Short course on General Relativity

Machinery of GR

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Lecture # 2

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Lecture 1: Recap

- Gravity interacts with all forms of matter including photons.
- Presence of Massive Object $\Longrightarrow$ Non-flat space-time
- Uniform gravitational fields are equivalent to accelerated frames Equivalence Principle (EP)
- Trajectories of freely falling particles in curved space-time are Geodesics
- Near-by geodesics can provide information about the nature of the space.

Overview of Lecture 2

- New Length scale: compactness parameter
- Gravitational Redshift and Newtonian potential
- Curved Space-time
- Riemann tensor
- Einstein's equations

Units and new length scale

Geometric Units

- In Special relativity, Space and time are no-different. We set $c = 1$.
- We will also set $G = 1$. Mass and Distance have same dimension!

New Length scale

- G and c are fundamental constants
- Using G, c and Mass, we can define length

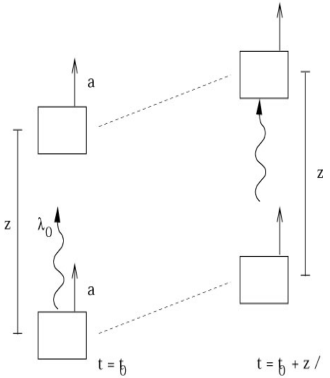
$$r_h = \frac{GM}{c^2}$$

- r_h/R is a measure to know when the GR effects need to included!

Object	r_h (m)	r_h/R
Earth	10^{-2}	10^{-8}
Sun	10^{3}	10^{-5}
Neutron Star	10^{3}	0.1
Black-holes	10^{3}	1

Redshift and Newtonian potential

Gravitational Redshift



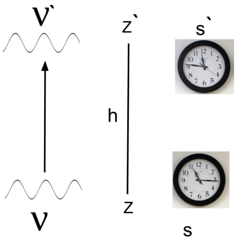
- Two boxes at constant distance z always!
- Both are at constant acceleration a
- $t = t_0$: emits photon at wavelength λ
- Light takes $\Delta t = z/c$ to reach upper box.
- During Δt, boxes pick additional velocity $\Delta v = a\Delta t = az/c$
- Photon when it reached upper box will be redshifted by Doppler effect

$$\frac{\Delta\lambda}{\lambda} = \frac{\delta v}{c} \simeq \frac{az}{c^2}$$

Credit: Carroll '04

Gravitational Redshift

Elevetor with constant g is same as static antenna's on Earth's surface (EP)



- Radio waves emitted from ground will be redshifted:

$$\frac{\Delta\lambda}{\lambda} = \frac{gz}{c^2}$$

- In generel, $g = \nabla\Phi$ (No -ve Sign??)

$$\frac{\Delta\lambda}{\lambda_0} = \frac{1}{c^2}\int \partial_z \Phi dz = \frac{\Delta\Phi}{c^2}$$

- Interms of frequency/clock time we have

$$\nu = \nu'\left(1 + \frac{\Delta\Phi}{c^2}\right) \quad \delta S' = \delta S\left(1 + \frac{\Delta\Phi}{c^2}\right)$$

Metric and Newtonian potential

- Time-dilation can be rewritten as

$$\frac{dS(z')}{dS(z)} = 1 + \frac{\Phi(z) - \Phi(z')}{c^2} \simeq \frac{1 + \Phi(z)/c^2}{1 + \Phi(z')/c^2}$$

$$(1 + \Phi(z')/c^2) dS(z') = (1 + \Phi(z)/c^2) dS(z)$$

- Clocks are stationary $\Longrightarrow d\vec{l}^2 = 0$
- Proper-time of each clock is

$$d\tau^2 = (1 + \Phi(z')/c^2)^2 dS^2(z') - d\vec{l}^2 = (1 + \Phi(z)/c^2)^2 dS^2(z) - d\vec{l}^2$$

- $g_{00}(z) \simeq 1 + 2\Phi(z)/c^2$ Space-time is curved!
- Newtonian potential appears as a component of $g_{\mu\nu}$.

Metric and Newtonian potential

- We are forced to consider general metric $g_{\mu\nu}$. Line-element:

$$ds^2 = g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu \qquad x^\alpha : \text{ arbitrary}$$

- A line-element specifies geometry. However, many line-elements may specify the same geometry

$$dl^2 = dx^2 + dy^2 + dz^2; \qquad dl^2 = dr^2 + r^2 d\theta^2 + dz^2$$

- Aim is to formulate laws in any coordiante system.

  Changing coordinates $x^\alpha \to \tilde{x}^\alpha$ should not change Physics!

