binom{n}{i}$.

Consider the following examples for a fractional L-intersecting family.

Example 2. Let $L = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \ \ldots, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$, where we omit fractions, like $\frac{2}{4}$, which are not irreducible. The collection of all the non-empty subsets of $[n]$ is a fractional

L-intersecting family of cardinality $2^n - 1$. Here, $|L| = s = \Theta(n^2)$. Since $t \geqslant s$, we can apply Statement (b) of Theorem 1 to get an upper bound of $c_1(2^n - 1)$ which is asymptotically tight. In general, when $L = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \ldots, \frac{1}{n-c}, \ldots, \frac{n-c-1}{n-c}\}$, where $c \geqslant 0$ is a constant, the set of all the non-empty subsets of $[n]$ of cardinality at most $n - c$ is an example which demonstrates that the bound given in Statement (b) of Theorem 1 is asymptotically tight.

Example 3. Let us now consider another example where s $(= |L|)$ is a constant. Let $L = \{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$. The collection of all the s-sized subsets of $[n]$ is a fractional L-intersecting family of cardinality $\binom{n}{s}$. In this case, the bound given by Theorem 1 is asymptotically tight up to a factor of $\frac{\ln^2 n}{\ln \ln n}$. We believe that if $\mathcal{F}$ is a fractional L-intersecting family of maximum cardinality, where s $(= |L|)$ is a constant, then $|\mathcal{F}| \in \Theta(n^s)$.

Coming back to the classical L-intersecting families, it is known that when $\mathcal{F}$ is an L-intersecting family where $|L| = s = 1$, the Fisher's Inequality (see Theorem 7.5 in [9]) yields $|\mathcal{F}| \leqslant n$. Study of such intersecting families was initiated by Ronald Fisher in 1940 (see [5]). This fundamental result of design theory is among the first results in the field of L-intersecting families. Analogously, consider the scenario when $L = \{\frac{a}{b}\}$ is a singleton set. Can we get a tighter bound (compared to Theorem 1) in this case? We show in Theorem 4 that if b is a constant prime we do have a tighter bound.

Theorem 4. *Let n be a positive integer. Let $\mathcal{G}$ be a fractional L-intersecting families of subsets of $[n]$, where $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0, 1)$, and b is a prime. Then, $|\mathcal{G}| \leqslant (b-1)(n+1)\lceil \frac{\ln n}{\ln b} \rceil + 1$.*

Assuming $L = \{\frac{1}{2}\}$, Examples 11 and 12 in Section 3 give fractional L-intersecting families on $[n]$ of cardinality $\frac{3n}{2} - 2$ thereby implying that the bound obtained in Theorem 4 is asymptotically tight up to a factor of $\ln n$ when b is a constant prime. We believe that the cardinality of such families is at most cn, where $c > 0$ is a constant.

The rest of the paper is organized in the following way: In Section 2.1, we give the proof of Theorem 1 after introducing some necessary lemmas in the beginning. In Theorem 9 in Section 2.2, we demonstrate that any fractional L-intersecting families on $[n]$ whose member sets are 'large enough' has size at most n. In Section 3, we consider the case when L is a singleton set and give the proof of Theorem 4. Later in this section, in Theorem 14, we consider the case when the cardinalities of the sets in the fractional L-intersecting family are restricted. Finally, we conclude with some remarks, some open questions, and a conjecture.

2 The general case

2.1 Proof of Theorem 1

Before we move to the proof of Theorem 1, we introduce a few lemmas that will be used in the proof.

2.1.1 A few auxiliary lemmas

The following lemma is popularly known as the 'Independence Criterion' or 'Triangular Criterion'.

Lemma 5 (Lemma 13.11 in [9], Proposition 2.5 in [3]). *For $i = 1, \dots, m$ let $f_i : \Omega \to \mathbb{F}$ be functions and $v_i \in \Omega$ elements such that*

(a) $f_i(v_i) \neq 0$ for all $1 \leqslant i \leqslant m$;

(b) $f_i(v_j) = 0$ for all $1 \leqslant j < i \leqslant m$.

Then $f_1, \dots, f_m$ are linearly independent members of the space $\mathbb{F}^\Omega$.

