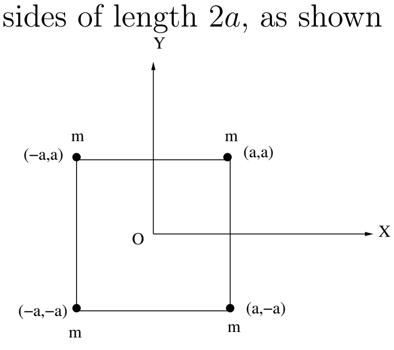
EP 222: Classical Mechanics
Tutorial Sheet 5: Solution

This tutorial sheet contains problems related to angular momentum, inertia tensor, and rigid body motion.

1. Three equal point masses m are located at $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. Find the principal moments of inertia about the origin and a set of principal axes.
Soln: By using the formulas for various components of inertia tensor, one can easily calculate it to be
$$I = \begin{pmatrix} 10ma^2 & 0 & 0 \\ 0 & 6ma^2 & -4ma^2 \\ 0 & -4ma^2 & 6ma^2 \end{pmatrix}$$
on setting up the characteristic polynomial
$$\det(I - \lambda \mathbb{I}) = 0,$$
leads to eigenvalues $\lambda = 2ma^2,\ 10ma^2,\ 10ma^2$, last two of which are degenerate. One can show that the corresponding eigenvectors are $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ leading to principle axes directions $\hat{u}_1 = \frac{1}{\sqrt{2}}\left(\hat{j} + \hat{k}\right)$, $\hat{u}_2 = \hat{\imath}$, and $\hat{u}_3 = \frac{1}{\sqrt{2}}\left(\hat{j} - \hat{k}\right)$

2. Obtain the inertia tensor of a system, consisting of four identical particles of mass m each, arranged on the vertices of a square of sides of length $2a$, with the coordinates of the four particles given by $(\pm a, \pm a, 0)$.
Soln: Consider four identical particles of mass m, arranged on the vertices of a square, with sides of length $2a$, as shown



Noting that for all particles, the z coordinate is zero ($z_i = 0$), we obtain
$$I_{xx} = \sum_{i=1}^{4} m_i(y_i^2 + z_i^2) = 4ma^2$$
$$I_{yy} = \sum_{i=1}^{4} m_i(x_i^2 + z_i^2) = 4ma^2$$
$$I_{zz} = \sum_{i=1}^{4} m_i(x_i^2 + y_i^2) = 8ma^2$$

Now

$$I_{xy} = -\sum_{i=1}^{4} m_i x_i y_i = -m\left(a^2 - a^2 - a^2 + a^2\right) = 0 = I_{yx}$$

$$I_{xz} = -\sum_{i=1}^{4} m_i x_i z_i = -m(0+0+0+0) = 0 = I_{zx}$$

$$I_{yz} = -\sum_{i=1}^{4} m_i y_i z_i = -m(0+0+0+0) = 0 = I_{zy}$$

Thus, tensor of inertia is diagonal here, due to the high symmetry of the problem

$$I = \begin{pmatrix} 4ma^2 & 0 & 0 \\ 0 & 4ma^2 & 0 \\ 0 & 0 & 8ma^2 \end{pmatrix}$$

3. A rigid body consists of three point masses of 2 kg, 1 kg, and 4 kg, connected by massless rods. These masses are located at coordinates (1, -1,1), (2,0,2), and (-1,1,0) in meters, respectively. Compute the inertia tensor of this system. What is the angular momentum vector of this body, if it is rotating with an angular velocity $\boldsymbol{\omega} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$?
Soln: We have

$$I_{xx} = \sum_i m_i(y_i^2 + z_i^2) = 2(1+1) + 1(0+4) + 4(1+0) = 12$$

$$I_{xy} = -\sum_i m_i x_i y_i = -2(-1) - 1(0) - 4(-1) = 6 = I_{yx}$$

$$I_{xz} = -\sum_i m_i x_i z_i = -2(1) - 1(4) - 4(0) = -6 = I_{zx}$$

$$I_{yy} = \sum_i m_i(x_i^2 + z_i^2) = 2(2) + 1(8) + 4(1) = 16$$

$$I_{yz} = -\sum_i m_i y_i z_i = -2(-1) - 1(0) - 4(0) = 2 = I_{zy}$$

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2) = 2(2) + 1(4) + 4(2) = 16$$

