Chapter 5: Motion Under the Influence of a Central Force
Question: What is a central force?
Answer: Any force which is directed towards a center, and depends only on the distance between the center and the particle in question.

Question: Any examples of central forces in nature?
Answer: Two fundamental forces of nature, gravitation, and Coulomb forces are central forces.

Question: But gravitation and Coulomb forces are two body forces, how could they be central?
Answer: Correct, these two forces are indeed two-body forces, but they can be reduced to central forces by a mathematical trick.
Aim and Scope

Kepler took the astronomical data of Tycho Brahe, and obtained three laws by clever mathematical fitting

- **Law 1:** Every planet moves in an elliptical orbit, with sun on one of its foci.
- **Law 2:** Position vector of the planet with respect to the sun, sweeps equal areas in equal times.
- **Law 3:** If $T$ is the time for completing one revolution around sun, and $A$ is the length of major axis of the ellipse, then $T^2 \propto A^3$.

We will be able to derive all these three laws based upon the mathematical theory we develop for central force motion.
Reduction of a two-body central force problem to a one-body problem

- Gravitational force acting on mass $m_1$ due to mass $m_2$ is

\[ \mathbf{F}_{12} = -\frac{Gm_1 m_2}{r_{12}^2} \hat{r}_{12}, \]

i.e., it acts along the line joining the two masses

- Similarly, the Coulomb force between two charges $q_1$ and $q_2$ is given by

\[ \mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}^2} \hat{r}_{12}. \]
An ideal central force is of the form

$$F(r) = f(r)\hat{r},$$

i.e., it is a one-body force depending on the coordinates of only the particle on which it acts.

But gravity and Coulomb forces are two-body forces, of the form

$$F(r_{12}) = f(r_{12})\hat{r}_{12}$$

Can they be reduced to a pure one-body form?

Yes, and this is what we do next.
Reduction of two-body problem...

- Relevant coordinates are shown in the figure

- We define

\[ r = r_1 - r_2 \]

\[ \Rightarrow r = |r| = |r_1 - r_2| \]

- Given \( \mathbf{F}_{12} = f(r)\mathbf{\hat{r}} \), we have

\[ m_1 \ddot{r}_1 = f(r)\mathbf{\hat{r}} \]

\[ m_2 \ddot{r}_2 = -f(r)\mathbf{\hat{r}} \]
Both the equations above are coupled, because both depend upon $r_1$ and $r_2$.

In order to decouple them, we replace $r_1$ and $r_2$ by $r = r_1 - r_2$ (called relative coordinate), and center of mass coordinate $R$

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

Now

$$\ddot{R} = \frac{m_1 \ddot{r}_1 + m_2 \ddot{r}_2}{m_1 + m_2} = \frac{f \dot{r} - f \dot{r}}{m_1 + m_2} = 0$$

$$\Rightarrow R = R_0 + V t,$$

above $R_0$ is the initial location of center of mass, and $V$ is the center of mass velocity.
This equation physically means that the center of mass of this two-body system is moving with constant velocity, because there are no external forces on it.

We also obtain

\[
\ddot{r}_1 - \ddot{r}_2 = f(r) \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \hat{r}
\]

\[
\implies \ddot{r} = \left( \frac{m_1 + m_2}{m_1 m_2} \right) f(r) \hat{r}
\]

\[
\mu \ddot{r} = f(r) \hat{r},
\]

where \( \mu = \frac{m_1 m_2}{m_1 + m_2} \), is called reduced mass.
Note that this final equation is entirely in terms of relative coordinate \( r \).

It is an effective equation of motion for a single particle of mass \( \mu \), moving under the influence of force \( f(r) \hat{r} \).

There is just one coordinate \( (r) \) involved in this equation of motion.

Thus the two body problem has been effectively reduced to a one-body problem.

This separation was possible only because the two-body force is central, i.e., along the line joining the two particles.

