Short course on General Relativity

What GR attempts to solve?

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Lecture # 1

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Overview

- Newtonian Gravity
- Thought experiments on Gravity — Schild '67
  - Gravitational redshift
  - Time Dilation
  - Non-flat geoemetry
  - Einstein's lift
- Equivalence principle

Overview

- Newtonian Gravity
- Thought experiments on Gravity — Schild '67
  - Gravitational redshift
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- Equivalence principle
- Relativity and Gravitation
  - Tides and geodesic deviation
  - There is no universal inertial frame
  - What does GR plan to do about it?

Geometric Units:

- In Special relativity, Space and time are no-different. We set $c = 1$.
- We will also set $G = 1$. Mass and Distance have same dimension!
- $[\text{Force}] = [\text{Velocity}] = [L]^0, [\text{Energy}] = L$

Newtonian Gravity

Newtonian Picture (1687)

Laws of motion

- Inertial frame: No Force $\Longrightarrow$ constant motion
- Inertial mass: Measures the reluctance to be set into motion

$$F = m_i a \qquad m_i : \text{inertial mass}$$

Newtonian Picture (1687)

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Law of Gravitation

Object's mass is determined by measuring how much gravity force it feels.

$$\vec{F}_g = \frac{-GM\, m_g}{r^2}\hat{r} \qquad m_g : \text{Gravitational mass}$$

Grav. potential $\Phi_g = -\dfrac{GM}{r} \Longrightarrow \vec{F}_g = -m_g \nabla \Phi_g$

Is $m_i = m_g$?

- m_i describes inertial properties of object regardless of origin of force

$$m_i = \frac{\text{External force}}{\text{Particle's resistence}}$$

- m_g describes the strength of the gravitational force
- Combining the two, we get $$a = \frac{GM}{r^2}\frac{m_g}{m_i}$$

- Galileo All objects fall in gravitational field in the same way.

$$\implies \quad m_g = m_i$$

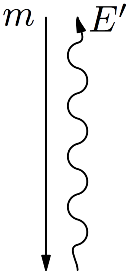
- Best test of $m_g = m_i$ comes from how Earth and Moon fall towards the Sun. The two agree to about 1 in 10^{-13}s.

This equality is elevated to a principle in General relativity

Thought experiments on Gravitation

1: Gravitational Redshift — Schild ’67

$$V = -\frac{M}{r}$$



$$m + mgh = E$$

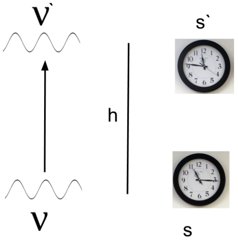
- Drop a ball from height h from earth.
- Assume, we convert all the energy into a Photon of energy E. Send it upwards towards the original position.
- Photon reaches the point with energy E' which we convert it back into particle. Repeat the process.
- Photon looses energy

$$E' = m = h\nu' = \frac{E}{1+gh} < h\nu$$

- Energy conservation $\Longrightarrow$ Photon looses energy as it climbs gravitational potential!
- Gravity interacts with all forms of energy!

Gravitational Redshift confirmed by Pound-Rebka experiment in 1960.

2: Time Dilation Schild '67



- Consider a radio station at ground constantly sending radio signal of frequency ν.
- A receiver at level h receives the signal at lower-frequency ν'.
- Entire set-up is stationary $\Longrightarrow$ Waves can not originate/disappear between sender and reciever $\Longrightarrow$ Change in frequency is not possible!
- Time flows at different rates at different levels of gravitational field
- Clock measure n crests in s time

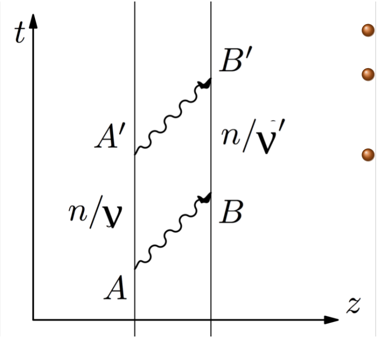
$$\nu = n/s \qquad \nu' = n/s'$$

Gravitational time dilation

$$\frac{s - s'}{s'} \propto h$$

3: $1 + 2 \Longrightarrow$ non-flat geometry Schild '67

$$V = -\frac{M}{r}$$

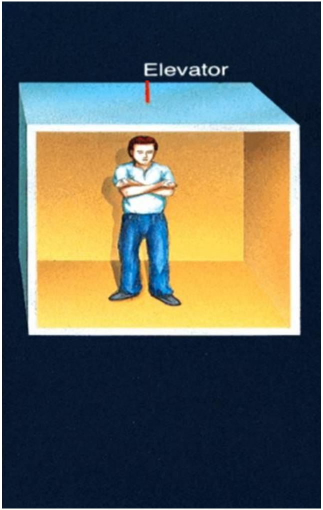


- Send EM waves from A to B. Photons will be redshifted to new frequency (ν').
- After some periods n, repeat the same expt.
- Both the waves follow same path. World lines AB and $A'B'$ are parallel.
- Because of the red-shift, the time intervals AA' and BB' measured by the local clocks are not the same! $AA' \equiv s \neq s' \equiv BB'$

$\Longrightarrow ABB'A'$ is not a parallelogram!