Lemma 6. *Let p be a prime; $\Omega = \{0,1\}^n$. Let $f \in \mathbb{F}_p^\Omega$ and let $i \in \mathbb{F}_p$. For any $A \subseteq [n]$, let $V_A \in \{0,1\}^n$ denote its 0-1 incidence vector and let $x_A = \Pi_{j\in A} x_j$. Assume $f(V_A) \neq 0$, for every $|A| \not\equiv i \pmod p$. Then, the set of functions $\{x_A f \ : \ |A| \not\equiv i \pmod p$ and $|A| < p\}$ is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over $\mathbb{F}_p$.*

Proof. Arrange every subset of $[n]$ of cardinality less than p in a linear order, say $\prec$, such that $A \prec B$ implies $|A| \leqslant |B|$. For any two distinct sets A and B with $|B| \leqslant |A|$, we know that $x_A(V_B)f(V_B) = 0$, where $x_A(V_B)$ denote the evaluation of the function x_A at V_B. Suppose $\sum_{A:|A|\not\equiv i \pmod p,\ |A|<p} \lambda_A x_A f = 0$ has a non-trivial solution. Then, identify the first set, say A_0, in the linear order $\prec$ for which λ_{A_0} is non-zero. Evaluate the functions on either side of the above equation at V_{A_0} to get $\lambda_{A_0} = 0$ which is a contradiction to our assumption. □

The following lemma is from [3] (see Lemma 5.38).

Lemma 7 (Lemma 5.38 in [3]). *Let p be a prime; $\Omega = \{0,1\}^n$. Let $f \in \mathbb{F}_p^\Omega$ be defined as $f(x) = \sum_{i=1}^n x_i - k$. For any $A \subseteq [n]$, let $V_A \in \{0,1\}^n$ denote its 0-1 incidence vector and let $x_A = \Pi_{j\in A} x_j$. Assume $0 \leqslant s, k \leqslant p-1$ and $s + k \leqslant n$. Then, the set of functions $\{x_A f \ : \ |A| \leqslant s-1\}$ is linearly independent in the vector space $\mathbb{F}_p^\Omega$ over $\mathbb{F}_p$.*

2.1.2 The proof

Proof of Theorem 1. Let p be a prime with $p > t$. We partition $\mathcal{F}$ into p parts, namely $\mathcal{F}_0, \dots, \mathcal{F}_{p-1}$, where $\mathcal{F}_i = \{A \in \mathcal{F} \ : \ |A| \equiv i \ (mod\ p)\}$.

Estimating $|\mathcal{F}_i|$, when $i > 0$.

Let $\mathcal{F}_i = \{A_1, \dots, A_m\}$ and let $V_1, \dots, V_m$ denote their corresponding 0-1 incidence vectors. Define m functions f_1 to f_m, where each $f_j \in \mathbb{F}_p^{\{0,1\}^n}$, in the following way.

$$f_j(x) = (\langle V_j, x\rangle - \frac{a_1}{b_1} i)(\langle V_j, x\rangle - \frac{a_2}{b_2} i) \cdots (\langle V_j, x\rangle - \frac{a_s}{b_s} i).$$

Note that since $|A_j| \equiv i \pmod p$, $\langle V_j, V_j \rangle \equiv i \pmod p$. Since $p > t$, for every $l \in [s]$, $i \not\equiv \frac{a_l}{b_l} i \pmod p$ unless $i \equiv 0 \pmod p$. So,

$$f_j(x) \begin{cases} \neq & 0, \text{ if } x = V_j \\ = & 0, \text{ otherwise.} \end{cases} \tag{1}$$

So, f_j's are linearly independent in the vector space ${\mathbb{F}_p}^{\{0,1\}^n}$ over $\mathbb{F}_p$ (by Lemma 5). Since $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$, $x_i^r = x_i$, for any positive integer r. Each f_j is thus an appropriate linear combination of distinct monomials of degree at most s. Therefore, $|\mathcal{F}_i| = m \leqslant \sum_{j=0}^{s} \binom{n}{j}$. We can improve this bound by using the "swallowing trick" in a way similar to the way it is used in the proof of Theorem 1.1 in [2]. Let $f \in \mathbb{F}_p^{\{0,1\}^n}$ be defined as $f(x) = \sum_{j \in [n]} x_j - i$. From Lemma 6, we know that the set of functions $\{x_A f \ : \ |A| \not\equiv i \pmod p \text{ and } |A| < s\}$ is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over $\mathbb{F}_p$.

Claim 8. $\{f_j \ : \ 1 \leqslant j \leqslant m\} \cup \{x_A f \ : \ |A| \not\equiv i \pmod p$ *and* $|A| < s\}$ *is a collection of functions that is linearly independent in the vector space* $\mathbb{F}_p^{\{0,1\}^n}$ *over* $\mathbb{F}_p$.

