

Note: Deadline: 13 March 2015 (12:30 PM)

1. Spin- J atoms

We have previously considered spins that can only take two values, up or down. Many of the atoms that act as spins in paramagnetic salts have spins larger than 1 and such a description is inadequate for quantitative comparison with experiment.

For a spin J atom, a quantum mechanical description implies that J_z can take discrete values m_z which are spaced between $-J$ and J in integral steps, i.e.

$$m_z = -J, -J + 1, \dots, J - 1, J,$$

corresponding to a total of $(2J + 1)$ different values. The energy for a single spin is

$$E = -g\mu_B \mathbf{J} \cdot \mathbf{B} = -\boldsymbol{\mu} \cdot \mathbf{B},$$

where $\boldsymbol{\mu} = g\mu_B \mathbf{J}$, \mathbf{J} is the magnetic moment, g is the g-factor of the atom and $\mu_B = e\hbar/(2m_e)$ is the Bohr magneton.

- For a magnetic field \mathbf{B} parallel to the z axis, use the canonical ensemble to calculate the partition function for a single atom.
- Write down an expression for the magnetization in the z direction $M_z = \langle \mu_z \rangle$.
- Consider the asymptotic behaviour of M_z as $B \rightarrow 0$ and for $B \ll 1$ and use this to sketch M_z as a function of B . Find an expression for the magnetic susceptibility:

$$\xi_m = \frac{\partial M}{\partial B}$$

in the limit that $B \rightarrow 0$. You should find that your answer is proportional to known as Curies Law.

2. 2-dimensions

In the case of 3 space dimensions, we saw that the density of states in \mathbf{k} -space is given by

$$g(k)dk = \frac{V}{2\pi^2}k^2dk \quad (1)$$

In the case of 2 space dimensions, the density of states in \mathbf{k} -space is

$$g(k)dk = \frac{A}{2\pi}kdk \quad (2)$$

where A is the area in 2-dimensions.

Use the above relation for the density of states for ripples (which are quantised capillary waves on the free surface of super fluid ${}^4\text{He}$) whose dispersion relation is

$$\omega(k) = \left(\frac{\gamma k^3}{\rho}\right)^2 \quad (3)$$

where γ is the surface tension and ρ is the mass density of liquid helium.

Calculate the thermal contribution to the surface energy per unit area due to ripples.

3. Ideal gas

- (a) Calculate the entropy, specific heat and equation of state of a two dimensional classical ideal gas at temperature T with N particles confined in an area $A = L^2$ and compare your answer to the results for a three dimensional classical ideal gas.
- (b) We have seen how to calculate C_V for an ideal gas. Calculate C_P for an ideal gas in three dimensions recalling that it can be written as a derivative of the enthalpy H via:

$$C_P = \left.\frac{\partial H}{\partial T}\right|_P = T \left.\frac{\partial S}{\partial T}\right|_P$$

4. A one dimensional lattice consists of linear array of N particles ($N \gg 1$) interacting via spring-like nearest neighbor forces. The normal mode frequencies are given by:

$$\omega_n = \bar{\omega} \sqrt{2(1 - \cos[2\pi n/N])} \quad (4)$$

where $\bar{\omega}$ is a constant and n an integer ranging $-N/2$ to $N/2$. The system is in equilibrium at temperature T . Let c_v be the constant length specific heat.

- (a) Compute c_v for the regime $T \rightarrow \infty$.

- (b) For $T \rightarrow 0$, $c_v \rightarrow A\omega^{-\alpha}T^\gamma$. What is A , α and γ ?

Hint: The problem is to be treated quantum mechanically.

5. Van der Waals gases

The partition function for an interacting gas is assumed to be:

$$Z = \left(\frac{V - Nb}{N} \right)^N \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3N/2} e^{N^2 a^2 / (Vk_B T)}$$

where a and b are constants. Show that the pressure is of the same form as Van der Waals equation.

6. Average energy for classical and quantum oscillator

Consider an oscillator of mass m and spring constant K . The total energy of the oscillator is

$$E = \frac{p^2}{2m} + \frac{Kx^2}{2}$$

The oscillator is in contact with a heat bath of temperature T .

- (a) View the oscillator as a classical oscillator. Show that the average kinetic energy $\langle p^2 / (2m) \rangle$ and the average potential energy $\langle \frac{Kx^2}{2} \rangle$ are both given by $\frac{k_B T}{2}$ regardless of the value of m and K . As a result, the average total energy is $\langle E \rangle = k_B T$.
- (b) View the oscillator as a quantum oscillator. Calculate the average total energy $\langle E \rangle$, and show that $\langle E \rangle = k_B T$ in high temperature limit. So the equipartition theorem is valid for a quantum oscillator in high temperature limit. Below what temperature we start to see a violation of equipartition theorem for the quantum oscillator? What is $\langle E \rangle$ in $T \rightarrow 0$ limit?

7. One-particle statistics

- (a) Let us consider a system to be a single quantum state of energy ϵ . Let us assume that this system can either have one or no particle in this energy state. Write down the partition function of this system. What is the average number of particles in this energy state? What does this distribution correspond to?
- (b) Let us consider another system to be a single quantum state of energy ϵ . Let us further assume that this system can have any number of such particles with the same energy. Write down the partition function of this system. What is the average number of particles in this energy state? What does this distribution correspond to?