Therefore

$$I = \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix}$$

Given angular velocity can be expressed as

$$\boldsymbol{\omega} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

We know that, in the matrix form, one can write

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

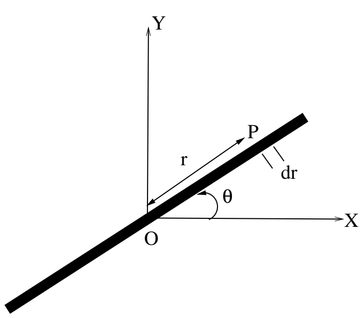
Thus, for this case

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 42 \end{pmatrix}$$

or

$$\mathbf{L} = -6\hat{\mathbf{j}} + 42\hat{\mathbf{k}}$$

4. Obtain the moment of inertia tensor of a thin uniform rod of length l, and mass M, assuming that the origin of the coordinate system is at the center of mass of the rod.
Soln: We assume that the rod is lying in the xy plane, making an angle θ with respect to the x axis, as shown in the figure below



Because the rod lies in the xy plane with $z = 0$, therefore, all its off-diagonal components involving z coordinate will be zero. Nonzero components are

$$\begin{aligned} I_{xx} &= \int dm(y^2 + z^2) = \int dm y^2 \\ I_{yy} &= \int dm(x^2 + z^2) = \int dm x^2 \\ I_{zz} &= \int dm(x^2 + y^2) \\ I_{xy} &= -\int dm xy \end{aligned}$$

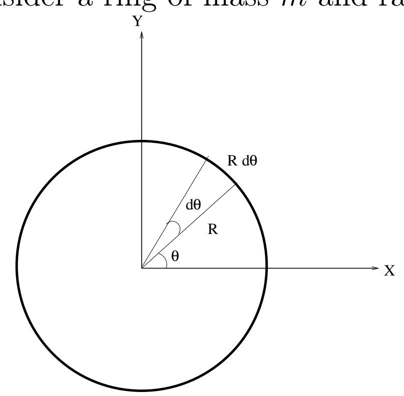
We define a linear density $\lambda = \frac{m}{l}$, so that $dm = \lambda dr$, $x = r\cos\theta$, and $y = r\sin\theta$. Note that here r varies from $-l/2$ to $l/2$, while θ is constant. Therefore,

$$I_{xx} = \lambda \sin^2\theta \int_{-\frac{l}{2}}^{\frac{l}{2}} r^2 dr = \frac{ml^2}{12}\sin^2\theta$$
$$I_{yy} = \lambda \cos^2\theta \int_{-\frac{l}{2}}^{\frac{l}{2}} r^2 dr = \frac{ml^2}{12}\cos^2\theta$$
$$I_{zz} = \lambda \int_{-\frac{l}{2}}^{\frac{l}{2}} r^2 dr = \frac{ml^2}{12}$$
$$I_{xy} = -\lambda \cos\theta\sin\theta \int_{-\frac{l}{2}}^{\frac{l}{2}} r^2 dr = -\frac{ml^2}{12}\sin\theta\cos\theta$$

So that

$$I = \begin{pmatrix} \frac{ml^2}{12}\sin^2\theta & -\frac{ml^2}{12}\sin\theta\cos\theta & 0 \\ -\frac{ml^2}{12}\sin\theta\cos\theta & \frac{ml^2}{12}\cos^2\theta & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{pmatrix}$$

5. Obtain the moment of inertia tensor of a thin uniform ring of radius R, and mass M, with the origin of the coordinate system placed at the center of the ring, and the ring lying in the xy plane.
Soln: Consider a ring of mass m and radius R, as shown



Let us compute I_{zz} for this

$$I_{zz} = \int dm(x^2 + y^2)$$

Mass is distributed uniformly on the ring, therefore, one can define a linear mass density λ

$$\lambda = \frac{M}{2\pi R}$$

Mass dm of a infinitesimal segment of ring which subtends angle $d\theta$ on the center is $dm = \lambda R d\theta = \frac{M}{2\pi} d\theta$. This element is on a circle of radius R, therefore, $x^2 + y^2 = R^2$. Thus

$$I_{zz} = \frac{M}{2\pi}\int R^2 d\theta = \frac{MR^2}{2\pi}\int_0^{2\pi} d\theta = MR^2$$

