Chapter 5: Motion Under the Influence of a Central Force

Introduction

- Question: What is a central force?
- Answer: Any force which is directed towards a center, and depends only on the distance between the center and the particle in question.
- Question: Any examples of central forces in nature?
- Answer: Two fundamental forces of nature, gravitation, and Coulomb forces are central forces
- Question: But gravitation and Coulomb forces are two body forces, how could they be central?
- Answer: Correct, these two forces are indeed two-body forces, but they can be reduced to central forces by a mathematical trick.

Aim and Scope

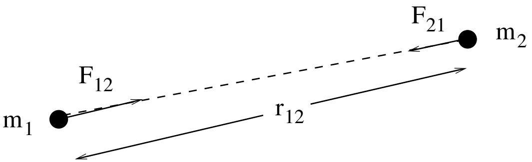
- Kepler took the astronomical data of Tycho Brahe, and obtained three laws by clever mathematical fitting
- Law 1: Every planet moves in an elliptical orbit, with sun on one of its foci.
- Law 2: Position vector of the planet with respect to the sun, sweeps equal areas in equal times.
- Law 3: If T is the time for completing one revolution around sun, and A is the length of major axis of the ellipse, then $T^2 \propto A^3$.
- We will be able to derive all these three laws based upon the mathematical theory we develop for central force motion

Reduction of a two-body central force problem to a one-body problem

- Gravitational force acting on mass m_1 due to mass m_2 is

$$\mathbf{F}_{12} = -\frac{Gm_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12},$$

i.e., it acts along the line joining the two masses



- Similarly, the Coulomb force between two charges q_1 and q_2 is given by

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2}\hat{\mathbf{r}}_{12}.$$

Reduction of two-body problem....

- An ideal central force is of the form

$$\mathbf{F}(r) = f(r)\hat{\mathbf{r}},$$

  i.e., it is a one-body force depending on the coordinates of only the particle on which it acts

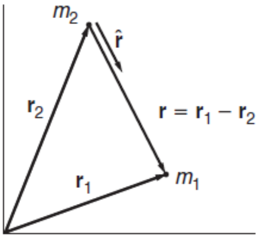
- But gravity and Coulomb forces are two-body forces, of the form

$$\mathbf{F}(r_{12}) = f(r_{12})\hat{\mathbf{r}}_{12}$$

- Can they be reduced to a pure one-body form?
- Yes, and this is what we do next

Reduction of two-body problem...

- Relevant coordinates are shown in the figure



- We define

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$
$$\Longrightarrow \quad r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$$

- Given $\mathbf{F}_{12} = f(r)\hat{\mathbf{r}}$, we have

$$m_1 \ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}}$$
$$m_2 \ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}$$

Decoupling equations of motion

- Both the equations above are coupled, because both depend upon $\mathbf{r}_1$ and $\mathbf{r}_2$.
- In order to decouple them, we replace $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ (called relative coordinate), and center of mass coordinate R

$$\mathsf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r_2}}{m_1 + m_2}$$

- Now

$$\begin{aligned} \ddot{\mathsf{R}} &= \frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2} = \frac{f\hat{\mathbf{r}} - f\hat{\mathbf{r}}}{m_1 + m_2} = 0 \\ \implies \mathsf{R} &= \mathsf{R}_0 + \mathbf{V}t, \end{aligned}$$

above R_0 is the initial location of center of mass, and $\mathbf{V}$ is the center of mass velocity.

Decoupling equations of motion...

- This equation physically means that the center of mass of this two-body system is moving with constant velocity, because there are no external forces on it.
- We also obtain

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = f(r)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{\mathbf{r}}$$

$$\implies \ddot{\mathbf{r}} = \left(\frac{m_1 + m_2}{m_1 m_2}\right) f(r)\hat{\mathbf{r}}$$

$$\mu\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}},$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$, is called reduced mass.

Reduction of two-body problem to one body problem

- Note that this final equation is entirely in terms of relative coordinate $\mathbf{r}$
- It is an effective equation of motion for a single particle of mass μ, moving under the influence of force $f(r)\hat{\mathbf{r}}$.
- There is just one coordinate ($\mathbf{r}$) involved in this equation of motion
- Thus the two body problem has been effectively reduced to a one-body problem
- This separation was possible only because the two-body force is central, i.e., along the line joining the two particles
- In order to solve this equation, we need to know the nature of the force, i.e., $f(r)$.