In order to prove the claim, assume $\sum_{j=1}^{m} \lambda_j f_j + \sum_{A : |A| \leqslant s-1,\ |A| \not\equiv i \pmod p} \mu_A x_A f = 0$ for some $\lambda_j, \mu_A \in \mathbb{F}_p$. Evaluating at V_j, all terms in the second sum vanish (since $f(V_j) = 0$) and by Equation 1, only the term with subscript j remains of the first sum. We infer that $\lambda_j = 0$, for every j. It then follows from Lemma 6 that every μ_A is zero thus proving the claim.

Since each function in the collection of functions in Claim 8 can be obtained as a linear combination of distinct monomials of degree at most s, we can infer that $m + \sum_{j \neq i, j=0}^{s-1} \binom{n}{j} \leqslant \sum_{j=0}^{s} \binom{n}{j}$. We thus have

$$|\mathcal{F}_i| \leqslant \begin{cases} \binom{n}{s} + \binom{n}{i}, \text{ if } i < s \\ \binom{n}{s}, \text{ otherwise} \end{cases} \tag{2}$$

Observe that $i \leqslant p - 1$. We will shortly see that the prime p we choose is always at most $2g(t,n)\ln(g(t,n))$, where $g(t,n) = \frac{(2t + \ln n)}{\ln(2t + \ln n)}$. So if $s \leqslant n + 1 - 2g(t,n)\ln(g(t,n))$, the condition $s + i \leqslant n$ (here i stands for the symbol k in Lemma 7) given in Lemma 7 is satisfied and therefore the more powerful Lemma 7 can be used instead of Lemma 6 while applying the swallowing trick. We can then claim that (proof of this claim is similar to the proof of Claim 8 and is therefore omitted) $\{f_j \ : \ 1 \leqslant j \leqslant m\} \cup \{x_A f \ : \ |A| < s\}$ (where $f(x) = \sum_{j=1}^{n} x_j - i$) is a collection of functions that is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over $\mathbb{F}_p$ which can be obtained as a linear combination of distinct monomials of degree at most s. It then follows that $|\mathcal{F}_i| \leqslant \binom{n}{s}$.

In the rest of the proof, we shall assume the general bound for $|\mathcal{F}_i|$ given by Inequality 2. (Using the $\binom{n}{s}$ upper bound for $|\mathcal{F}_i|$ in place of Inequality 2 when $s \leqslant n + 1 - 2g(t,n)\ln(g(t,n))$ in the rest of the proof will yield the tighter bound for $|\mathcal{F}|$ given in Statement (a) in the theorem.)

Observe that we still do not have an estimate on $|\mathcal{A}_0|$ since $i \equiv \frac{a_l}{b_l} i \pmod p$ when $i \equiv 0 \pmod p$. To overcome this problem, consider the collection $P = \{p_{q+1}, \ldots, p_r\}$ of $r - q$ smallest primes with $p_q \leqslant t < p_{q+1} < \cdots < p_r$ (p_j denotes the j-th prime; $p_1 = 2, p_2 = 3$, and so on) such that for every $A \in \mathcal{F}$, there exists a prime $p \in P$ with $p \nmid |A|$. Note that if we repeat the steps done above for each $p \in P$, we obtain the following upper bound.

$$\begin{aligned} |\mathcal{F}| &\leqslant (p_{q+1} + \cdots + p_r - (r-q)) \binom{n}{s} + (r-q) \sum_{j=1}^{s-1} \binom{n}{j} \\ &< (r-q) \left(p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right) \end{aligned}$$

To obtain a small cardinality set P of the desired requirement, we choose the minimum r such that $p_{q+1} p_{q+2} \cdots p_r > n$. If $t > n - c_1$, for some positive integer constant c_1, then $P = \{p_{q+1}, \ldots, p_{q+c_1}\}$ satisfies the desired requirements of P. We thus have,

$$|\mathcal{F}| < \begin{cases} c_1 \left(p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right), \text{ if } t > n - c_1 \text{ (here } c_1 \text{ is a +ve integer constant)} \\ r \left(p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right), \text{ otherwise} \end{cases} \quad (3)$$

The product of the first k primes is the *primorial function* $p_k\#$ and it is known that $p_k\# = e^{(1+o(1))k \ln k}$. Given a natural number N, let $N\#$ denote the product of all the primes less than or equal to N (some call this the primorial function). It is known that $N\# = e^{(1+o(1))N}$. Since $\frac{p_r\#}{t\#} = p_{k+1} p_{k+2} \cdots p_r$, setting $\frac{e^{(1+o(1))r \ln r}}{e^{(1+o(1))t}} > n$, we get, $r \leqslant \frac{2(2t + \ln n)}{\ln(2t + \ln n)} = g(t, n)$. Using the prime number theorem (see Section 5.1 of [15]), the rth prime p_r is at most $2r \ln r$. Thus, we have $p_r \leqslant 2g(t,n) \ln(g(t,n))$. Substituting for r and p_r in Inequality 3 gives the theorem. □

2.2 When the sets in $\mathcal{F}$ are 'large enough'

In the following theorem, we show that when the sets in a fractional L-intersecting $\mathcal{F}$ are 'large enough', then $|\mathcal{F}|$ is at most n.

