Indian Institute of Science Education and Research, Trivandrum

PHY-321 **Stat-Mech Mid sem 2** **9.03.2015**

Instructions

1. This is a closed book exam. **Total of 20 points.**
2. Be clear; Be specific; Be neat.
3. Useful formulas are given at the end of the last question.

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1. **[3 points]**

   Consider a three level single particle system with five microstates with energies $0,\ \epsilon,\ \epsilon,\ \epsilon$ and 2ϵ. What is the mean energy of the system if it is equilibrium with a heat bath at temperature T ?

2. **Density of States in 1 and 2-dimensions** **[3 points]**

   In the case of 3 space dimensions, we saw that the density of states of a non-relativistic particle in $\mathbf{k}-$space is given by

   $$g(k)dk = \frac{V}{2\pi^2}k^2 dk \tag{1}$$

   In the case of 1 and 2 space dimensions, what is the density of states in $\mathbf{k}-$space?

3. **Root mean square fluctuations** **[4 points]**

   A one-dimensional quantum harmonic oscillator (whose ground state energy is $\hbar\omega/2$) is in thermal equilibrium with a heat bath at temperature T.

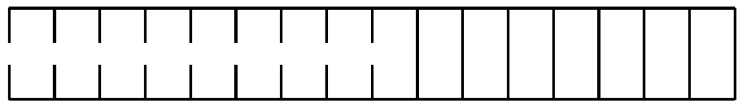
   (a) What is the mean value of the oscillator's energy $\langle E\rangle$, as a function of temperature T? **(1 point)**

   (b) What is the value of ΔE, the root-mean-square fluctuation in energy about $\langle E\rangle$? **(2 points)**

   (c) How do $\langle E\rangle$ and ΔE behave in the limits $k_B T \ll \hbar\omega$ and $k_B T \gg \hbar\omega$? **(1 point)**

4. **Modeling DNA** **[5 points]**

   The unwinding of double-stranded DNA is like unzipping a zipper. The DNA has N links, each of them can be in one of two states: a closed state with energy 0 and an open state with energy Δ. A link can be in the open state only when all the links to its left are open (see the figure below)



(a) Show that the partition function of the DNA chain has a form **3 points**

$$Z = \frac{1 - e^{-(N+1)\beta\Delta}}{1 - e^{-\Delta\beta}} \tag{2}$$

(b) Find the average number of open links in $k_B T \ll \Delta$ limit. **2 points**

5. **Cooling by adiabatic demagnetization** **[5 points]**

(a) Consider N spin-1/2 spins in a magnetic field B. Initially, the system has a temperature T. If we slowly reduce the magnetic field to $B/2$, what is the corresponding temperature of the system? If we slowly reduce the magnetic field to zero, what is the temperature of the system? (Hint: this is an adiabatic process.) **2 points**

(b) Let us again consider N spin-1/2 spins in a magnetic field B. The spin system is in thermal contact with an ideal gas of N particles in a volume V. Initially, the two systems have a temperature T. Assume $\mu_B B \gg k_B T$. If we slowly reduce the magnetic field to zero, what is the temperature of the gas? **3 points**

Note: (i) The energy of the spins is $E = \mu_B B \sum_i S_i$.

(ii) The entropy of the ideal gas is

$$S_{\text{ideal}} = N\,k_B \left(\frac{5}{2} + \ln\left[\frac{V}{N} \frac{1}{\lambda_D^3} \right] \right) \tag{3}$$

where λ_D is the thermal de Broglie wavelength.

Useful formulae

1. Stirling's approximation $\ln n! \simeq n \ln n - n$
2. Useful constants: $k_B = 1.380650310^{-23} m^2\,kg\,s^{-2} K^{-1}$; $\hbar = 1.05457148 \times 10^{-34} m^2 kg s^{-1}$
3. $$\frac{1}{T} = \frac{\partial S}{\partial U}; \qquad \frac{P}{T} = \frac{\partial S}{\partial V}; \qquad C_V = \left.\frac{\partial U}{\partial T}\right|_{V,N}$$