In order to solve this equation, we need to know the nature of the force, i.e., \( f(r) \).
Two-body central force problem continued

- We have already solved the equation of motion for the center-of-mass coordinate $R$
- Therefore, once we solve the “reduced equation”, we can obtain the complete solution by solving the two equations

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$
$$r = r_1 - r_2$$

- Leading to

$$r_1 = R + \left( \frac{m_2}{m_1 + m_2} \right) r$$
$$r_2 = R - \left( \frac{m_1}{m_1 + m_2} \right) r$$

- Next, we discuss how to approach the solution of the reduced equation
General Features of Central Force Motion

- Before attempting to solve $\mu \ddot{r} = f(r)\hat{r}$, we explore some general properties of central force motion.
- Let $L = r \times p$ be angular momentum corresponding to the relative motion.
- Then clearly
  
  \[
  \frac{dL}{dt} = \frac{dr}{dt} \times p + r \times \frac{dp}{dt} = v \times p + r \times F
  \]

- But $v$ and $p = \mu v$ and parallel, so that $v \times p = 0$.
- And for the central force case, $r \times F = f(r)r \times \hat{r} = 0$, so that
  
  \[
  \frac{dL}{dt} = 0
  \]

  \[\implies L = \text{constant}\]

- Thus, in case of central force motion, the angular momentum is conserved, both in direction, and magnitude.
Conservation of angular momentum

- Conservation of angular momentum implies that the relative motion occurs in a plane.

- Direction of $\mathbf{L}$ is fixed, and because $\mathbf{r} \perp \mathbf{L}$, so $\mathbf{r}$ must be in the same plane.

- So, we can use plane polar coordinates $(r, \theta)$ to describe the motion.
Equations of motion in plane-polar coordinates

- We know that in plane polar coordinates
  \[ \mathbf{a} = \ddot{r} = \left( \ddot{r} - r \dot{\theta}^2 \right) \mathbf{\hat{r}} + \left( 2 \dot{r} \dot{\theta} + r \ddot{\theta} \right) \mathbf{\hat{\theta}} \]

- Therefore, the equation of motion \( \mu \ddot{r} = f(r) \mathbf{\hat{r}} \), becomes
  \[ \mu (\ddot{r} - r \dot{\theta}^2) \mathbf{\hat{r}} + \mu (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \mathbf{\hat{\theta}} = f(r) \mathbf{\hat{r}} \]

- On comparing both sides, we obtain following two equations
  \[ \mu (\ddot{r} - r \dot{\theta}^2) = f(r) \]
  \[ \mu (2 \dot{r} \dot{\theta} + r \ddot{\theta}) = 0 \]

- By multiplying second equation on both sides by \( r \), we obtain
  \[ \frac{d}{dt} (\mu r^2 \dot{\theta}) = 0 \]
This equation yields

\[ \mu r^2 \dot{\theta} = L \text{ (constant)}, \]

we called this constant \( L \) because it is nothing but the angular momentum of the particle about the origin. Note that \( L = I \omega \), with \( I = \mu r^2 \).

As the particle moves along the trajectory so that the angle \( \theta \) changes by an infinitesimal amount \( d\theta \), the area swept with respect to the origin is

\[ dA = \frac{1}{2} r^2 d\theta \]

\[ \implies \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} = \text{constant}, \]

because \( L \) is constant.

Thus constancy of areal velocity is a property of all central forces, not just the gravitational forces.

And it holds due to conservation of angular momentum.
Conservation of Energy

- Kinetic energy in plane polar coordinates can be written as

$$K = \frac{1}{2} \mu \mathbf{v} \cdot \mathbf{v}$$

$$= \frac{1}{2} \mu \left( \mathbf{r} \dot{\mathbf{r}} + r \dot{\theta} \dot{\mathbf{\hat{r}}} \right) \cdot \left( \mathbf{r} \dot{\mathbf{r}} + r \dot{\theta} \dot{\mathbf{\hat{r}}} \right)$$

$$= \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\theta}^2$$

- Potential energy $V(r)$ can be obtained by the basic formula

$$V(r) - V(r_O) = -\int_{r_O}^{r} f(r) dr,$$

where $r_O$ denotes the location of a reference point.
Total energy $E$ from work-energy theorem

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\theta}^2 + V(r) = \text{constant}$$

We have

$$L = \mu r^2 \dot{\theta}$$

$$\Rightarrow \frac{1}{2} \mu r^2 \dot{\theta}^2 = \frac{L^2}{2\mu r^2}$$

So that

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$$

We can write

$$E = \frac{1}{2} \mu \dot{r}^2 + V_{\text{eff}}(r)$$

with $V_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + V(r)$
This energy is similar to that of a 1D system, with an effective potential energy $V_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + V(r)$