Two-body central force problem continued

- We have already solved the equation of motion for the center-of-mass coordinate $\mathbf{R}$
- Therefore, once we solve the "reduced equation", we can obtain the complete solution by solving the two equations

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

- Leading to

$$\mathbf{r}_1 = \mathbf{R} + \left(\frac{m_2}{m_1 + m_2} \right) \mathbf{r}$$
$$\mathbf{r}_2 = \mathbf{R} - \left(\frac{m_1}{m_1 + m_2} \right) \mathbf{r}$$

- Next, we discuss how to approach the solution of the reduced equation

General Features of Central Force Motion

- Before attempting to solve $\mu\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}}$, we explore some general properties of central force motion
- Let $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ be angular momentum corresponding to the relative motion
- Then clearly

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F}$$

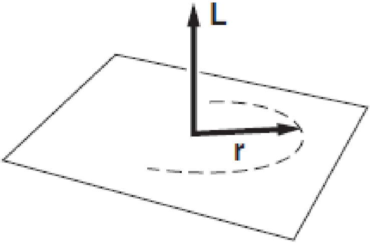
- But $\mathbf{v}$ and $\mathbf{p} = \mu\mathbf{v}$ and parallel, so that $\mathbf{v} \times \mathbf{p} = 0$
- And for the central force case, $\mathbf{r} \times \mathbf{F} = f(r)\mathbf{r} \times \hat{\mathbf{r}} = 0$, so that

$$\frac{d\mathbf{L}}{dt} = 0$$
$$\implies \mathbf{L} = \text{constant}$$

- Thus, in case of central force motion, the angular momentum is conserved, both in direction, and magnitude

Conservation of angular momentum

- Conservation of angular momentum implies that the relative motion occurs in a plane



- Direction of **L** is fixed, and because $\mathbf{r} \perp \mathbf{L}$, so **r** must be in the same plane
- So, we can use plane polar coordinates (r, θ) to describe the motion

Equations of motion in plane-polar coordinates

- We know that in plane polar coordinates

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

- Therefore, the equation of motion $\mu\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}}$, becomes

$$\mu(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \mu(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = f(r)\hat{\mathbf{r}}$$

- On comparing both sides, we obtain following two equations

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r)$$
$$\mu(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

- By multiplying second equation on both sides by r, we obtain

$$\frac{d}{dt}(\mu r^2\dot{\theta}) = 0$$

Equations of motion

- This equation yields
$$\mu r^2 \dot{\theta} = L\,(\text{constant}),$$
we called this constant L because it is nothing but the angular momentum of the particle about the origin. Note that $L = I\omega$, with $I = \mu r^2$.
- As the particle moves along the trajectory so that the angle θ changes by an infinitesimal amount $d\theta$, the area swept with respect to the origin is
$$dA = \frac{1}{2} r^2 d\theta$$
$$\implies \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} = \text{constant},$$
because L is constant.
- Thus constancy of areal velocity is a property of all central forces, not just the gravitational forces.
- And it holds due to conservation of angular momentum

Conservation of Energy

- Kinetic energy in plane polar coordinates can be written as

$$
\begin{aligned}
K &= \frac{1}{2}\mu \mathbf{v}\cdot\mathbf{v} \\
&= \frac{1}{2}\mu\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\right).\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\right) \\
&= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2
\end{aligned}
$$

- Potential energy $V(r)$ can be obtained by the basic formula

$$V(r) - V(r_O) = -\int_o^r f(r)dr,$$

where r_O denotes the location of a reference point.

Conservation of Energy...

- Total energy E from work-energy theorem

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + V(r) = \text{constant}$$

- We have

$$L = \mu r^2 \dot{\theta}$$

$$\Longrightarrow \frac{1}{2}\mu r^2 \dot{\theta}^2 = \frac{L^2}{2\mu r^2}$$

- So that

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$$

- We can write

$$E = \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r)$$

$$\text{with } V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$$

Conservation of energy contd.

- This energy is similar to that of a 1D system, with an effective potential energy $V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$
- In reality $\frac{L^2}{2\mu r^2}$ is kinetic energy of the particle due to angular motion
- But, because of its dependence on position, it can be treated as an effective potential energy

Integrating the equations of motion

- Energy conservation equation yields

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)}$$

- Leading to the solution

$$\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)}} = t - t_0, \tag{1}$$

which will yield r as a function of t, once $f(r)$ is known, and the integral is performed

Integration of equations of motion...