In the presence of static massive object, geometry of space-time is not flat.

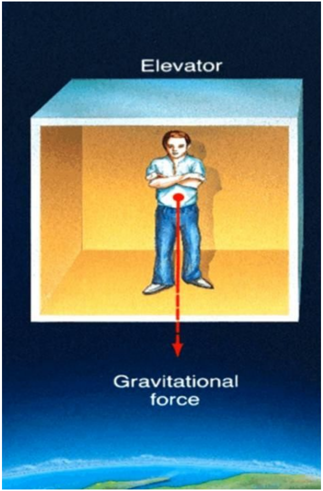
Einstein's lift



Stage 1

- Lift is placed in a far away region
  $\Longrightarrow$ No gravity
- Internal observer measures that free bodies move with uniform motion and have no acceleration.

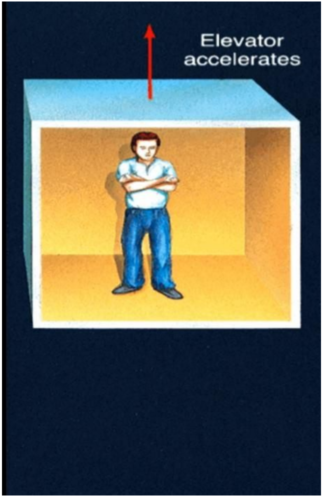
Einstein's lift



Stage 2

- Place the elevator, in a gravitational field, and let it fall freely.
- Acceleration of gravity is the same for all bodies, including the wall of the elevator
- the internal observer can not distinguish this set-up from the Stage 1!

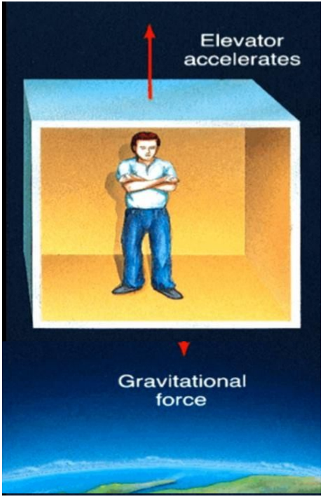
Einstein's lift



Stage 3

- Elevator is brought again into empty space. It is uniformly accelerated by a rocket engine.
- All bodies inside will appear accelerated by an acceleration a.
- This will be exactly opposite to that of the elevator and is common to all bodies.

Einstein's lift



Stage 4

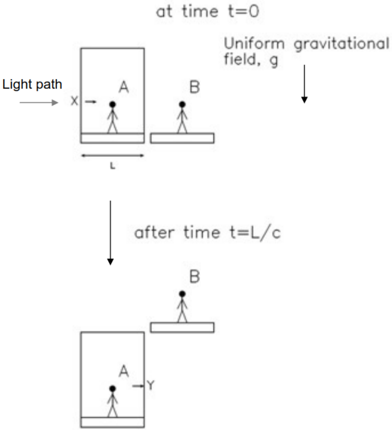
- Let us now hang the elevator in a gravitational field with $g = a$
- Internal observer will again find motions in no way distinguishable from those of the third experiment.

Einstein's lift

Key Inference

- Shows degree of equivalence between inertia and gravity
- Two equivalent Inertial frames:
    1. being far away from gravitating matter
    2. freely falling in a gravitational field
- Equivalence Principle
  Uniform gravitational fields are equivalent to frames that accelerate uniformly relative to inertial frames.
- We know that presence of massive object makes geometry non-flat
  $\Longrightarrow$ Accelerated frames also make geometry non-flat.

Equivalence principle and light



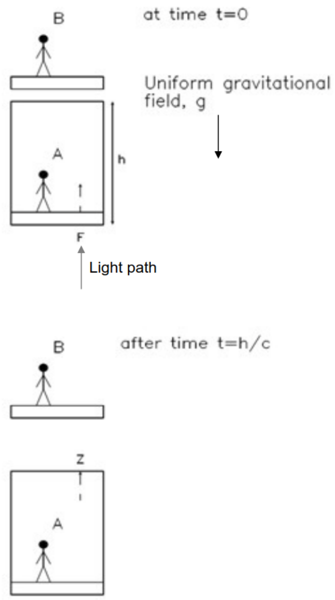
- $t = 0$
  Light enters lift horizontally at X.
  Lift also falls free at that time.