Theorem 9. *Let n be a positive integer. Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0,1)$ is an irreducible fraction. Let $\frac{a}{b} = \max(\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s})$. Let $\mathcal{F}$ be a fractional L-intersecting family of subsets of $[n]$ such that for every $A \in \mathcal{F}$, $|A| > \alpha n$, where $\alpha = \max(\frac{1}{2}, \frac{4a-b}{2b})$. Then, $|\mathcal{F}| \leqslant n$.*

Proof. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$. For every $A_i \in \mathcal{F}$, we define its $(+1,-1)$-incidence vector as:

$$X_{A_i}(j) = \begin{cases} +1, & \text{if } j \in A_i \\ -1, & \text{if } j \notin A_i. \end{cases} \quad (4)$$

We prove the theorem by proving the following claim.

Claim 10. *$X_{A_1}, \ldots, X_{A_m}$ are linearly independent in the vector space $\mathbb{R}^n$ over $\mathbb{R}$.*

Assume for contradiction that $X_{A_1}, \ldots, X_{A_m}$ are linearly dependent in the vector space $\mathbb{R}^n$ over $\mathbb{R}$. Then, we have some reals $\lambda_{A_1}, \ldots, \lambda_{A_m}$ where not all of them are zeroes such that

$$\lambda_{A_1} X_{A_1} + \cdots + \lambda_{A_m} X_{A_m} = 0. \tag{5}$$

It is given that, for every $A_i \in \mathcal{F}$, $|A_i| > \frac{n}{2}$. Let $u = (1, 1, \ldots, 1) \in \mathbb{R}^n$ be the all ones vector. Then, $\langle X_{A_i}, u \rangle > 0$, for every $A_i \in \mathcal{F}$. Therefore, if all non-zero λ_{A_i}s in Equation (5) are of the same sign, say positive, then the inner product of u with the L.H.S of Equation (5) would be non-zero which is a contradiction. Hence, we can assume that not all λ_{A_i}s are of the same sign. We rewrite Equation (5) by moving all negative λ_{A_i}s to the R.H.S. Without loss of generality, assume $\lambda_{A_1}, \ldots, \lambda_{A_k}$ are non-negative and the rest are negative. Thus, we have

$$v = \lambda_{A_1} X_{A_1} + \cdots + \lambda_{A_k} X_{A_k} = -(\lambda_{A_{k+1}} X_{A_{k+1}} + \cdots + \lambda_{A_m} X_{A_m}),$$

where v is a non-zero vector.

For any two distinct sets $A, B \in \mathcal{F}$, $\exists \frac{a_i}{b_i} \in L$ such that

$$\langle X_A, X_B \rangle = \begin{cases} n - 2|A| + \frac{4a_i - 2b_i}{b_i}|B|, & \text{if } |A \cap B| = \frac{a_i}{b_i}|B|, \\ n - 2|B| + \frac{4a_i - 2b_i}{b_i}|A|, & \text{otherwise (that is, if } |A \cap B| = \frac{a_i}{b_i}|A|). \end{cases} \tag{6}$$

Since $\frac{a}{b} = \max(\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s})$, we have $\langle X_A, X_B \rangle \leqslant n - 2|A| + \frac{4a-2b}{b}|B|$ or $\langle X_A, X_B \rangle \leqslant n - 2|B| + \frac{4a-2b}{b}|A|$. Applying the fact that the cardinality of every set S in $\mathcal{F}$ satisfies $\alpha n < |S| \leqslant n$, where $\alpha = \max(\frac{1}{2}, \frac{4a-b}{2b})$, we get $\langle X_A, X_B \rangle < 0$. This implies that $\langle v, v \rangle = \langle \lambda_{A_1} X_{A_1} + \cdots + \lambda_{A_k} X_{A_k}, -(\lambda_{A_{k+1}} X_{A_{k+1}} + \cdots + \lambda_{A_m} X_{A_m}) \rangle < 0$ which is a contradiction. This proves the claim and thereby the theorem. □

3 *L* is a singleton set

As explained in Section 1, Fisher's Inequality is a special case of the classical L-intersecting families, where $|L| = 1$. In this section, we study fractional L-intersecting families with $|L| = 1$; a fractional variant of the Fisher's inequality.