Curved space-time

Curved space-time

- $g_{\mu\nu}$ has 10 independent components

- EP: We can define a local inertial frame. We can perform a coordinate transformation (like in a free fall)

$$x^{\alpha} \to \tilde{x^{\alpha}} \qquad g_{\mu\nu} \to \tilde{g}_{\mu\nu}|_P = \eta_{\mu\nu}$$

What happens in the neighbourhood of P?

- We can always choose $\tilde{x^{\alpha}}$ such that first derivative of metric vanish!

$$\tilde{g}_{\mu\nu}|_P = \eta_{\mu\nu} \qquad \left.\frac{\partial \tilde{g}_{\mu\nu}}{\partial \tilde{x}^{\alpha}}\right|_P = 0$$

- What happens to second derivative of the metric?

Curved space-time

- Under $x^\alpha \to \tilde{x}^\alpha$, $g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x})$ such that

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \sum_{\alpha\beta} g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu}$$

- Consider a coordinate transformation in the vicinity of P:

$$x^\mu(\tilde{x}) = x^\mu(\tilde{x}_P) + \left.\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\right|_P \delta x^\alpha + \frac{1}{2} \left.\frac{\partial^2 x^\mu}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta}\right|_P \delta\tilde{x}^\alpha \delta\tilde{x}^\beta + \frac{1}{6} \left.\frac{\partial^3 x^\mu}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta \partial \tilde{x}^\gamma}\right|_P \delta\tilde{x}^\alpha \delta\tilde{x}^\beta \delta\tilde{x}^\gamma$$

- In the vicinity of P, the transformation is specified by constants $\partial x^\mu / \partial \tilde{x}^\alpha|_P$

Curved space-time

1. To obtain $\tilde{g}_{\mu\nu}|_P = \eta_{\mu\nu}$. We need 10 conditions.

   $\partial x^\mu / \partial \tilde{x}^\alpha|_P$ provide 16 constants. 6 extra corresponds to 3 boosts and 3 rotations (Minkowski space-time).

2. To obtain $\left.\frac{\partial \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\alpha}\right|_P = 0$, we need to impose $10 \times 4 = 40$ conditions.

   $\left.\partial^2 x^\mu / \partial \tilde{x}^\alpha \partial \tilde{x}^\beta\right|_P$ gives exactly 40 numbers. Completely detemined!

3. To obtain $\left.\frac{\partial^2 \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\alpha \tilde{x}^\beta}\right|_P = 0$. We need to impose $10 \times 10 = 100$ conditios.

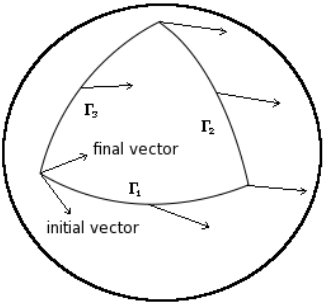
   $\left.\frac{\partial^3 x^\mu}{\partial \tilde{x}^\alpha \partial \tilde{x}^\beta \partial \tilde{x}^\gamma}\right|_P$ has only 80 numbers. Short of 20!

Locally Inertial frame has deviations from Minkowski that depends on the second derivative of the metric!

20 "degrees of freedom" are the 20 independent components of Riemann curvature tensor!

Riemann tensor

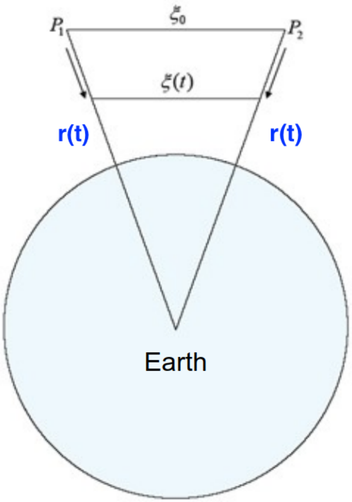
Riemann tensor



$$R^{\beta}_{\lambda\nu\alpha} = \frac{D^{\beta}}{dx^{\lambda} dx^{\nu} dx^{\alpha}}$$

- Take a vector V^{σ} around a closed loop.
- In general the initial vector and the final vector will not be the same.
- Initially, V points in the direction j with magnitude dx^j.
- At each step, do not turn V that you carry, even though the coordinates may themselves turn. Coordinate description of V change.
- Difference between initial and final vector will have components D^1, D^2, D^3, D^4.
- $R^{\beta}_{\lambda\nu\alpha}$ is a (1, 3) tensor known as the Riemann tensor (or simply "curvature tensor").