- $t = L/c$
  Observer A sees the light reach opposite wall at Y in a straight line.

- Observer B will see, from X to Y, the light path curved.

  Interpretation: Gravitational field

Equivalence principle and light



- $t = 0$
  Light enters lift vertically at F.
  Lift also falls free at that time.

- $t = h/c$
  Observer A sees the light reach ceiling at Z with same frequency.

- Observer B will see, the light redshifted.

  Interpretation: Gravitational field

Recap

- Presence of Massive Object $\Longrightarrow$ Non-flat space-time
- Uniform gravitational fields are equivalent to accelerated frames
- Trajectories of freely falling particles in curved space-time are Geodesics

Geodesic Deviation: Newtonian context

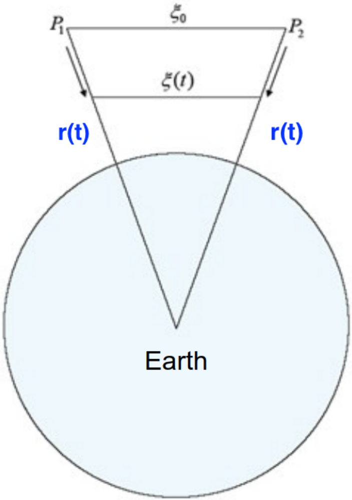
Consider 2-D Surfaces

Flat Surface: Zero deviation $\Longrightarrow$ zero curvature

2-Sphere: Geodesics Converge $\Longrightarrow$ Positive curvature

Hyperboloid: Geodesics diverge $\Longrightarrow$ Negative curvature

Geodesic Deviation: Newtonian context



- Consider two objects at P_1 and P_2. freely falling towards the Earth.
- At $r(t)$, they are separated by $\xi(t)$.
- Two similar triangles have the property

$$\frac{\xi(t)}{r(t)} = \frac{\xi_0}{r_0} = k \Longrightarrow \ddot{\xi} = k\ddot{r} = -k\,\frac{GM}{r^2}$$

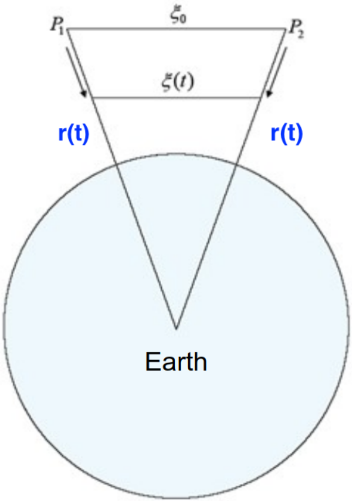
- We have

$$\ddot{\xi} = -\frac{GM}{r^3}\xi \quad \text{or} \quad \frac{d^2\xi}{d(ct)^2} = -\frac{\xi}{\mathcal{R}^2}$$

where

$$\frac{1}{\mathcal{R}^2} = \frac{GM}{r^3 c^2}$$

Geodesic Deviation: Newtonian context



- Inertial frames attached to freely falling particles approach each other at an increasing speed!
- What is $\mathcal{R}$? $\mathcal{R} = \left(\frac{GM}{r^3c^2}\right)^{-1/2}$
  - It has the dimension of [L].
  - Near Earth: $\mathcal{R} \sim 10^{11} m \gg R_{\mathrm{Earth}}$
- $\mathcal{R}$ represents radius of curvature of spacetime near Earth's surface.

  $\Longrightarrow$ Space-time is nearly flat near Earth's surface!

$\mathcal{R}$ for different objects

Object	Mass	Radius	$\mathcal{R}$	$\mathcal{R}/\mathrm{radius}$
Earth	10^{24} Kg	10^{6} m	10^{11} m	10^{5}
Sun	10^{30} Kg	10^{9} m	10^{14} m	10^{5}
Neutron Star	10^{30} Kg	10^{4}m	10^{7} m	10^{3}
Black-holes	10^{30}	10^{3} m	10^{5} m	10^{2}

What next?

- Presence of gravity $\Longrightarrow$ Curved space-time
- STR requires existence of inertial frames.
- Curved space-time corresponds to inequivalent Inertial frames
- Cannot use STR to calculate what the other inertial observer would measure in their frame
- Can not use STR + Newtonian Gravity
- Need to come-up with completely different formalism
  $\Longrightarrow$ General Relativity