- Once $r(t)$ is known, to obtain $\theta(t)$, we use conservation of angular momentum

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

$$\theta - \theta_0 = \frac{L}{\mu}\int_{t_0}^{t}\frac{dt}{r^2}$$

- We can obtain the shape of the trajectory $r(\theta)$, by combining these two equations

$$\frac{d\theta}{dr} = \left(\frac{\frac{d\theta}{dt}}{\frac{dr}{dt}}\right) = \frac{\frac{L}{\mu r^2}}{\sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)}}$$

- Leading to

$$\theta - \theta_0 = L\int_{r_0}^{r}\frac{dr}{r^2\sqrt{2\mu\left(E - V_{eff}(r)\right)}} \qquad (2)$$

Integration of equations of motion contd.

- Thus, by integrating these equations, we can obtain $r(t)$, $\theta(t)$, and $r(\theta)$
- This will complete the solution of the problem
- But, to make further progress, we need to know what is $f(r)$
- Next, we will discuss the case of gravitational problem such as planetary orbits

Case of Planetary Motion: Keplerian Orbits

- We want to use the theory developed to calculate the orbits of different planets around sun
- Planets are bound to sun because of gravitational force
- Therefore

$$f(r) = -\frac{GMm}{r^2}$$

- So that

$$V(r) = -\frac{GMm}{r} = -\frac{C}{r}, \tag{3}$$

above, $C = GMm$, where G is gravitational constant, M is mass of the Sun, and m is mass of the planet in question.

Derivation of Keplerian orbits

- On substituting $V(r)$ from Eq. 3 into Eq. 2, we have

$$\theta - \theta_0 = L \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{2\mu(E - \frac{L^2}{2\mu r^2} + \frac{C}{r})}}$$
$$= L \int \frac{dr}{r\sqrt{2\mu E r^2 + 2\mu C r - L^2}} \qquad (4)$$

- We converted the definite integral on the RHS to an indefinite one, because θ_0 is a constant of integration in which the constant contribution of the lower limit $r = r_0$ can be absorbed. This orbital integral can be done by the following substitution

$$r = \frac{1}{s - \alpha} \qquad (5)$$
$$\Longrightarrow \; dr = -\frac{ds}{(s-\alpha)^2}$$
$$\Longrightarrow \; \frac{dr}{r} = -\frac{ds}{(s-\alpha)} \qquad (6)$$

Orbital integral....

- Substituting Eqs. 5 and 6, in Eq. 4, we obtain

$$\begin{aligned}\theta - \theta_0 &= -L \int \frac{ds}{(s-\alpha)\sqrt{\frac{2\mu E}{(s-\alpha)^2} + \frac{2\mu C}{s-\alpha} - L^2}} \\ &= -L \int \frac{ds}{\sqrt{2\mu E + 2\mu C(s-\alpha) - L^2(s-\alpha)^2}} \\ &= -L \int \frac{ds}{\sqrt{2\mu E + 2\mu Cs - 2\mu C\alpha - L^2 s^2 + 2L^2 \alpha s - L^2\alpha^2}}\end{aligned}$$

- The integrand is simplified if we choose $\alpha = -\frac{\mu C}{L^2}$, leading to

$$\begin{aligned}\theta - \theta_0 &= -L \int \frac{ds}{\sqrt{2\mu E + 2\frac{(\mu C)^2}{L^2} - L^2 s^2 - \frac{(\mu C)^2}{L^2}}} \\ &= -L \int \frac{ds}{\sqrt{2\mu E + \frac{(\mu C)^2}{L^2} - L^2 s^2}}\end{aligned}$$

- Finally, the integral is

$$\theta - \theta_0 = -L^2 \int \frac{ds}{\sqrt{2\mu EL^2 + (\mu C)^2 - L^4 s^2}}$$
$$= -\int \frac{ds}{\sqrt{\frac{2\mu EL^2 + (\mu C)^2}{L^4} - s^2}}$$

- On substituting $s = a \sin\phi$, where $a = \sqrt{\frac{2\mu EL^2 + (\mu C)^2}{L^4}}$, the integral transforms to

$$\theta - \theta_0 = -\phi = -\sin^{-1}\left(\frac{s}{a}\right)$$
$$s = -a\sin(\theta - \theta_0)$$
$$\implies \frac{1}{r} + \alpha = -a\sin(\theta - \theta_0)$$
$$\implies r = \frac{1}{-\alpha - a\sin(\theta - \theta_0)}$$

