EP 222: Classical Mechanics
Tutorial Sheet 4: Solution

This tutorial sheet contains problems related to rigid body kinematics.

1. (a) By examining the eigenvalues of an antisymmetric 3×3 real matrix A, show that $I \pm A$ is nonsingular, where I is the identity matrix.
Soln: Let us examine the nature of eigenvalues of A, which satisfies $A^T = -A$. If λ is an eigenvalue with eigenvector X, then

$$\begin{aligned} AX &= \lambda X \\ \implies (AX)^\dagger &= \lambda^* X^\dagger \\ \implies X^\dagger A^T &= \lambda^* X^\dagger \\ \implies X^\dagger A^T X &= \lambda^* X^\dagger X \\ \implies -X^\dagger A X &= \lambda^* X^\dagger X \\ \implies -\lambda X^\dagger X &= \lambda^* X^\dagger X \\ \implies \lambda &= -\lambda^* \end{aligned}$$

This implies that the eigenvalues of A are either purely imaginary or zero. Thus the eigenvalues of $I \pm A$ will be $1 \pm \lambda$, which will never be zero. Thus $\det(I \pm A)$ will never be zero, i.e., matrix $I \pm A$ is nonsingular

(b) Show then that under the same conditions the matrix

$$B = (I + A)(I - A)^{-1}$$

is orthogonal.
Soln: If B is orthogonal, it must satisfy

$$B^T B = B B^T = I$$

(i) Check $B^T B$

$$\begin{aligned} B^T B &= \{(I + A)(I - A)^{-1}\}^T (I + A)(I - A)^{-1} \\ &= \{(I - A)^T\}^{-1} (I + A)^T (I + A)(I - A)^{-1} \\ &= (I + A)^{-1}(I - A)(I + A)(I - A)^{-1} \end{aligned}$$

But $(I - A)(I + A) = (I - A + A - A^2) = (I - A^2) = (I + A)(I - A)$. Therefore,

$$B^T B = (I + A)^{-1}(I + A)(I - A)(I - A)^{-1} = I$$

(ii) Check BB^T

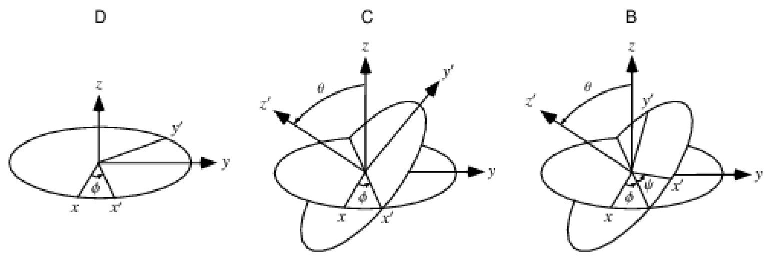
$$\begin{aligned} BB^T &= (I + A)(I - A)^{-1}\{(I + A)(I - A)^{-1}\}^T \\ &= (I + A)(I - A)^{-1}(I + A)^{-1}(I - A) \\ &= (I + A)\{(I + A)(I - A)\}^{-1}(I - A) \\ &= (I + A)\{(I - A)(I + A)\}^{-1}(I - A) \\ &= (I + A)(I + A)^{-1}(I - A)^{-1}(I - A) = I \end{aligned}$$

Hence B is an orthogonal matrix.

2. Show that the components of the angular velocity along the space set of axes are given in terms of the Euler angles by

$$
\begin{aligned}
\omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi, \\
\omega_y &= \dot{\theta}\sin\phi - \dot{\ }\ \sin\theta\cos\psi\phi \\
\omega_z &= \dot{\psi}\cos\theta + \dot{\phi}
\end{aligned}
$$

Soln:



The angular velocities corresponding to three Euler angles, along with their directions are:

(a) $\dot{\phi}$ about z axis

(b) $\dot{\theta}$ about x' axis in the middle figure, so that it has components $\dot{\theta}\cos\phi$ about x axis, and $\dot{\theta}\sin\phi$ about the y axis.

(c) $\dot{\psi}$ about the z' axis, so that in the primed axis, it can be expressed as

$$\omega' = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

. We can find its components along the x, y, and z by applying the inverse of $A = BCD$ (see notes, or book for an explanation), i.e., $A^{-1} = A^T$ which yields

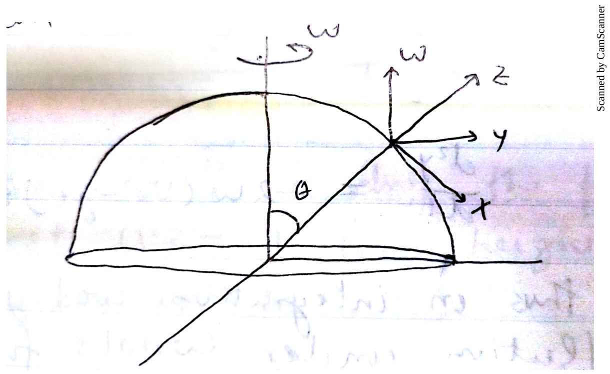
$$A^T\omega' = \begin{pmatrix} \sin\theta\sin\phi\dot{\psi} \\ -\sin\theta\cos\phi\dot{\psi} \\ \cos\theta\dot{\psi} \end{pmatrix}$$

Combining the conclusions of (a), (b), and (c), we obtain the final result

$$
\begin{aligned}
\omega_x &= \dot{\theta}\cos\phi + \sin\theta\sin\phi\dot{\psi} \\
\omega_y &= \dot{\theta}\sin\phi - \sin\theta\cos\phi\dot{\psi} \\
\omega_z &= \dot{\phi} + \cos\theta\dot{\psi}
\end{aligned}
$$