In reality $\frac{L^2}{2\mu r^2}$ is kinetic energy of the particle due to angular motion

But, because of its dependence on position, it can be treated as an effective potential energy
Integrating the equations of motion

- Energy conservation equation yields
  \[
  \frac{dr}{dt} = \sqrt{\frac{2}{\mu}} (E - V_{\text{eff}}(r))
  \]

- Leading to the solution
  \[
  \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu} (E - V_{\text{eff}}(r))}} = t - t_0, \quad (1)
  \]

which will yield \( r \) as a function of \( t \), once \( f(r) \) is known, and the integral is performed
Once $r(t)$ is known, to obtain $\theta(t)$, we use conservation of angular momentum

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

$$\theta - \theta_0 = \frac{L}{\mu} \int_{t_0}^{t} \frac{dt}{r^2}$$

We can obtain the shape of the trajectory $r(\theta)$, by combining these two equations

$$\frac{d\theta}{dr} = \left( \frac{d\theta}{dt} \right) = \frac{L}{\mu r^2} \sqrt{\frac{2}{\mu} \left( E - V_{\text{eff}}(r) \right)}$$

Leading to

$$\theta - \theta_0 = L \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{2\mu(E - V_{\text{eff}}(r))}} \quad (2)$$
Thus, by integrating these equations, we can obtain \( r(t) \), \( \theta(t) \), and \( r(\theta) \).

This will complete the solution of the problem.

But, to make further progress, we need to know what is \( f(r) \).

Next, we will discuss the case of gravitational problem such as planetary orbits.
We want to use the theory developed to calculate the orbits of different planets around sun.

Planets are bound to sun because of gravitational force.

Therefore

\[ f(r) = -\frac{GMm}{r^2} \]

So that

\[ V(r) = -\frac{GMm}{r} = -\frac{C}{r}, \quad (3) \]

above, \( C = GMm \), where \( G \) is gravitational constant, \( M \) is mass of the Sun, and \( m \) is mass of the planet in question.
Derivation of Keplerian orbits

On substituting $V(r)$ from Eq. 3 into Eq. 2, we have

$$\theta - \theta_0 = L \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{2\mu \left( E - \frac{L^2}{2\mu r^2} + \frac{C}{r} \right)}}$$

$$= L \int \frac{dr}{r \sqrt{2\mu Er^2 + 2\mu Cr - L^2}} \quad (4)$$

We converted the definite integral on the RHS to an indefinite one, because $\theta_0$ is a constant of integration in which the constant contribution of the lower limit $r = r_0$ can be absorbed. This orbital integral can be done by the following substitution

$$r = \frac{1}{s - \alpha} \quad (5)$$

$$\Rightarrow dr = -\frac{ds}{(s - \alpha)^2}$$

$$\Rightarrow \frac{dr}{r} = -\frac{ds}{(s - \alpha)} \quad (6)$$
Substituting Eqs. 5 and 6, in Eq. 4, we obtain

\[ \theta - \theta_0 = -L \int \frac{ds}{(s - \alpha) \sqrt{\frac{2\mu E}{(s-\alpha)^2} + \frac{2\mu C}{s-\alpha} - L^2}} \]

\[ = -L \int \frac{ds}{\sqrt{2\mu E + 2\mu C(s - \alpha) - L^2(s - \alpha)^2}} \]

\[ = -L \int \frac{ds}{\sqrt{2\mu E + 2\mu Cs - 2\mu C\alpha - L^2s^2 + 2L^2\alpha s - L^2\alpha^2}} \]

The integrand is simplified if we choose \( \alpha = -\frac{\mu C}{L^2} \), leading to

\[ \theta - \theta_0 = -L \int \frac{ds}{\sqrt{2\mu E + 2\left(\frac{\mu C}{L^2}\right)^2 - L^2s^2 - \left(\frac{\mu C}{L^2}\right)^2}} \]

\[ = -L \int \frac{ds}{\sqrt{2\mu E + \left(\frac{\mu C}{L^2}\right)^2 - L^2s^2}} \]
Finally, the integral is

\[ \theta - \theta_0 = -L^2 \int \frac{ds}{\sqrt{2\mu EL^2 + (\mu C)^2 - L^4 s^2}} \]

\[ = - \int \frac{ds}{\sqrt{2\mu EL^2 + (\mu C)^2 - L^4 s^2}} \]

