EP 222: Classical Mechanics
Tutorial Sheet 3

This tutorial sheet contains exercises related to the central force problem.

1. Suppose a satellite is moving around a planet in a circular orbit of radius r_0. Due to a collision with another object, satellite's orbit gets perturbed. Show that the radial position of the satellite will execute simple harmonic motion with $\omega = \frac{l}{mr_0^2}$, where l is the initial angular momentum of the satellite.
   Soln: Because it is a small perturbation, we can Taylor expand the potential energy of the satellite around $r_0 = r_{min}$

$$V'(r) = V_{min} + (r - r_{min}) \left.\frac{dV'}{dr}\right|_{r=r_{min}} + \frac{1}{2}(r - r_{min})^2 \left.\frac{d^2V'}{dr^2}\right|_{r=r_{min}} + \cdots$$

Noting that

$$\left.\frac{dV'}{dr}\right|_{r=r_{min}} = 0$$
$$\left.\frac{d^2V'}{dr^2}\right|_{r=r_{min}} = \frac{3l^2}{mr_{min}^4} - \frac{2k}{r_{min}^3} = \frac{3l^2}{m}\left(\frac{m^4k^4}{l^8}\right) - 2k\left(\frac{m^3k^3}{l^6}\right) = \frac{m^3k^4}{l^6}$$

But, balance of force condition for the circular orbit yields

$$\frac{mv^2}{r_{min}} = \frac{k}{r_{min}^2},$$

from which, using the fact that $v = l/mr_{min}$, we obtain $k = l^2/mr_{min}$. Therefore,

$$\left.\frac{d^2V'}{dr^2}\right|_{r=r_{min}} = \frac{m^3}{l^6}\left(\frac{l^8}{m^4r_{min}^4}\right) = \frac{l^2}{mr_{min}^4}$$

but $r_{min} = r_0$

$$V'(r) \approx V_{min} + \frac{l^2}{2mr_0^4}(r - r_0)^2$$

Radial equation of motion of the perturbed orbit

$$m\ddot{r} = -\frac{dV'(r)}{dr} = -\frac{l^2}{mr_0^4}(r - r_0)$$

Define $x = r - r_0$, we obtain from above

$$\ddot{x} + \omega^2 x,$$

where $\omega = \frac{l}{mr_0^2}$, and equation above denotes simple harmonic motion about $r = r_0$, with frequency ω.

2. A particle of mass m is moving under the influence of a central force $\mathbf{F}(\mathbf{r}) = -\frac{C}{r^3}\hat{\mathbf{r}}$, with $C > 0$. Find the nonzero values of angular momentum l for which the particle will move in a circular orbit.
Soln: For this, the potential energy can be obtained as

$$V(r) = -\int_{\infty}^{r} F(r')dr' = C\int_{\infty}^{r} \frac{dr'}{r'^3} = -\frac{C}{2r^2}\bigg|_{\infty}^{r} = -\frac{C}{2r^2}.$$

The effective potential energy for this case

$$V^{'}(r) = \frac{l^2}{2mr^2} - \frac{C}{2r^2}.$$

We know that for the circular orbit, the total energy must be equal to the minimum of the effective potential energy, which can be found by

$$\frac{dV^{'}(r)}{dr} = -\frac{l^2}{mr^3} + \frac{C}{r^3} = 0$$
$$\implies l = \sqrt{mC}.$$

Thus, if the system has this angular momentum, circular orbit of any radius is possible.

3. In the lectures, we obtained the equation of the orbit (an equation connecting r and θ) for the Kepler's problem ($V(r) = -k/r$), by solving a second order differential equation for the variable $u = 1/r$. Show that one gets the same result if one directly integrates the integral connecting the r and θ coordinates, derived in the lectures.
Soln: In the lectures it was shown that the equation of the orbit is given by

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}},$$

where $u = 1/r$. For the Kepler problem $V(r) = -k/r \implies V(u) = -ku$, so that the integral becomes

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u^2}}. \tag{1}$$

Integral of Eq. (1) can be performed using the standard integral

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1} - \left(\frac{\beta + 2\gamma x}{\sqrt{q}}\right), \tag{2}$$

where, $q = \beta^2 - 4\alpha\gamma$, and identifying

$$\alpha = \frac{2mE}{l^2}, \quad \beta = \frac{2mk}{l^2}, \quad \gamma = -1,$$

we obtain in Eq. (1)

$$\theta - \theta_0 = -\cos^{-1}\left[-\left\{\frac{\frac{2mk}{l^2} - 2u}{\sqrt{\frac{4m^2k^2}{l^4} + \frac{8mE}{l^2}}}\right\}\right],$$

which can be simplified to

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + e\cos(\theta - \theta')\right),$$

where

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}.$$

Clearly, the orbit is a conic section of eccentricity e.