Geodesic Deviation



- Consider two objects at P_1 and P_2. freely falling towards the Earth.
- In Newtonian theory, we have

$$\ddot{x}^i = -\frac{\partial \Phi(x^j)}{\partial x^i} \quad \ddot{x}^i + \ddot{\xi}^i = -\frac{\partial \Phi(x^j + \xi^j)}{\partial x^i}$$

$$\ddot{\xi}^i = -\frac{\partial^2 \Phi}{\partial x^i \partial x^j} \xi^j \simeq -\frac{GM}{r^3} \xi^i$$

- In GR, we have

$$\frac{d^2 \xi^\alpha}{dt^2} = R^\alpha_{\mu\beta\nu} \frac{dx^\mu}{dt} \frac{dx^\beta}{dt} \xi^\nu$$

Einstein's equations

Einstein's equations

Geodesic Deviation equation in Newtonian and GR

$$\ddot{\xi}^i = -\frac{\partial^2 \Phi}{\partial x^i \partial x^j}\xi^j \qquad \frac{d^2\xi^\alpha}{dt^2} = R^\alpha_{\mu\beta\nu}\frac{dx^\mu}{dt}\frac{dx^\beta}{dt}\xi^\nu$$

- In Newtonian Gravity, Φ satisfies

$$\nabla^2\Phi = -4\pi G\rho(x) \quad \text{(Poisson)} \quad \nabla^2\Phi = 0 \quad \text{(Laplace)}$$

- Comparing the two equations suggest

$$R^\alpha_{\mu\beta\nu}\frac{dx^\mu}{dt}\frac{dx^\beta}{dt} \quad \text{analogous to} \quad \Phi_{,ij}$$

- Particle velocities are arbitrary.

$$\nabla^2\Phi \quad \text{in Poisson is analogous to} \quad R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \quad \text{in GR}$$

- Guess: Relativistic analogue of Laplace equation is

$$R_{\mu\nu} = 0 \qquad R_{\mu\nu} \text{ is Ricci tensor}$$

Einstein's equations

- General relativity replaces the time-independent Poisson equation

$$\nabla^2\Phi = -4\pi\, G\, \rho_{\mathrm{mass}}$$

by set of 10 dynamical equations (like Maxwell's equations)

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{Space-time geometry}} \quad = \quad \underbrace{\frac{8\pi G}{c^4} T_{\mu\nu}}_{\text{Energy-momentum density of matter}}$$

(includes all forms of energy and matter)

- Coupled non-linear partial differential equations; solutions can be obtained only for simple geometries.

Full machinery of GR

Metric : Distance between neighbouring points in space-time.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Example: Schwarzschild – Metric around a spherical mass m

$$ds^2 = (1 - 2m/r)dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Full machinery of GR

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Geodesics: Use minimum distance $\delta \int ds = 0$ to obtain geodesics

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \qquad \text{Geodesic Equation}$$

Connection coeff.; $\Gamma^i_{jk} \sim$ Inertial forces from reference frame's motion

$$\gamma^\mu_{\alpha\beta} = \frac{g^{\mu\gamma}}{2}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma})$$

Full machinery of GR

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Curvature: Geodesic deviation is governed by the Curvature

$$\frac{d^2\xi^\alpha}{dt^2} = R^\alpha_{\mu\beta\nu}\frac{dx^\mu}{dt}\frac{dx^\beta}{dt}\xi^\nu$$

Riemann Curvature $R^\alpha_{\beta\gamma\delta}$ have $20/256$ independent components.

$$R^\alpha_{\beta\gamma\delta} = \frac{\partial}{\partial x^\gamma}\Gamma^\alpha_{\beta\delta} - \frac{\partial}{\partial x^\delta}\Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\epsilon}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma}$$

Full machinery of GR

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Einstein's Equations: $G_{\mu\nu} === 8\pi G T_{\mu\nu}$ $\qquad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$

$G_{\mu\nu} \downarrow$ Einstein tensor
Spacetime dynamics

$T_{\mu\nu} \downarrow$ Stress energy tensor
Distribution of matter field

Full machinery of GR

Einstein's Equations: $G_{\mu\nu} === 8\pi G T_{\mu\nu}$ $\quad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

$$\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} === 8\pi G \underbrace{T_{\mu\nu}}_{\text{Stress energy tensor}}$$

Einstein tensor
Spacetime dynamics

Stress energy tensor
Distribution of matter field

$$\begin{tikzcd}
\text{Matter field distribution} \arrow[r] \arrow[d] & \text{Geometry/Spacetime dynamics} \arrow[dl] \\
\text{Motion along the geodesics} &
\end{tikzcd}$$