Short course on General Relativity

Exact Solution: Schwarzschild Black-holes

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Lecture # 3

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Recap

- Space-time is curved by an amount that depends on the mass/energy in the vicinity.

- Space-time is locally flat

- freely falling objects follow the shortest path — geodesics — in the curved space-time.

- information about gravity propagates at the speed of light (Part 12 lectures).

- Einstein's Equations $\underbrace{G_{\mu\nu}}_{\substack{\text{Einstein tensor} \\ \text{Spacetime dynamics}}} === 8\pi G \underbrace{T_{\mu\nu}}_{\substack{\text{Stress energy tensor} \\ \text{Distribution of matter field}}}$ (Earlier Lectures)

Overview of Lecture 3

- Schwarzschild metric and consequences
- Black-holes
- Black-hole properties
- Black-hole thermodynamics

The first surprise solution to Einstein's equations

Über das Gravitationsfeld eines Massenpunktes nach der EINSTEINschen Theorie.

Von K. SCHWARZSCHILD.

- The first exact solution to Einstein's equations is by Karl Schwarzschild.
- The solution describes the space-time around a non-rotating spherical mass.
- Solution describes the commonly occurring situations that one is interested in also in Solar system. Example: advance of the perihelion of Mercury.

Schwarzschild metric

Properties and conditions imposed

- Since it is spherically symmetric, we use spherical coordinates (r, θ, ϕ) to describe the spatial part of the metric.
- Since this solution is outside of the mass, Einstein's equation takes the form, $R_{\mu\nu} = 0$.
- No matter outside does not mean that the curvature $R^{\alpha}_{\mu\beta\nu}$ components vanish!
- Condition 1: As we take the distance from this mass out to infinity $(r \to \infty)$ the metric should approach the flat spacetime metric:

$$ds^2 = dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Condition 2: As the mass is taken to zero, we should again regain the flat space-time metric.

Schwarzschild metric

Using the above properties and conditions, Einstein's equations lead to:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Properties

- Line-element gives the geometry of spacetime outside of a single massive object (Earth, Sun, or a black hole by inserting the appropriate mass)
- Radial distance between two points measured simultaneously $(dt = d\theta = d\phi = 0)$

$$d\ell^2 = \sqrt{-ds^2} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

$\Longrightarrow$ proper distance measured at r is greater than the coordinate distance measured at ∞.

Schwarzschild metric

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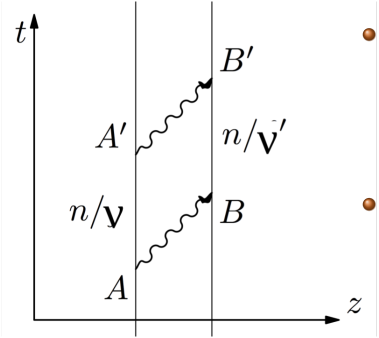
Schwarzschild radius

- G and c are fundamental constants
- Using G, c and Mass, we can define length

$$r_h = \frac{2GM}{c^2}$$

Object	r_h (m)	r_h/R
Earth	10^{-2}	10^{-8}
Sun	10^{3}	10^{-5}
Neutron Star	10^{3}	0.1
Black-holes	10^{3}	1

Consequence 1: Gravitational redshift



- Let observers A, B be at r_1, r_2.
- Send EM Waves from A to B.
- Rays will travel along

$$\left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = 0$$

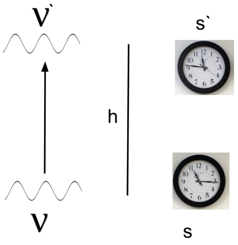
- Note that the Coordinate time δt measured is the same! Propertime is different:

$$\delta\tau_i = \sqrt{1 - \frac{2M}{r_i}}\delta t$$

- Ratio of frequencies

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{1 - 2M/r_2}{1 - 2M/r_1}}$$

Consequence 2: Time dilatation



$d\tau_{\mathrm{Earth}} < d\tau_{\mathrm{Orbit}}$

Gravitational Time dilatation

- Consider an observer sitting at the surface of the Earth, $r_E = R, \theta_E = const = \phi_E$.
- Observer can construct the local Minkowski coordinates: $ds^2 = d\tau_E^2 - dx^2 - dy^2 - dz^2$
- ds^2 is invariant — local Minkowski and Schwarzschild $d\tau_E = dt\sqrt{1 - 2M/R}$
- Another observer in Geostationary orbit at height h. $d\tau_O = dt\sqrt{1 - 2M/(R + h)}$
- When both observers measure the same process that start at the same coordinate time t and ends at $t + dt$:

$$d\tau_E = d\tau_0\sqrt{\frac{1 - 2M/R}{1 - 2M/(R + h)}}$$

Consequence 3: Recovering Newton's law of Gravitation

- Near surface of the Earth $\frac{GM}{c^2 r} \ll 1$. Weak Gravity limit

$$ds^2 \simeq \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Non-relativistic limit: $\frac{\vec{v}}{c} \to 0 \Longrightarrow \frac{1}{c}\frac{dx^r}{ds} \to 0$ $\qquad \frac{GMm}{r} = \frac{mv^2}{2}$

$$ds^2 \simeq \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Lagrangian of the particle moving in this background is

$$L[x(s)] = m\, c \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}$$

- Euler-Lagrange equation leads to

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^2}\hat{e}_r \qquad \text{Newton's law of Gravitation}$$

Black-holes

Appendix A
Translation of an essay by Peter Simon Laplace†

Proof of the theorem, that the attractive force of a heavenly body could be so large, that light could not flow out of it.‡

- Escape velocity to leave an object of mass M and radius R

$$\frac{GMm}{r} = \frac{1}{2} m v_{\mathrm{es}}^2 \quad \Longrightarrow \quad r = \left.\frac{2GM}{v_{\mathrm{es}}^2}\right|_{v_{\mathrm{es}}=c} = \frac{2GM}{c^2} = r_h$$

- For an object of mass M and radius $R = r_h$, $v_{\mathrm{es}} = c$
- For any object $R < r_h$, the escape velocity is greater.
  $\Longrightarrow$ no object, nor information can escape from such an object.

Early Idea of Black-holes — Laplace 1798

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However, there is a difference between this classical idea and what GR predicts.

What happens at Schwarzschild radius?

I: Time stops at Schwarzschild radius

- Consider two observers. One at r and other at $r + h$. Time dilatation equation becomes

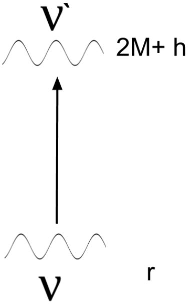
$$d\tau = d\tau_0 \sqrt{\frac{1 - 2M/r}{1 - 2M/(r + h)}}$$

- If the observer is at rest at $r = 2M \Longrightarrow d\tau = 0$

Proper time stops at event horizon

What happens at Schwarzschild radius?

II: Black-holes are black



No radiation or signals can ever come out of the event horizon, black hole is indeed black!

- Consider an excited atom, deexcites and emits a photon just before it passes through the event horizon.
- These photons are emitted radially outwards and detected by a stationary observer at $2M + h$. Gravitational redshift:

$$\lambda_{\rm detected} = \lambda_{emitted}\sqrt{\frac{1 - 2M/(2M+h)}{1 - 2M/r}}$$

- If the atom is at $r = r_h + \delta(\delta \ll 1)$; the wavelength is finite, but extremely large.
- If the atom is at $r = r_h$, the detected wavelength of the photon becomes infinite!
- Photons have to "climb up" infinite potential when they start at horizon.

What happens at Schwarzschild radius?

III. What happens if an observer falls through the event-horizon?

- Consider observer in Minkowski background: $ds^2 = dt^2 - d\ell^2$
- Dividing this by propertime $d\tau$, we have

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 = \eta_{\mu\nu} u^\mu u^\nu$$

- 4-velocity u^μ of the observer is time-like and magnitude is constant
- Observer can change space velocity arbitrarily, cannot have zero velocity through time.

Same effects exists in curved space-time

What happens at Schwarzschild radius?

III. What happens if an observer falls through the event-horizon?

- Let us look at Schwarzschild background:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Whether an observer will always have non-zero velocity through time?
- $r > 2M$: Like Minkowski background. u^μ of the observer is time-like.
- $r < 2M$: the coefficients in front of dt^2 and dr^2 change signs!
- t and r switch their roles, and consequently the observer can adjust velocity through time, but cannot have zero radial velocity.