4. Two particles move about each other in circular orbits under the influence of gravitational forces, with a period τ. Their motion is suddenly stopped at a given instant of time, and they are then released and allowed to fall into each other. Prove that they collide after a time $\tau/4\sqrt{2}$.
Solution: Since the two bodies are interacting under a central force (gravitational force), their motion can be seen as that of a single particle of mass $\mu = \frac{m_i m_2}{m_i + m_2}$, about the center of mass. So, here we have for circular orbits

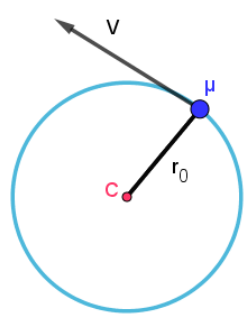

Figure 1:

$$\frac{\mu v^2}{r_0} = \frac{k}{r_0^2},$$

$$v = \sqrt{\frac{k}{r_0 \mu}},$$

So, the time period

$$\tau = \frac{2\pi r_0}{v}$$

$$\tau = 2\pi\sqrt{\frac{r_0^3\mu}{k}}, \tag{3}$$

$$r_0 = \left(\frac{k\tau^2}{4\pi^2\mu}\right)^{\frac{1}{3}}.$$

If the particle are momentarily stopped at this position and then released, the law of conservation of energy for subsequent motion tell us (motion is purely radial)

$$\begin{aligned}
\frac{1}{2}\mu\dot{r}^2 - \frac{k}{r} &= -\frac{k}{r_0}, \\
\frac{dr}{dt} &= -\sqrt{\frac{2k}{\mu}\left(\frac{1}{r} - \frac{1}{r_0}\right)},
\end{aligned}$$

negative sign for radial speed implies that radial distance is decreasing with time. Thus the time taken for the particle to collide with each other is

$$\begin{aligned}
T &= -\sqrt{\frac{\mu}{2k}}\int_{r_0}^{0} \frac{dr}{\sqrt{\left(\frac{1}{r} - \frac{1}{r_0}\right)}}, \\
T &= -\sqrt{\frac{\mu}{2k}r_0}\int_{r_0}^{0} \frac{\sqrt{r}dr}{\sqrt{r_0 - r}},
\end{aligned}$$

substituting $r = r_0 \sin^2\theta$ and $dr = 2r_0 \sin\theta\cos\theta d\theta$. So that

$$\begin{aligned}
-\int_{r_0}^{0} \frac{\sqrt{r}dr}{\sqrt{r_0 - r}} &= \int_{0}^{\frac{\pi}{2}} \frac{r_0^{\frac{1}{2}} \sin\theta 2r_0 \sin\theta\cos\theta d\theta}{r_0^{\frac{1}{2}} \cos\theta}, \\
&= 2r_0 \int_{0}^{\frac{\pi}{2}} \sin^2\theta d\theta, \\
&= r_0 \int_{0}^{\frac{\pi}{2}} (1 - \cos 2\theta)\,\theta d\theta, \\
&= \left[\left(\theta - \frac{1}{2}\sin 2\theta\right) r_0\right]_0^{\frac{\pi}{2}}, \\
&= \frac{\pi}{2}r_0.
\end{aligned}$$

