EP 222: Classical Mechanics
Tutorial Sheet 1

This tutorial sheet contains problems on the Newton's laws of motion and Lagrangian formalism.

1. Show that for a single particle with a constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

3. Show that the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j,$$

can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

These are sometimes called the Nielsen form of Lagrange equations.

4. If L is a Lagrangian for a system of n degrees of freedom satisfying the Lagrange equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, \ldots, q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

5. Obtain the Lagrange equations of motion for a spherical pendulum, i.e., a point mass suspended by a rigid weightless rod.

6. Obtain the Lagrangian and equations of motion for a double pendulum, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2, confined to move in a plane.

7. If we want to obtain the equations of motion for a charged particle of mass m, moving in an electromagnetic field ($\mathbf{E}$, $\mathbf{B}$), the potential in the Lagrangian has to be velocity dependent $U = q\phi - q\mathbf{A}\cdot\mathbf{v}$, where q is the charge of the particle, and ϕ, and $\mathbf{A}$, respectively, are the scalar and vector potentials of the electromagnetic field so that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Show that using this Lagrangian, we obtain the correct equations of motion for the particle.

8. The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \nabla\psi(\mathbf{r}, t), \\ \phi &\rightarrow \phi - \frac{\partial \psi}{\partial t}, \end{aligned}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a moving particle in the electromagnetic field? Is the equation of motion affected?