3.1 Proof of Theorem 4

Statement of Theorem 4: Let n be a positive integer. Let $\mathcal{G}$ be a fractional L-intersecting families of subsets of $[n]$, where $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0, 1)$, and b is a prime. Then, $|\mathcal{G}| \leqslant (b-1)(n+1)\lceil \frac{\ln n}{\ln b} \rceil + 1$.

Proof. It is easy to see that if $a = 0$, then $|\mathcal{G}| \leqslant n$ with the set of all singleton subsets of $[n]$ forming a tight example to this bound. So assume $a \neq 0$. Let $\mathcal{F} = \mathcal{G} \setminus \mathcal{H}$, where $\mathcal{H} = \{A \in \mathcal{G} \ : \ b \nmid |A|\}$. From the definition of a fractional $\frac{a}{b}$-intersecting family it is clear that $|\mathcal{H}| \leqslant 1$. The rest of the proof is to show that $|\mathcal{F}| \leqslant (b-1)(n+1)\lceil \frac{\ln n}{\ln b} \rceil$. We do this by partitioning $\mathcal{F}$ into $(b-1)\lceil \log_b n \rceil$ parts and then showing that each part is of size at most $n+1$. We define F_i^j as

$$\mathcal{F}_i^j = \{A \in \mathcal{F} \mid |A| \equiv j \pmod{i}\}.$$

Since b divides $|A|$, for every $A \in \mathcal{F}$, under this definition $\mathcal{F}$ can be partitioned into families $\mathcal{F}_{b^k}^{ib^{k-1}}$, where $2 \leqslant k \leqslant \lceil \log_b n \rceil$ and $1 \leqslant i \leqslant b-1$. We show that, for every $i \in [b-1]$ and for every $2 \leqslant k \leqslant \lceil \log_b n \rceil$, $|\mathcal{F}_{b^k}^{ib^{k-1}}| \leqslant n+1$.

In order to estimate $|\mathcal{F}_{b^k}^{ib^{k-1}}|$, for each $A \in \mathcal{F}_{b^k}^{ib^{k-1}}$, create a vector X_A as follows:

$$X_A(j) = \begin{cases} \frac{1}{\sqrt{b^{k-2}}}, \text{ if } j \in A; \\ 0, \text{ otherwise.} \end{cases}$$

Note that, for $A, B \in \mathcal{F}_{b^k}^{ib^{k-1}}$

$$\langle X_A, X_B \rangle \equiv \begin{cases} b \pmod{b^2}, \text{ if } A = B, \\ ai \pmod{b}, \text{ if } A \neq B, \end{cases} \tag{7}$$

Let $|\mathcal{F}_{b^k}^{ib^{k-1}}| = m$. Let $M_{k,i}$ denote the $m \times n$ matrix formed by taking X_As as rows for each $A \in \mathcal{F}_{b^k}^{ib^{k-1}}$. Then, $|\mathcal{F}_{b^k}^{ib^{k-1}}| \leqslant n+1$ can be proved by considering $B = M_{k,i} \times M_{k,i}^T$ and showing that $B - aiJ$ (, where J is the $m \times m$ all 1 matrix,) has full rank; determinant of $B - aiJ$ is non-zero since the only term not divisible by the prime b in the expansion of its determinant comes from the product of all the diagonals (note that $a < b$, $i < b$, and since b is a prime, we have $b \nmid ai$). □

We shall call $\mathcal{F}$ a *bisection closed family* if $\mathcal{F}$ is a fractional L-intersecting family where $L = \{\frac{1}{2}\}$. We have two different constructions of families that are bisection closed and are of cardinality $\frac{3n}{2} - 2$ on $[n]$.

Example 11. Let n be an even positive integer. Let $\mathcal{B}$ denote the collection of 2-sized sets that contain only 1 as a common element in any two sets, i.e. $\{1,2\}, \{1,3\}, \ldots, \{1,n\}$; and let $\mathcal{C}$ denote collection of 4-sized sets that contain only $\{1,2\}$ as common elements, i.e. $\{1,2,3,4\}$, $\{1,2,5,6\}, \ldots, \{1,2,n-1,n\}$. It is not hard to see that $\mathcal{B} \cup \mathcal{C}$ is indeed bisection closed.