Observer outside the horizon can choose not to travel through space, but invariably must travel through time. Observer inside the horizon can choose not to travel through time and can have zero orbital velocity, but invariably must fall toward the center.

Black-hole properties

Black-hole properties

- Black holes have no hair.

Symmetry	$T_{\mu\nu}$	Parameter	solution
Spherical	vacuum	M	Schwarzschild
	E-M field	M & Q	Reissner-Norstrom
Axial	vacuum	M & J	Kerr
	E-M field	M, J & Q	Kerr-Newmann

  Entirely defined by their mass M, rotation rate J, and charge Q. All memory of how the hole was made is lost. Almost like an elementary particle.

- Gravitational field of a black hole close to the event horizon is complicated, but by the time you are several Schwarzschild radii away, it is indistinguishable from that of an ordinary star.

Black-holes in nature

Black holes in nature — end points of stellar evolution

- In our galaxy alone, theory suggests 50 million black holes (2 Supernova per century for 10^{10} years. Quarter of which make black holes)

- Most massive galaxies have massive black holes at their centers (10^9 galaxies)

- LIGO-VIRGO have detected many black-hole binaries!

Black-hole thermodynamics

Black-hole thermodynamics

- Laws of black-hole mechanics are analogous to thermodynamics

	Black-hole	Thermodynamics
I law	$dM = \frac{\kappa}{2\pi} d\mathcal{A}_{\mathrm{H}} + \mathrm{work}$	$dE = TdS + \mathrm{work}$
II law	$\mathcal{A}_{\mathrm{H}}$ increases	S increases
III law	$\kappa \not\to 0$	$T \not\to 0$

Black-holes	Mass (M)	Horizon-area ($\mathcal{A}_{\mathrm{H}}$)	surface gravity (κ)
Thermodynamics	Energy (E)	entropy (S)	temperature (T)

Black-hole thermodynamics

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Black-holes	Mass (M)	Horizon-area ($\mathcal{A}_{\rm H}$)	surface gravity (κ)
Thermodynamics	Energy (E)	entropy (S)	temperature (T)

Classically, these laws are strictly formal with no direct physical implications

Black-hole thermodynamics

Semi-classical limit [gravity classical; matter fields quantum]

$$T_{\rm H} = \left(\frac{\hbar c}{k_B}\right)\left(\frac{\kappa}{2\pi}\right)$$ κ surface gravity Hawking radiation

$$\mathcal{S}_{\rm BH} = \left(\frac{k_B}{4}\right)\left(\frac{\mathcal{A}_{\rm H}}{\ell_P^2}\right)$$ $\mathcal{A}_{\rm H}$ Horizon area $\ell_P^2 \equiv \frac{\hbar G}{c^3}$

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Properties

- Unlike ideal gas [$S \propto V$], black-hole entropy is not extensive
- Hawking temperature $T_{\rm H}$ is tiny $T_{\rm H} = 3.68 \times 10^{-52}\left(\frac{M_\odot}{M}\right)$

Black-hole thermodynamics

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κ surface gravity — Hawking radiation

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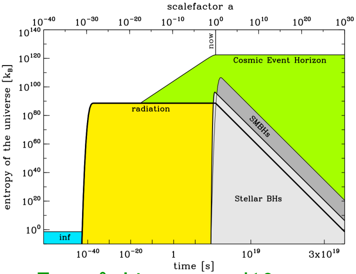

Egan & Lineweaver '10

- Black-hole entropy is large!!!

$$\mathcal{S}_{\rm BH} \sim 10^{77}\,\frac{M^2}{M_\odot^2} \qquad S_{\rm Univ}^{\rm radiation} \sim 10^{88}$$

- Supermassive BHs ($10^8 M_\odot$) dominate entropy contribution to the Universe

Black-holes: Recap

- A region of space-time from which no information carrying signals can escape to a distant observer.
- Simplest macroscopic physical objects

  Described by few parameters like mass, charge, angular momentum

  No-hair theorem
- Classical entropy of black-hole is infinite.
- Semiclassically, black-holes emit radiation — Hawking Radiation