Let us calculate other elements

$$I_{xx} = \int dm(y^2 + z^2)$$

Ring is in the xy plane, so $z = 0$, and using plane polar coordinates $y = R\sin\theta$, so that

$$I_{xx} = \frac{MR^2}{2\pi}\int_0^{2\pi} \sin^2\theta d\theta = \frac{MR^2}{4\pi}\int_0^{2\pi}(1-\cos 2\theta)d\theta = \frac{MR^2}{2}$$

Similarly, we can show

$$I_{yy} = \frac{MR^2}{2} = I_{xx}$$

All the off-diagonal elements in this case are zero, for example

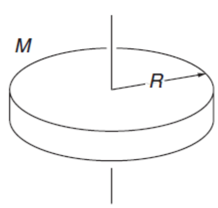
$$\begin{aligned} I_{xy} &= -\int dmxy = -\frac{M}{2\pi}\int (R\cos\theta)(R\sin\theta)d\theta \\ &= -\frac{MR^2}{4\pi}\int_0^{2\pi} \sin 2\theta d\theta = 0 \end{aligned}$$

$I_{xz} = I_{yz} = 0$ trivially, because everywhere on the ring $z = 0$. Thus, the inertia tensor of a uniform ring of mass M and radius R lying in the xy plane, with its center treated as origin

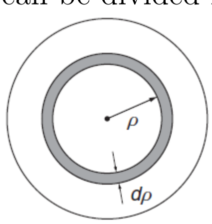
$$I = \begin{pmatrix} \frac{MR^2}{2} & 0 & 0 \\ 0 & \frac{MR^2}{2} & 0 \\ 0 & 0 & MR^2 \end{pmatrix}.$$

Because the inertia tensor is diagonal, therefore, x, y, and z axes are its principal axes.

6. Obtain the moment of inertia tensor of a thin uniform disk of radius R, and mass M, with the origin of the coordinate system placed at the center of the disk, and the disk lying in the xy plane.
Soln: Consider a uniform circular disk of mass M and radius R, as shown



A disk can be divided into a large number of rings with radii $0 \le \rho \le R$



Because mass M is distributed uniformly, we can define a surface mass density (mass per unit area) σ

$$\sigma = \frac{M}{\pi R^2}$$

Therefore, mass dm of the ring of radius ρ, and width $d\rho$ is

$$dm = \sigma 2\pi\rho d\rho = \frac{2M}{R^2}\rho d\rho$$

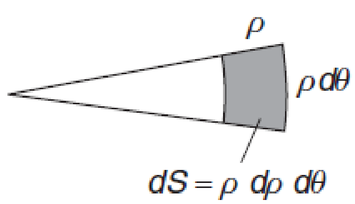
Assuming that the disk lies in the xy plane, with origin at its center, so that $x^2+y^2 = \rho^2$

$$I_{zz} = \int dm(x^2+y^2) = \frac{2M}{R^2}\int_0^R \rho^3 d\rho = \frac{2M}{R^2}\left(\frac{R^4}{4}\right) = \frac{MR^2}{2}$$

Other diagonal elements, keeping in mind that $z = 0$, on the disk

$$I_{yy} = \int dm(x^2+z^2) = \int dm x^2$$

Now we consider a area element located at (ρ, θ) as shown



So that

$$dm = \sigma\rho d\rho d\theta = \frac{M}{\pi R^2}\rho d\rho d\theta$$

Given that $x = \rho\cos\theta$, we have

$$I_{yy} = \int dm x^2 = \frac{M}{\pi R^2}\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi}\rho^3\cos^2\theta d\theta = \frac{MR^2}{4}$$

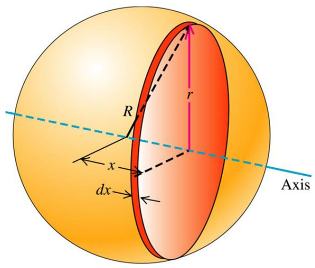
Similarly we can prove $I_{xx} = I_{yy} = \frac{MR^2}{4}$, and all off-diagonal elements are zero, so that the inertia tensor is diagonal

$$I = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}.$$

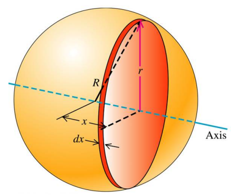
Thus the chosen coordinate axes are also the principal axis of the disk.