Example 12. The second example of a bisection closed family of cardinality $\frac{3n}{2} - 2$ comes from *Recursive Hadamard matrices.* A Recursive Hadamard matrix $H(k)$ of size $2^k \times 2^k$ can be obtained from $H(k-1)$ of size $2^{k-1} \times 2^{k-1}$ as follows

$$H(k) = \begin{bmatrix} H(k-1) & H(k-1) \\ H(k-1) & -H(k-1) \end{bmatrix},$$

where $H(0) = 1$. Now consider the matrix:

$$M(k) = \begin{bmatrix} H(k-1) & H(k-1) \\ H(k-1) & -H(k-1) \\ H(k-1) & J(k-1) \end{bmatrix}$$

, where $J(k-1)$ denotes the $2^{k-1} \times 2^{k-1}$ all 1s' matrix.

Let $M'(k)$ be the matrix obtained from $M(k)$ by removing the first and the (2^k+1)th rows and replacing the -1's by 1's and 1's by 0's. $M'(k)$ is clearly bisection closed and has cardinality $\frac{3n}{2}-2$, where $n=2^k$.

3.2 Restricting the cardinalities of the sets in $\mathcal{F}$

When $L=\{\frac{a}{b}\}$, where b is a prime, Theorem 4 yields an upper bound of $O(\frac{b}{\log b} n \log n)$ for $|\mathcal{F}|$. However, we believe that when $|L|=1$, the cardinality of any fractional L-intersecting family on $[n]$ would be at most cn, where $c>0$ is a constant. To this end, we show in Theorem 14 that when the sizes of the sets in $\mathcal{F}$ are restricted, we can achieve this.

The following lemma is crucial to the proof of Theorem 14.

Lemma 13. *[1, 4] Let A be an $m\times m$ real symmetric matrix with $a_{i,i}=1$ and $|a_{i,j}|\leqslant \epsilon$ for all $i\neq j$. Let $tr(A)$ denote the trace of A, i.e., the sum of the diagonal entries of A. Let $rk(A)$ denote the rank of A. Then,*

$$rk(A) \geqslant \frac{(tr(A))^2}{tr(A^2)} \geqslant \frac{m}{1+(m-1)\epsilon^2}.$$

Proof. Let $\lambda_1,\dots,\lambda_m$ denote the eigenvalues of A. Since only $rk(A)$ eigenvalues of A are non-zero, $(tr(A))^2 = (\sum_{i=1}^{m}\lambda_i)^2 = (\sum_{i=1}^{rk(A)}\lambda_i)^2 \leqslant rk(A)\sum_{i=1}^{rk(A)}\lambda_i^2 = rk(A)tr(A^2)$, where the inequality follows from the Cauchy-Schwartz Inequality. Thus, $rk(A)\geqslant \frac{(tr(A))^2}{tr(A^2)}$. Substituting $tr(A)=m$ and $tr(A^2)=m+m(m-1)\epsilon^2$ in the above inequality proves the theorem. □

Theorem 14. *Let n be a positive integer and let $\delta>1$. Let $\mathcal{F}$ be a fractional L-intersecting family of subsets of $[n]$, where $L=\{\frac{a}{b}\}$, $\frac{a}{b}\in[0,1)$ is an irreducible fraction and for every $A\in\mathcal{F}$, $|A|$ in an integer in the range $\left[\frac{b}{4(b-a)}n-\frac{b}{4a\delta}\sqrt{n}, \frac{b}{4(b-a)}n+\frac{b}{4a\delta}\sqrt{n}\right]$. Then, $|\mathcal{F}|<\frac{\delta^2}{\delta^2-1}n$.*

Proof. For any $A\in\mathcal{F}$, let $Y_A\in\mathbb{R}^n$ be a vector defined as:

$$Y_A(j)=\begin{cases}+\frac{1}{\sqrt{n}}, & \text{if } j\in A\\ -\frac{1}{\sqrt{n}}, & \text{if } j\notin A.\end{cases}$$

Clearly, $\langle Y_A,Y_A\rangle=1$. For any two distinct sets $A,B\in\mathcal{F}$, we have

$$\langle Y_A,Y_B\rangle=\begin{cases}\frac{n-2|A|+\frac{4a-2b}{b}|B|}{n}, & \text{if } |A\cap B|=\frac{a}{b}|B|,\\ \frac{n-2|B|+\frac{4a-2b}{b}|A|}{n}, & \text{otherwise (that is, if } |A\cap B|=\frac{a}{b}|A|).\end{cases} \tag{8}$$