3. A particle is thrown up vertically with initial speed v_0, reaches a maximum height and falls back to ground. Show that the Coriolis deflection when it again reaches the

ground is opposite in direction, and four times greater in magnitude, than the Coriolis deflection when it is dropped at rest from the same maximum height.
Soln: Let us use the coordinate system shown below:



Above, θ is the co-latitude, and the x, y, and z axis are in the local south, east, and vertical directions, respectively. In this coordinate system, the earth's angular velocity $\boldsymbol{\omega}$ at the location is given by

$$\boldsymbol{\omega} = -\omega \sin\theta \hat{i} + \omega \cos\theta \hat{k}$$

Case (a): When the particle is thrown up from the ground with initial speed v_0, its velocity $\mathbf{v} = (v_0 - gt)\hat{k}$, so that the Coriolis force on it will be

$$\begin{aligned}
\mathbf{F}_{cor} &= -2m(\boldsymbol{\omega} \times \mathbf{v}) = -2m\omega \sin\theta (v_0 - gt)\hat{j} \\
\frac{d^2y}{dt^2} &= -2\omega \sin\theta (v_0 - gt) \\
\implies \Delta y &= -\omega \sin\theta (v_0 t^2 - \frac{1}{3} g t^3),
\end{aligned}$$

where Δy is the deflection for time of flight t, obtained by integrating the acceleration equation. Because total time of flight will be $t = 2v_0/g$, so the deflection is

$$\Delta y_1 = -\frac{4}{3} \frac{v_0^3 \omega \sin\theta}{g^2}$$

Case (b): When the particle is dropped from the rest from the same height, then $\mathbf{v} = -gt\hat{k}$, so that we have

$$\begin{aligned}
\mathbf{F}_{cor} &= -2m(\boldsymbol{\omega} \times \mathbf{v}) = 2m\omega \sin\theta g t \hat{j} \\
\frac{d^2y}{dt^2} &= 2\omega \sin\theta g t \\
\implies \Delta y &= \frac{1}{3} \omega \sin\theta g t^3.
\end{aligned}$$

Here the time of flight $t = v_0/g$, so that the deflection is

$$\Delta y_2 = \frac{1}{3} \frac{v_0^3 \omega \sin\theta}{g^2}.$$

Thus $\Delta y_1 = -4\Delta y_2$.

4. A projectile is fired horizontally along Earth's surface. Show that to a first approximation the angular deviation from the direction of fire resulting from the Coriolis effect varies linearly with time at a rate $\omega\cos\theta$, where ω is the angular frequency of Earth's rotation and θ is co-latitude, the direction of deviation being to the right in the northern hemisphere.
Soln: Using the same coordinate system as above, we assume that the projectile is initially fired horizontally towards east with initial speed v_0, so that

$$\mathbf{v} = v_0\hat{\jmath}.$$

Now the Coriolis force on the projectile will be

$$\mathbf{F}_{cor} = -2m(\boldsymbol{\omega} \times \mathbf{v}),$$

using $\boldsymbol{\omega} = -\omega\sin\theta\hat{\imath} + \omega\cos\theta\hat{k}$, we obtain

$$\mathbf{F}_{cor} = 2m\omega v_0(\cos\theta\hat{\imath} + \sin\theta\hat{k}).$$

Clearly, z component of the force is insignificant as compared to gravity, so we ignore it. Then the Coriolis force is in the x direction which is to the right of the direction of motion. Now, deviation Δx due to the Coriolis force can be calculated as

$$m\frac{d^2x}{dt^2} = 2m\omega v_0\cos\theta$$
$$\implies \Delta x = \omega v_0 t^2\cos\theta.$$

Ignoring air friction, displacement in the y direction in time t is

$$\Delta y = v_0 t.$$

If ϕ is the angular deviation from the original direction of motion, then clearly

$$\tan\phi = \frac{\Delta x}{\Delta y}$$
$$\implies \tan\phi \approx \phi = \omega t\cos\theta$$

5. A wagon wheel with spokes is mounted on a vertical axis so it is free to rotate in the horizontal plane. The wheel is rotating with an angular speed of $\omega = 3.0$ radians/s. A bug crawls out on one of the spokes of the wheel with a velocity of 0.5 cm/s holding on the spoke with a coefficient of friction $\mu = 0.30$. How far can the bug crawl along the spoke before it starts to slip?
Soln: In the rotating frame we use cylindrical coordinates. If the bug is moving with an outward velocity v_0, we have

$$\mathbf{v} = v_0\hat{\mathbf{r}}$$
$$\boldsymbol{\omega} = \omega\hat{k}$$

So that

$$\mathbf{F}_{cor} = -2m(\boldsymbol{\omega} \times \mathbf{v}) = -2m\omega v_0 \hat{\boldsymbol{\theta}}$$

so that the normal force due to the Coriolis force is $\mathbf{N}_{cor} = -\mathbf{F}_{cor} = 2m\omega v_0 \hat{\boldsymbol{\theta}}$. Normal force due to gravity is $\mathbf{N}_g = mg\hat{k}$, so that the total normal force is

$$\mathbf{N} = \mathbf{N}_{cor} + \mathbf{N}_g = 2m\omega v_0 \hat{\boldsymbol{\theta}} + mg\hat{k}$$
$$\Longrightarrow \; N = m\sqrt{g^2 + 4\omega^2 v_0^2}.$$

When the centrifugal force experienced by the bug exceeds the net frictional force, the bug will no longer be able to crawl. The limiting condition is

$$m\omega^2 r_{max} = \mu N = m\sqrt{g^2 + 4\omega^2 v_0^2}$$
$$r_{max} = \frac{\sqrt{g^2 + 4\omega^2 v_0^2}}{\omega^2}$$