So,

$$T = \sqrt{\frac{\mu}{2k}r_0}\left(\frac{\pi}{2}r_0\right),$$

$$T = \frac{\pi}{2\sqrt{2}}\sqrt{\frac{\mu}{k}r_0^3},$$

using Eq. 1, we get

$$T = \frac{\tau}{4\sqrt{2}}.$$

5. Show that the motion of a particle in the potential field

$$V(r) = -\frac{k}{r} + \frac{h}{r^2}$$

is the same as that of the motion under the Kepler potential alone when expressed in terms of a coordinate system rotating or precessing around the center of force. For negative total energy, show that if the additional potential term is very small compared to the Kepler potential, then the angular speed of precession of the elliptical orbit is

$$\dot{\Omega} = \frac{2\pi mh}{l^2\tau}.$$

The perihelion of Mercury is observed to precess (after correction for known planetary perturbations) at the rate of $40''$ of arc per century. Show that this precession could be accounted for classically if the dimensionless quantity

$$\eta = \frac{h}{ka}$$

(which is a measure of the perturbing inverse-square potential relative to the gravitational potential) were as small as 7×10^{-8}. (The eccentricity of Mercury's orbit is 0.206, and its period is 0.24 year.)
Solution: The equation of the orbit is given by

$$\theta = \theta_0 - \int\limits_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}},$$

since

$$V(r) = -\frac{k}{r} + \frac{h}{r^2},$$
$$V(u) = -ku + hu^2,$$

So from above we get

$$\theta = \theta_0 - \int\limits_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - \left(\frac{2mh}{l^2} + 1\right)u^2}},$$

but we know that

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}}\cos^{-1} - \left(\frac{\beta + 2\gamma x}{\sqrt{q}}\right),$$

where, $q = \beta^2 - 4\alpha\gamma$. Here, we have

$$\alpha = \frac{2mE}{l^2}, \quad \beta = \frac{2mk}{l^2},$$

$$\gamma = -\left(\frac{2mh + l^2}{l^2}\right),$$

so,

$$q = \beta^2 - 4\alpha\gamma = \frac{4m^2k^2}{l^4} + \frac{8mE\left(2mh + l^2\right)}{l^4},$$

$$q = \frac{4m^2k^2 + 16m^2Eh + 8mEl^2}{l^4}.$$

So, we get

$$\theta = \theta_0 - \frac{l}{\sqrt{2mh + l^2}} \cos^{-1}\left[-\frac{2mk - 2\left(2mh + l^2\right)u}{\sqrt{4m^2k^2 + 16m^2Eh + 8mEl^2}}\right],$$

$$\theta = \theta_0 - \frac{l}{\sqrt{2mh + l^2}} \cos^{-1}\left[-\frac{\left(\left(\frac{2mh+l^2}{mk}\right)u - 1\right)}{\sqrt{\frac{2E}{mk^2}\left(2mh + l^2\right) + 1}}\right],$$

$$\sqrt{\frac{1 + 2mh}{l^2}}\left(\theta - \theta_0\right) = -\cos^{-1}\left[\frac{\left(\left(\frac{2mh+l^2}{mk}\right)u - 1\right)}{\sqrt{\frac{2E}{mk^2}\left(2mh + l^2\right) + 1}}\right],$$

or by further simplifying we get,

$$\frac{1}{r} = c\left[1 + \epsilon\cos\alpha\theta\right], \tag{4}$$

where

$$\begin{aligned} c &= \frac{2mk}{2mh + l^2}, \\ \epsilon &= \sqrt{\left(1 + \frac{2E\left(2mh + l^2\right)}{mk^2}\right)}, \\ \alpha &= \sqrt{1 + \frac{2mh}{l^2}}. \end{aligned}$$

Eq. 2 is the equation of an ellipse, where the particle makes an angle θ from a fixed direction in space, but it makes an angle $\alpha\theta$ from the semi-major axis.

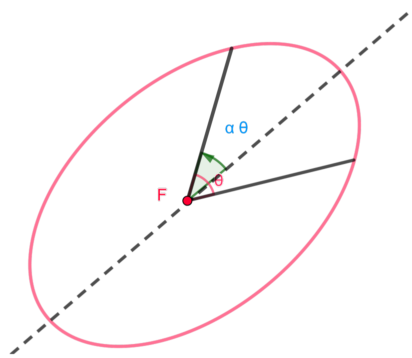

Figure 2:

Clearly, this corresponds to a semi-major axis that is rotating in space, or the ellipse is precessing. This is can be written as

$$\alpha\theta(t) = \theta(t) - \dot{\Omega}t,$$

where, $\dot{\Omega}$ is angular velocity of precession of the semi-major axis.