Suppose $\mathcal{F}=\{A_1,\dots,A_m\}$. Let B be the $m\times n$ matrix with $Y_{A_1},\dots,Y_{A_m}$ as its rows. Then, from Equation 8, it follows that BB^T is an $m\times m$ real symmetric matrix with the diagonal entries being 1 and the absolute value of any other entry being at most $\frac{1}{\delta\sqrt{n}}$. Applying Lemma 13, we have $n\geqslant rk(BB^T)\geqslant \frac{m}{1+\frac{m-1}{\delta^2 n}} > \frac{m}{1+\frac{m}{\delta^2 n}}$. Thus, $n+\frac{m}{\delta^2}>m$ or $m<\frac{\delta^2}{\delta^2-1}n$. □

4 Discussion

In Theorem 1, we gave a general upper bound for $|\mathcal{F}|$, where $\mathcal{F}$ is a fractional L-intersecting family. In Section 1, we also gave an example to show that this bound is asymptotically tight up to a factor of $\frac{\ln^2 n}{\ln \ln n}$, when s $(= |L|)$ is a constant. However, when s is a constant, we believe that $|\mathcal{F}| \in \Theta(n^s)$.

Consider the following special case for a fractional L-intersecting family $\mathcal{F}$, where $L = \{\frac{1}{2}\}$. We call such a family a bisection-closed family (see definition in Section 3).

Conjecture 15. If $\mathcal{F}$ is a bisection-closed family, then $|\mathcal{F}| \leqslant cn$, where $c > 0$ is a constant.

We have not been able to find an example of a bisection-closed family of size $2n$ or more.

The problem of determining a linear sized upper bound for the size of any bisection-closed family leads us to pose the following question:

Open Problem 16. Suppose $0 < a_1 \leqslant \cdots \leqslant a_n$ are n distinct reals. Let $\mathcal{M}_n(a_1, \ldots, a_n)$ denote the set of all symmetric matrices M satisfying $m_{ij} \in \{a_i, a_j\}$ for $i \neq j$ and $m_{ii} = 0$ for all i. Then, does there exist an absolute constant $c > 0$ such that $rk(M) \geqslant cn$, for all $M \in \mathcal{M}_n(a_1, \ldots, a_n)$?

To see how this question ties in with our problem, suppose that a family $\mathcal{F} \subset \mathcal{P}([n])$ is a bisection closed family, i.e., for $A, B \in \mathcal{F}$ and $A \neq B$ then $|A \cap B| \in \{|A|/2, |B|/2\}$. For simplicity, let us write $\mathcal{F} = \{A_1, \ldots, A_m\}$ and denote $|A_i| = a_i$ where the a_i are arranged in ascending order. We say A bisects B if $|A \cap B| = |B|/2$. For each $A \in \mathcal{F}$, let $\mathbf{u}_A \in \mathbb{R}^n$ where $\mathbf{u}_A(i) = 1$ if $i \in A$ and -1 if $i \notin A$. Then note that

$$\begin{aligned} \langle \mathbf{u}_A, \mathbf{u}_B \rangle &= n - 2|A| \qquad \text{if } A \text{ bisects } B, \\ &= n - 2|B| \qquad \text{if } B \text{ bisects } A, \\ \| \mathbf{u}_A \|^2 &= n. \end{aligned}$$

Consider the $m \times m$ matrix M whose rows and columns are indexed by the members of $\mathcal{F}$, with $M_{A,B} = \langle \mathbf{u}_A, \mathbf{u}_B \rangle$. Then, since M is a Gram matrix of vectors in $\mathbb{R}^n$, it follows that $rk(M) \leqslant n$. If $\mathcal{X} = \frac{1}{2}(nJ - M)$, where J is the all ones matrix of order m, then $rk(\mathcal{X}) \leqslant n + 1$. But note that $\mathcal{X} \in \mathcal{M}(a_1, \ldots, a_m)$. So, if the answer to the aforementioned open problem is 'yes', then $rk(\mathcal{X}) \geqslant cm$. This gives $cm \leqslant r(\mathcal{X}) \leqslant n + 1$ which in turn gives $m \leqslant c^{-1}(n + 1)$.

The problem of determining the maximum size of a fractional L-intersecting family is far from robust in the following sense. Suppose $L = \{1/2\}$ and we consider the problem of determining the size of an 'ε-approximately fractional L-intersecting family,' i.e., for any $A \neq B$ we have that at least one of $\frac{|A \cap B|}{|A|}, \frac{|A \cap B|}{|B|} \in (1/2 - \varepsilon, 1/2 + \varepsilon)$ for small $\varepsilon > 0$, then such families can in fact be exponentially large in size. Let each set A_i be chosen uniformly and independently at random from $\mathcal{P}([n])$. Then since each $|A_i|$ and $|A_i \cap A_j|$ are independent binomial $B(n, 1/2)$ and $B(n, 1/4)$ respectively, by standard Chernoff bounds (see [12], chapter 5), it follows (by straightforward computations) that one can get such a family of cardinality at least $e^{2\varepsilon^2 n/75}$. In fact this same construction gives super-polynomial sized families even if $\varepsilon = n^{-1/2+\delta}$ for any fixed $\delta > 0$.