On substituting \( s = a \sin \phi \), where \( a = \sqrt{\frac{2\mu EL^2 + (\mu C)^2}{L^4}} \), the integral transforms to

\[ \theta - \theta_0 = -\phi = -\sin^{-1} \left( \frac{S}{a} \right) \]

\[ s = -a \sin(\theta - \theta_0) \]

\[ \Rightarrow \frac{1}{r} + \alpha = -a \sin(\theta - \theta_0) \]

\[ \Rightarrow r = \frac{1}{-\alpha - a \sin(\theta - \theta_0)} \]
Keplerian Orbit

We define $r_0 = -\frac{1}{\alpha} = \frac{L^2}{\mu C}$, to obtain

$$r = \frac{r_0}{1 - \sqrt{1 + \frac{2EL^2}{\mu C^2} \sin(\theta - \theta_0)}}$$

Conventionally, one takes $\theta_0 = -\pi/2$, and we define

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}$$

To obtain the final result

$$r = \frac{r_0}{1 - \varepsilon \cos \theta}$$

We need to probe this expression further to find which curve it represents.
Curves such as circle, parabola, ellipse, and hyperbola are called conic sections.

We will show that the curve \( r = \frac{r_0}{1 - \varepsilon \cos \theta} \) in plane polar coordinates, denotes different conic sections for various values of \( \varepsilon \), which is nothing but the eccentricity.
Using the fact that \( r = \sqrt{x^2 + y^2} \), and \( \cos \theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \), we obtain

\[
\sqrt{x^2 + y^2} = \frac{r_0}{1 - \frac{\varepsilon x}{\sqrt{x^2 + y^2}}}
\]

\[
\Rightarrow \sqrt{x^2 + y^2} = r_0 + \varepsilon x
\]

\[
\Rightarrow x^2(1 - \varepsilon^2) - 2r_0\varepsilon x + y^2 = r_0^2
\]

Case I: \( \varepsilon = 1 \), which means \( E = 0 \), we obtain

\[
y^2 = 2r_0x + r_0^2
\]

which is nothing but a parabola. This is clearly an open or unbound orbit. This is typically the case with comets.
Nature of orbits: hyperbolic and circular orbits

- Case II: $\epsilon > 1 \implies E > 0$, let us define $A = \epsilon^2 - 1$. With this, the equation of the orbit is
  \[ y^2 - Ax^2 - 2r_0\sqrt{1 + Ax} = r_0^2 \]
  Here, the coefficients of $x^2$ and $y^2$ are opposite in sign, therefore, the curve is unbounded, i.e., open. It is actually the equation of a hyperbola. Therefore, whenever $E > 0$, the particles execute unbound motion, and some comets and asteroids belong to this class.

- Case III: $\epsilon = 0$, we have
  \[ x^2 + y^2 = r_0^2 \]
  which denotes a circle of radius $r_0$, with center at the origin. This is clearly a closed orbit, for which the system is bound.

  \[ \epsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}} = 0 \implies E = -\frac{\mu C^2}{2L^2} < 0. \]
  Satellites launched by humans are put in circular orbits many times, particularly the geosynchronous ones.

Chapter 5: Motion Under the Influence of a Central Force
Nature of orbits: elliptical orbits

Case IV: $0 < \varepsilon < 1 \implies E < 0$, here we define $A = (1 - \varepsilon^2) > 0$, to obtain

$$Ax^2 - 2r_0\sqrt{1-Ax} + y^2 = r_0^2$$

Because coefficients of $x^2$ and $y^2$ are both positive, orbit will be closed (i.e. bound), and will be an ellipse.

To summarize, when $E \geq 0$, orbits are unbound, i.e., hyperbola or parabola

When $E < 0$, orbits are bound, i.e., circle or ellipse.
Time Period of Elliptical orbit

- There are two ways to compute the time needed to go around its elliptical orbit once
- First approach involves integration of the equation

\[
t_b - t_a = \int_{r_a}^{r_b} \frac{dr}{\sqrt{\frac{2}{\mu} \left( E - \frac{L^2}{2\mu r^2} + \frac{C}{r} \right)}}
\]

\[
= \mu \int_{r_a}^{r_b} \frac{rdr}{\sqrt{\left(2\mu Er^2 + 2\mu Cr - L^2\right)}}
\]

- When this is integrated with the limit \( r_b = r_a \), one obtains that time period \( T \) satisfies

\[
T^2 = \frac{\pi^2 \mu}{2C} A^3,
\]

where \( A \) is semi-major axis of the elliptical orbit. This result is nothing but Kepler’s third law.
Time period of the elliptical orbit...