7. Obtain the moment of inertia tensor of a uniform solid sphere of radius R, and mass M, with the origin of the coordinate system placed at the center of the sphere. Note that this problem can be done by dividing the sphere into a large number of infinitesimally thin disks.
Soln: Because of high symmetry, any three mutually perpendicular axes passing

through the center will be its principle axes, and moment of inertia about any of those axes will be the same. Let us calculate the moment of inertia about one such axis



We divide the sphere into a large number of disks of thickness dx, and integrate their



contribution

Moment of inertia of the the disk shown, with respect to the axis

$$dI = \frac{1}{2} dm r^2$$

dm is the mass of the given disk, given by

$$dm = \rho dV = \frac{M}{\frac{4}{3}\pi R^3} \pi r^2 dx = \frac{3M}{4R^3} r^2 dx,$$

So that, using $r^2 = R^2 - x^2$, we have

$$dI = \frac{3M}{8R^3} r^4 dx = \frac{3M}{8R^3} (R^2 - x^2)^2 dx$$

On integrating over $x \in [-R, R]$, we obtain the required moment of inertia

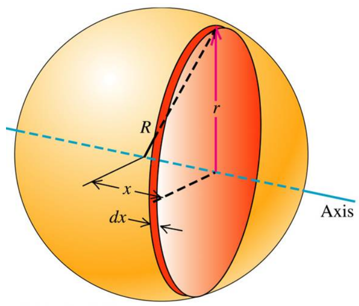
$$\begin{aligned} I &= \frac{3M}{8R^3} \int_{-R}^{R} (R^2 - x^2)^2 dx = \frac{3M}{4R^3} \int_{0}^{R} (R^2 - x^2)^2 dx \\ &= \frac{3M}{4R^3} \int_{0}^{R} (R^4 - 2R^2 x^2 + x^4) dx \\ &= \frac{3M}{4R^3} (R^5 - \frac{2R^5}{3} + \frac{R^5}{5}) = \frac{3M}{4R^3} \left(\frac{8R^3}{15} \right) = \frac{2}{5} M R^2 \end{aligned}$$

So that its inertia tensor is

$$I = \begin{pmatrix} \frac{2MR^2}{5} & 0 & 0 \\ 0 & \frac{2MR^2}{5} & 0 \\ 0 & 0 & \frac{2MR^2}{5} \end{pmatrix}.$$

8. Obtain the moment of inertia tensor of a uniform hollow sphere of radius R, and mass M, with the origin of the coordinate system placed at the center of the sphere. Note that this problem can be done by dividing the sphere into a large number of infinitesimally thin rings.

Soln: For a spherical shell, all three Cartesian directions are the same, therefore, it will have only one unique value of moment of inertia, and all off-diagonal elements will be zero. Its moment of inertia can be calculated by dividing the spherical shell into a large number of infinitesimally thin rings as shown



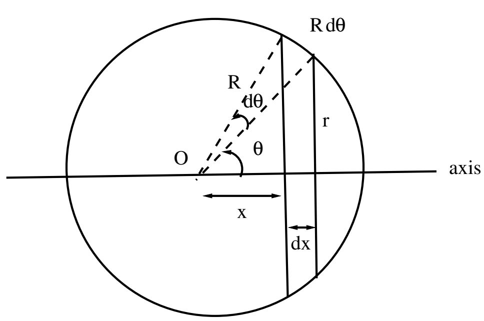
For the shell we define a surface mass density σ

$$\sigma = \frac{M}{4\pi R^2}$$

If the mass of each ring is dm, its moment of inertia dI about the axis will be

$$dI = dmr^2.$$

Because, area of each ring is $dA = 2\pi r dx$, we have $dm = \sigma 2\pi r dx = \frac{M}{2R^2} r dx$, so that



Assuming that the radius R shown in the figure makes an angle θ from the axis, and the ring width dx subtends angle $d\theta$ on the center, we obtain

$$dx = Rd\theta$$
$$r = R\sin\theta$$

So that

$$I = \frac{MR^2}{2}\int_0^{\pi} \sin^3\theta d\theta = \frac{MR^2}{2} \times \frac{4}{3} = \frac{2}{3}MR^2$$

9. Consider an asymmetric rigid body with principal moments of inertia $I_1 \neq I_2 \neq I_3$. Assuming that it is initially rotating about the principle axis $\hat{\mathbf{e}}_1$, with angular velocity ω_1, without any external torque. Suddenly, an external torque is applied to it for a brief time. Using the Euler equations show that

   (a) If I_1 is the smallest or the largest of I_1, I_2, I_3, the rotation of the rigid body will continue about the $\hat{\mathbf{e}}_1$ axis, in a stable manner

   (b) Otherwise, if the value of I_1 is intermediate compared to I_2 and I_3, then after the application of the external torque, the body will spin out of control.