Keplerian Orbit

- We define $r_0 = -\frac{1}{\alpha} = \frac{L^2}{\mu C}$, to obtain

$$r = \frac{r_0}{1 - \sqrt{1 + \frac{2EL^2}{\mu C^2}} \sin(\theta - \theta_0)}$$

- Conventionally, one takes $\theta_0 = -\pi/2$, and we define

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}$$

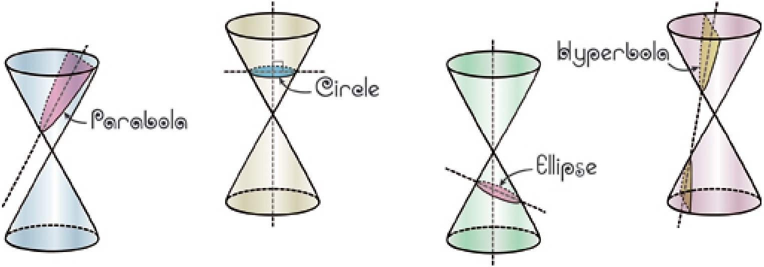
- To obtain the final result

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

- We need to probe this expression further to find which curve it represents.

A Brief Review of Conic Sections

- Curves such as circle, parabola, ellipse, and hyperbola are called conic sections



- We will show that the curve $r = \frac{r_0}{1-\varepsilon\cos\theta}$ in plane polar coordinates, denotes different conic sections for various values of ε, which is nothing but the eccentricity

Nature of orbits: parabolic orbit

- Using the fact that $r = \sqrt{x^2 + y^2}$, and $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$, we obtain

$$\sqrt{x^2 + y^2} = \frac{r_0}{1 - \frac{\epsilon x}{\sqrt{x^2+y^2}}}$$
$$\implies \sqrt{x^2 + y^2} = r_0 + \epsilon x$$
$$\implies x^2(1-\epsilon^2) - 2r_0\epsilon x + y^2 = r_0^2$$

- Case I: $\epsilon = 1$, which means $E = 0$, we obtain

$$y^2 = 2r_0 x + r_0^2$$

which is nothing but a parabola. This is clearly an open or unbound orbit. This is typically the case with comets.

Nature of orbits: hyperbolic and circular orbits

- Case II: $\varepsilon > 1 \implies E > 0$, let us define $A = \varepsilon^2 - 1 >$. With this, the equation of the orbit is

$$y^2 - Ax^2 - 2r_0\sqrt{1+A}x = r_0^2$$

  Here, the coefficients of x^2 and y^2 are opposite in sign, therefore, the curve is unbounded, i.e., open. It is actually the equation of a hyperbola. Therefore, whenever $E > 0$, the particles execute unbound motion, and some comets and asteroids belong to this class.
- Case III: $\varepsilon = 0$, we have

$$x^2 + y^2 = r_0^2$$

  which denotes a circle of radius r_0, with center at the origin. This is clearly a closed orbit, for which the system is bound. $\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}} = 0 \implies E = -\frac{\mu C^2}{2L^2} < 0$. Satellites launched by humans are put in circular orbits many times, particularly the geosynchronous ones.

Nature of orbits: elliptical orbits

- Case IV: $0 < \varepsilon < 1 \implies E < 0$, here we define $A = (1-\varepsilon^2) > 0$, to obtain

$$Ax^2 - 2r_0\sqrt{1-A}x + y^2 = r_0^2$$

  Because coefficients of x^2 and y^2 are both positive, orbit will be closed (i.e. bound), and will be an ellipse.
- To summarize, when $E \geq 0$, orbits are unbound, i.e., hyperbola or parabola
- When $E < 0$, orbits are bound, i.e., circle or ellipse.

Time Period of Elliptical orbit

- There are two ways to compute the time needed to go around its elliptical orbit once
- First approach involves integration of the equation

$$
\begin{aligned}
t_b - t_a &= \int_{r_a}^{r_b} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} + \frac{C}{r}\right)}} \\
&= \mu \int_{r_a}^{r_b} \frac{rdr}{\sqrt{(2\mu Er^2 + 2\mu Cr - L^2)}}
\end{aligned}
$$

- When this is integrated with the limit $r_b = r_a$, one obtains that time period T satisfies

$$T^2 = \frac{\pi^2 \mu}{2C} A^3,$$

where A is semi-major axis of the elliptical orbit. This result is nothing but Kepler's third law.