$$\alpha\dot{\theta} = \dot{\theta} - \dot{\Omega},$$

$$\dot{\Omega} = (1-\alpha)\,\dot{\theta} = \dot{\theta}\left(1 - \sqrt{1+\frac{2mh}{l^2}}\right),$$

for small h,

$$\dot{\Omega} = \frac{\dot{\theta}mh}{l^2}.$$

For small h, $\dot{\Omega}$ will be small and hence we can get a good estimate of it by putting average value of $\dot{\theta}$, i.e.

$$\dot{\theta} = \frac{2\pi}{\tau},$$

$$\dot{\Omega} = \frac{2\pi mh}{l^2\tau}.$$

Because h is small, we can approximate ϵ by e, the eccentricity of the perfect ellipse

$$\epsilon \approx e = \sqrt{1+\frac{2El^2}{mk^2}},$$

and using the fact that for perfect elliptical orbit $E = -k/2a$, we obtain from above

$$\frac{m}{l^2} = \frac{1}{(1-\epsilon^2)\,ka},$$

leading to

$$\dot{\Omega} = \frac{2\pi}{\tau}\left(\frac{h}{ka}\right)\frac{1}{1-\epsilon^2},$$

$$\dot{\Omega} = \frac{2\pi}{\tau}\eta\frac{1}{1-\epsilon^2},$$

$$\eta = \frac{\dot{\Omega}\left(1-\epsilon^2\right)\tau}{2\pi},$$

where $\epsilon = 0.206$, $\tau = 0.24$ year and $\dot{\Omega} = 40''$ per century. By substituting all these above, we obtain

$$\eta = 7.0 \times 10^{-8}.$$

6. A geostationary orbit is one in which a satellite moves in a circular orbit at the given height in the equatorial plane, so that its angular velocity of rotation around earth is same as earth's angular velocity, thereby, making it look stationary when seen from a point on equator. Assuming that the earth's rotational velocity, and radius, respectively, are $\Omega_e = \frac{2\pi}{86400}$ rad/s, and$R_e = 6400$ km, calculate the altitude of the satellite, and its orbital velocity.

   Soln: The radius of the circular orbit is obtained by the force condition

$$\frac{GM_e m}{r^2} = \frac{mv^2}{r}$$

$$\implies r = \frac{GM_e}{v^2}$$

   For geostationary satellite $v = \Omega_e r$, therefore,

$$r = \frac{GM_e}{\Omega_e^2 r^2}$$

$$\implies r = \left(\frac{GM_e}{\Omega_e^2}\right)^{1/3}$$

   But $r = h + R_e$, where h is the needed altitude, and R_e is the radius of the earth, and $GM_e = gR_e^2$, therefore

$$h = \left(\frac{gR_e^2}{\Omega_e^2}\right)^{1/3} - R_e.$$

   Using the values $g = 9.8$ m/s^2, $R_e = 6.4 \times 10^6$ m, and $\Omega_e = \frac{2\pi}{86400}$ s^{-1}, we obtain $h \approx 35850$ km. And orbital speed of the satellite $v = r\Omega_e = (35850 + 6400) \times 10^6 \times \frac{2\pi}{86400} = 3070\, m/s$

7. A space company wants to launch a satellite of mass $m = 2000$ kg, in an elliptical orbit around earth, so that the altitude of the satellite above earth at perigee is 1100 kms, and at apogee it is 35,850 kms. Assuming that the launch takes place at the equator, calculate: (a) energy of the satellite in the elliptical orbit, (b) energy required to launch the satellite, (c) eccentricity of the orbit, (d) angular momentum of the satellite, and (e) speeds of the satellite at apogee and perigee. Use the values of R_e and Ω_e specified in the previous problem.(a) We showed in the lectures that for the gravitational potential energy of the form

$$V(r) = -\frac{k}{r},$$

the energy of a mass moving in an elliptical orbit is

$$E = -\frac{k}{A},$$

where A is the major axis of the ellipse. In this case $k = GM_e m = R_e^2 gm$, where m is the mass of the satellite. This elliptical orbit is about earth, with earth's center as one of its foci. Thus, A will be sum of earth's diameter, altitude at perigee, and altitude at apogee

$$A = (1100 + 2 \times 6400 + 35,8500) \times 10^3 = 5 \times 10^7 m.$$

Therefore,

$$E_{orb} = -\frac{9.8 \times (6.4 \times 10^6)^2 \times 2000}{5 \times 10^7} = -1.61 \times 10^{10} J$$