Another interesting facet of the fractional intersection notion is the following extension of l-avoiding families [6, 10] [1]. A set B *bisects* another set A if $|A \cap B| = \frac{|A|}{2}$. A family $\mathcal{F}$ of even subsets of $[n]$ is called *fractional* $(\frac{1}{2})$*-avoiding* (or *bisection-free*) if for every $A, B \in \mathcal{F}$, neither B bisects A nor A bisects B (if we allow odd subsets in the definition of a fractional $(\frac{1}{2})$-avoiding family, then the set of all the odd-sized subsets on $[n]$ is an example of one such family). Let $\bar{\vartheta}(n)$ denote the maximum cardinality of a fractional $(\frac{1}{2})$-avoiding family on $[n]$. Let $A, B \subseteq [n]$ such that $|A| > \frac{2n}{3}$ and $|B| > \frac{2n}{3}$. It is not very hard to see that $|A \cap B| > n/3$ whereas $|A \cap ([n] \setminus B)| < n/3$. So, neither A can bisect B nor B can bisect A. Therefore, if we construct a family $\mathcal{F} = \{A \subseteq [n] \mid |A| > \frac{2n}{3},\ |A| \text{ is even.}\}$, $\mathcal{F}$ is fractional $(\frac{1}{2})$-avoiding. Moreover, $|\mathcal{F}| = \sum_{2|i, i=0}^{\frac{n}{3}-1} \binom{n}{i} > 1.88^n$, for sufficiently large n (using Stirling's formula). Let us now try to find an upper bound to the cardinality of a fractional $(\frac{1}{2})$-avoiding family. An application of a result of Frankl and Rödl [6, Corollary 1.6] gives the following theorem for the cardinalities of l-avoiding families as a corollary (see [10, Theorem 1.1]).

Theorem 17. *[6, 10] Let $\alpha, \epsilon \in (0, 1)$ with $\epsilon \leqslant \frac{\alpha}{2}$. Let $k = \lfloor \alpha n \rfloor$ and $l \in [\max(0, 2k - n) + \epsilon n, k - \epsilon n]$. Then any l-avoiding family $A \subseteq \binom{[n]}{k}$ satisfies $|A| \leqslant (1 - \delta)^n \binom{n}{k}$ where $\delta = \delta(\alpha, \epsilon) > 0$.*

For any fractional $(\frac{1}{2})$-avoiding family $\mathcal{F}$, any $\mathcal{F}' \subseteq \mathcal{F}$ consisting of sets of cardinality l is $\frac{l}{2}$-avoiding. So, given any fractional $(\frac{1}{2})$-avoiding family $\mathcal{F}$, split $\mathcal{F}$ into families $\mathcal{F}_{\leqslant \frac{n}{3}-1}, \mathcal{F}_{\frac{n}{3}}, \ldots, \mathcal{F}_{2\frac{n}{3}}, \mathcal{F}_{\geqslant \frac{2n}{3}+1}$. From Theorem 17, we know that each $\mathcal{F}_i$ has a cardinality at most $(1 - \delta_i)^n \binom{n}{i}$ for $\frac{n}{3} \leqslant i \leqslant \frac{2n}{3}$. Let $\delta = \min(\delta_{\frac{n}{3}}, \ldots, \delta_{\frac{2n}{3}})$. Then $\sum_{i=\frac{n}{3}}^{\frac{2n}{3}} |\mathcal{F}_i| \leqslant ((1 - \delta)2)^n$. Further, $|\mathcal{F}_{\leqslant \frac{n}{3}-1}| \leqslant \sum_{i=0}^{\frac{n}{3}-1} \binom{n}{i}$ and $|\mathcal{F}_{\geqslant \frac{2n}{3}+1}| \leqslant \sum_{i=0}^{\frac{n}{3}-1} \binom{n}{i} < 2^{nH(\frac{1}{3})} < 1.89^n$, where $H(\nu) = -\nu \log_2 \nu - (1 - \nu) \log_2 (1 - \nu)$ is the binary entropy function. Thus, for sufficiently large values of n, $1.88^n \leqslant \bar{\vartheta}(n) \leqslant ((1 - \epsilon)2)^n$, for some $0 < \epsilon \leqslant 0.06$.

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