- Now we use an easier approach to calculate the time period
- We use the constancy of angular momentum

\[ L = \mu r^2 \frac{d\theta}{dt} \]

\[ \Rightarrow \quad \frac{L}{2\mu} \ dt = \frac{1}{2} r^2 \ d\theta \]

R.H.S. of the previous equation is nothing but the area element swept as the particle changes its position by \( d\theta \)

- Now, the integrals on both sides can be carried out to yield

\[ \frac{LT}{2\mu} = \text{area of ellipse} = \pi ab. \]
Time period of the orbit contd.

- $a$ and $b$ in the equation are semi-major and semi-minor axes of the ellipse as shown

![Diagram of an ellipse with labels $a$ and $b$.]

- Now, we have

![Diagram of an ellipse with labels $r_{\text{min}}$, $r_{\text{max}}$.]

- Therefore

\[ a = \frac{A}{2} = \frac{r_{\text{min}} + r_{\text{max}}}{2} \]
Using the orbital equation $r = \frac{r_0}{1 - \varepsilon \cos \theta}$, we have

$$a = \frac{1}{2} \left( \frac{r_0}{1 - \varepsilon \cos \pi} + \frac{r_0}{1 - \varepsilon \cos 0} \right) = \frac{r_0}{2} \left( \frac{1}{1 + \varepsilon} + \frac{1}{1 - \varepsilon} \right) = \frac{r_0}{1 - \varepsilon^2}$$

Calculation of $b$ is slightly involved. Following diagram is helpful.
\( x_0 \) is the distance between the focus and the center of the ellipse, thus

\[
x_0 = a - r_{\text{min}} = \frac{r_0}{1 - \varepsilon^2} - \frac{r_0}{1 + \varepsilon} = \frac{r_0 \varepsilon}{1 - \varepsilon^2}
\]

In the diagram \( b = \sqrt{r^2 - x_0^2} \), and for \( \theta \), we have \( \cos \theta = \frac{x_0}{r} \), which on substitution in orbital equation yields

\[
r = \frac{r_0}{1 - \varepsilon \cos \theta} = \frac{r_0}{1 - \frac{\varepsilon x_0}{r}}
\]

\[\implies r = r_0 + \varepsilon x_0 = r_0 + \frac{r_0 \varepsilon^2}{1 - \varepsilon^2} = \frac{r_0}{1 - \varepsilon^2}
\]

So that

\[
b = \sqrt{r^2 - x_0^2} = \sqrt{\frac{r_0^2}{(1 - \varepsilon^2)^2} - \frac{r_0^2 \varepsilon^2}{(1 - \varepsilon^2)^2}} = \frac{r_0}{\sqrt{1 - \varepsilon^2}}
\]
Now

\[ 1 - \varepsilon^2 = 1 - \left(1 + \frac{2EL^2}{\mu C^2}\right) = -\frac{2EL^2}{\mu C^2} \]

Using \( r_0 = \frac{L^2}{\mu C} \), we have

\[ A = 2a = \frac{2r_0}{1 - \varepsilon^2} = \frac{2L^2}{\mu C} \times \left(-\frac{\mu C^2}{2EL^2}\right) = -\frac{C}{E} \]

\[ b = \frac{r_0}{\sqrt{1 - \varepsilon^2}} = \frac{L^2}{\mu C} \times \sqrt{-\frac{\mu C^2}{2EL^2}} = L \sqrt{-\frac{1}{2\mu E}} \]

Using this, we have

\[ T = \frac{2\pi \mu}{L} ab = \frac{2\pi \mu}{L} \times \left(-\frac{C}{2E}\right) \times L \sqrt{-\frac{1}{2\mu E}} = \pi \sqrt{\frac{\mu}{2C}} \left(-\frac{C}{E}\right)^{3/2} \]
Kepler’s Third Law

Which can be written as

\[ T = \pi \sqrt[3]{\frac{\mu}{2C}} A^{3/2} \]

\[ \implies T^2 = \frac{\pi^2 \mu}{2C} A^3, \]

which is nothing but Kepler’s third law.