(b) The energy of the satellite just before the launch is nothing but its gravitational potential energy at the surface of the earth, and kinetic energy due to rotation of the earth at the equator

$$\begin{aligned}
E_{ground} = V(r) + K &= -\frac{GM_e m}{R_e} + \frac{1}{2} m(\Omega_e R_e)^2 \\
&= -mgR_e + \frac{1}{2} m(\Omega_e R_e)^2 \\
&= -2000 \times 9.9 \times 6.4 \times 10^6 + 0.5 \times 2000 \times (6.4 \times 10^6)^2 \times (\frac{2\pi}{86400})^2 \\
&= -1.25 \times 10^{11} J.
\end{aligned}$$

Therefore, energy required to launch the satellite will be

$$\Delta E = E_{orb} - E_{ground} = 1.09 \times 10^{11} J$$

(c) From the equation of the orbit $r = r_0/(1 + e\cos\theta)$, the radial distances from the focus corresponding to perigee (r_{min}) and apogee (r_{max}) are given by

$$\begin{aligned}
r_{min} &= \frac{r_0}{1+e} \\
r_{max} &= \frac{r_0}{1-e}
\end{aligned}$$

These equations lead to

$$\begin{aligned}
r_0 &= r_{min}(1+\epsilon) = r_{max}(1-\epsilon) \\
\implies \epsilon &= \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{(35850 + 6400) - (1100 + 6400)}{(35850 + 6400) + (1100 + 6400)} = 0.7
\end{aligned}$$

(d) To obtain the angular momentum we use the formula for eccentricity derived in the lectures

$$e^2 = 1 + \frac{2E_{orb} l^2}{mk^2},$$

which on using various values yields

$$l = 1.43 \times 10^{14} kg\text{-}m^2/s$$

(e) We know that at perigee and apogee the velocity of the satellite will be perpendicular to the radial distance from the earth's center, thus

$$l = mr_p v_p = mr_a v_a,$$

where subscripts p and a denote, perigee and apogee respectively, $m = 2000$ kg, $r_p = r_{min} = 1100 + 6400 = 7.5 \times 10^6\ m$, $r_o = r_{max} = 35850 + 6400 = 4.225 \times 10^7\ m$. With this we obtain

$$v_a = \frac{l}{mr_a} = 1690\, m/s$$
$$v_p = \frac{l}{mr_p} = 9530\, m/s$$

8. The ultimate aim of the space company of the previous problem is to put the satellite in a geostationary orbit. Therefore, after launching it in the elliptical orbit, the company wants to transfer it in a geostationary orbit by firing rockets at the apogee to increase its speed to the required one. How much change in speed is needed to put the satellite in the geostationary orbit, and how much energy will be required to achieve that change?
Soln: Recalling that in problem 5 we obtained that the radius of the geostationary orbit is $R_{geo} = 35850\ km + 6400\ km = 4.225 \times 10^7\ m$, which is identical to the radial distance at the apogee r_o for the elliptical orbit. Thus, it is best to fire the rockets at the apogee of the elliptical orbit, to provide it the energy needed for a geostationary orbit. Now, energy required will be

$$\Delta E = -\frac{k}{A_{geo}} - E_{orb},$$

where E_{orb} is the energy of the elliptical orbit computed in the last problem, while A_{geo} is the major axis corresponding to the geostationary orbit. But, because geostationary orbit is a circular one, therefore, its major axis is nothing but its diameter, so that $A_{geo} = 2R_{geo} = 8.45 \times 10^7 m$. Using this we obtain

$$\Delta E = 6.6 \times 10^9\ J.$$

To compute the change in speed, we note that change in energy ΔE, changes only the kinetic energy of the satellite because during the rocket firing, the location of the satellite does not change, and hence its potential energy remains constant. Thus, if v_f is the final speed of the satellite after the rocket is fired, we have