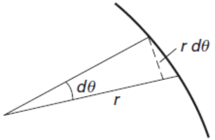
Time period of the elliptical orbit...

- Now we use an easier approach to calculate the time period
- We use the constancy of angular momentum

$$L = \mu r^2 \frac{d\theta}{dt}$$

$$\implies \frac{L}{2\mu} dt = \frac{1}{2} r^2 d\theta$$

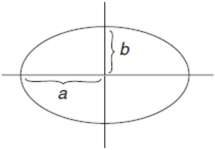
- R.H.S. of the previous equation is nothing but the area element swept as the particle changes its position by $d\theta$



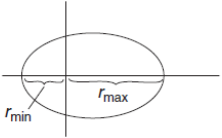
- Now, the integrals on both sides can be carried out to yield

$$\frac{LT}{2\mu} = \text{area of ellipse } = \pi ab.$$

- a and b in the equation are semi-major and semi-minor axes of the ellipse as shown



- Now, we have



- Therefore

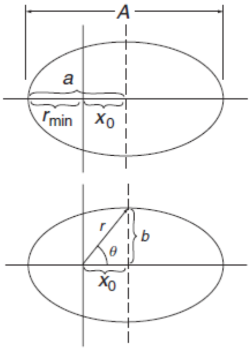
$$a = \frac{A}{2} = \frac{(r_{min} + r_{max})}{2}$$

Time period of the orbit....

- Using the orbital equation $r = \frac{r_0}{1-\varepsilon\cos\theta}$, we have

$$a = \frac{1}{2}\left(\frac{r_0}{1-\varepsilon\cos\pi} + \frac{r_0}{1-\varepsilon\cos 0}\right) = \frac{r_0}{2}\left(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right) = \frac{r_0}{1-\varepsilon^2}$$

- Calculation of b is slightly involved. Following diagram is helpful



- x_0 is the distance between the focus and the center of the ellipse, thus

$$x_0 = a - r_{min} = \frac{r_0}{1-\varepsilon^2} - \frac{r_0}{1+\varepsilon} = \frac{r_0\varepsilon}{1-\varepsilon^2}$$

- In the diagram $b = \sqrt{r^2 - x_0^2}$, and for θ, we have $\cos\theta = \frac{x_0}{r}$, which on substitution in orbital equation yields

$$r = \frac{r_0}{1-\varepsilon\cos\theta} = \frac{r_0}{1-\frac{\varepsilon x_0}{r}}$$

$$\implies r = r_0 + \varepsilon x_0 = r_0 + \frac{r_0\varepsilon^2}{1-\varepsilon^2} = \frac{r_0}{1-\varepsilon^2}$$

- So that

$$b = \sqrt{r^2 - x_0^2} = \sqrt{\frac{r_0^2}{(1-\varepsilon^2)^2} - \frac{r_0^2\varepsilon^2}{(1-\varepsilon^2)^2}} = \frac{r_0}{\sqrt{1-\varepsilon^2}}$$

- Now

$$1-\varepsilon^2 = 1-\left(1+\frac{2EL^2}{\mu C^2}\right) = -\frac{2EL^2}{\mu C^2}$$

- Using $r_0 = \frac{L^2}{\mu C}$, we have

$$A = 2a = \frac{2r_0}{1-\varepsilon^2} = \frac{2L^2}{\mu C}\times\left(-\frac{\mu C^2}{2EL^2}\right) = -\frac{C}{E}$$

$$b = \frac{r_0}{\sqrt{1-\varepsilon^2}} = \frac{L^2}{\mu C}\times\sqrt{-\frac{\mu C^2}{2EL^2}} = L\sqrt{-\frac{1}{2\mu E}}$$

- Using this, we have

$$T = \frac{2\pi\mu}{L}ab = \frac{2\pi\mu}{L}\times\left(-\frac{C}{2E}\right)\times L\sqrt{-\frac{1}{2\mu E}} = \pi\sqrt{\frac{\mu}{2C}}\left(-\frac{C}{E}\right)^{3/2}$$

Kepler's Third Law

- Which can be written as

$$T = \pi\sqrt{\frac{\mu}{2C}}A^{3/2}$$
$$\implies T^2 = \frac{\pi^2\mu}{2C}A^3,$$

which is nothing but Kepler's third law.