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_a^2 = \Delta E = 6.6 \times 10^9$$
$$\implies v_f = \sqrt{\frac{2\Delta E + mv_a^2}{m}}$$

Above v_a is the speed of the satellite at the apogee, calculated in the previous problem. Using values of various quantities, we obtain the required change in speed

$$\Delta v = v_f - v_a = \sqrt{\frac{2 \times 6.6 \times 10^9 + 2000 \times (1690)^2}{2000}} - 1690$$
$$= 3070 - 1690 = 1110\, m/s$$

9. Examine the scattering produced by a repulsive central force $f = kr^{-3}$. Show that the differential cross section is given by

$$\sigma(\Theta)d\Theta = \frac{k}{2E}\frac{(1-x)dx}{x^2(2-x)^2\sin\pi x},$$

where $x = \Theta/\pi$, and E is energy.
Solution: The repulsive force is given as,

$$f = \frac{k}{r^3},$$

$$V = \frac{2k}{r^2} = 2ku^2,$$

So the equation of orbit is

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{4mku^2}{l^2} - u^2}},$$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \left(1 + \frac{4mk}{l^2}\right)u^2}},$$

but we know that

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}}\cos^{-1} - \left(\frac{\beta + 2\gamma x}{\sqrt{q}}\right),$$

where, $q = \beta^2 - 4\alpha\gamma$. Here, we have

$$\alpha = \frac{2mE}{l^2}, \quad \beta = 0,$$

and

$$\gamma = -\left(1 + \frac{4mk}{l^2}\right).$$

So, we get

$$(\theta - \theta_0) = -\frac{l}{\sqrt{l^2 + 4mk}}\cos^{-1} u\sqrt{\frac{1 + \frac{4mk}{l^2}}{\frac{2mE}{l^2}}},$$

If we measure angles from periapsis we can set $\theta_0 = 0$, so that

$$u\sqrt{\frac{l^2}{2mE} + \frac{2k}{E}} = \cos\sqrt{1 + \frac{4mk}{l^2}}\theta,$$

$$\frac{1}{r} = \frac{1}{\sqrt{\frac{2k}{E} + \frac{l^2}{2mE}}}\cos\sqrt{1 + \frac{4mk}{l^2}}\theta,$$

for $r \to \infty$, $\theta \to \left(\frac{\pi}{2} - \frac{\Theta}{2}\right)$,

$$\cos\sqrt{1 + \frac{4mk}{l^2}}\left(\frac{\pi}{2} - \frac{\Theta}{2}\right) = 0,$$

$$\sqrt{1 + \frac{4mk}{l^2}}\left(\frac{\pi}{2} - \frac{\Theta}{2}\right) = \frac{\pi}{2},$$

define $x = \frac{\Theta}{\pi}$ and using $l = s\sqrt{2mE}$,we get

$$1 + \frac{2k}{s^2 E} = \frac{1}{(1-x)^2},$$

$$s = \sqrt{\frac{2k}{E}}\frac{(1-x)}{\sqrt{x(2-x)}}.$$

Now,

$$\frac{ds}{d\Theta} = \frac{ds}{\pi dx} = \frac{1}{\pi}\sqrt{\frac{2k}{E}}\left\{\frac{-\sqrt{x(2-x)} - \frac{(1-x)(2-2x)}{2\sqrt{x(2-x)}}}{x(2-x)}\right\},$$

$$|\frac{ds}{d\Theta}| = \frac{1}{\pi}\sqrt{\frac{2k}{E}}\frac{1}{x^{\frac{3}{2}}(2-x)^{\frac{3}{2}}}.$$

Now,

$$\sigma(\Theta)d\Theta = \frac{s}{\sin\Theta}|\frac{ds}{d\Theta}|d\Theta,$$

So,

$$\sigma(\Theta)d\Theta = \frac{1}{\pi}\sqrt{\frac{2k}{E}}\frac{(1-x)}{\sqrt{x(2-x)}\sin\pi x}\sqrt{\frac{2k}{E}}\frac{\pi dx}{x^{\frac{3}{2}}(2-x)^{\frac{3}{2}}},$$

$$\sigma(\Theta)d\Theta = \frac{2k}{E}\frac{(1-x)\,dx}{x^2(2-x)^2\sin\pi x}$$