   Soln: Consider a totally asymmetric rigid body with $I_1 \neq I_2 \neq I_3$ on which no external torque is applied ($\mathbf{N} = 0$). Assume that initially it is rotating with a constant angular velocity is in the 1-direction $\boldsymbol{\omega} = \omega_1 \hat{\mathbf{e}}_1$. Then it is perturbed by an external agency for a short period, at the end of which it has small components of $\boldsymbol{\omega}$ in the other two directions also

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{e}}_1 + \omega_2 \hat{\mathbf{e}}_2 + \omega_3 \hat{\mathbf{e}}_3,$$

with ω_2, $\omega_3 \ll \omega_1$. Euler equations immediately after the perturbation ends

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = 0 \tag{1}$$
$$I_2 \dot{\omega}_2 - \omega_1 \omega_3 (I_3 - I_1) = 0 \tag{2}$$
$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = 0 \tag{3}$$

We can neglect the second term of the first equation, because $\omega_2 \omega_3 \approx 0$, leading to

$$I_1 \dot{\omega}_1 = 0 \implies \omega_1 = \text{constant}.$$

Keeping this in mind, we differentiate Eq. 3, w.r.t. t, to obtain

$$I_2 \ddot{\omega}_2 - \omega_1 \dot{\omega}_3 (I_3 - I_1) = 0$$

Using the value of $\dot{\omega}_3$ from Eq. 4, we obtain

$$\ddot{\omega}_2 + \frac{(I_1 - I_3)(I_1 - I_2)\omega_1^2}{I_2 I_3} \omega_2 = 0$$

Or

$$\ddot{\omega}_2 + A\omega_2 = 0, \tag{4}$$

where $A = \frac{(I_1 - I_3)(I_1 - I_2)\omega_1^2}{I_2 I_3}$

$$A = \frac{(I_1 - I_3)(I_1 - I_2)\omega_1^2}{I_2 I_3}$$

Similarly, by taking time derivative of Eq. 4, and then eliminating $\dot{\omega}_2$ term using Eq. (3), we obtain identical equation for ω_3

$$\ddot{\omega}_3 + A\omega_3 = 0. \tag{5}$$

Consider two possibilities: Case I $I_1 > I_2 > I_3$ or $I_1 < I_2 < I_3$: here clearly $A > 0$, therefore Eqs. (5) and (6) denote simple harmonic motion, leading to bound oscillatory solutions for ω_2 and ω_3 of the form $e^{\pm it\sqrt{A}}$. This means motion about principal axis 1 is stable, if I_1 is either smallest or the largest moment of inertia.
Case II: $I_2 > I_1 > I_3$ or $I_3 > I_1 > I_2$: here clearly $A < 0$, therefore Eqs. (5) and (6) denote exponential type motion, leading to unbound solutions for ω_2 and ω_3 of the form $e^{\pm t\sqrt{A}}$. This means that the values of ω_2 and ω_3 will grow exponentially with time after motion about principal axis 1 is perturbed. Thus, motion about the principal axis corresponding to intermediate moment of inertia is unstable. To summarize the torque free rotation of a perfectly asymmetric rigid body: (a) Rotation about the principal axes corresponding to max/min moment of inertia is stable, (b) Rotation about the principal axis corresponding to intermediate moment of inertia is unstable.

10. Consider a rigid body with cylindrical symmetry so that its moments of inertia with respect to the principle axes are $I_1 = I$, $I_2 = I_3 = I_\perp$. If this body is rotating about a general axis without any external torque, write down its Euler's equations, and solve them.
Soln: Euler's equations for a rigid body are

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= \tau_1 \\ I_2\dot{\omega}_2 - \omega_1\omega_3(I_3 - I_1) &= \tau_2 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= \tau_3 \end{aligned}$$

Here no external torque is acting on the body, therefore, $\tau_1 = \tau_2 = \tau_3 = 0$, and using the given values $I_1 = I$, $I_2 = I_3 = I_\perp$,

$$I\dot{\omega}_1 = 0 \tag{6}$$

$$I_\perp\dot{\omega}_2 - \omega_1\omega_3(I_\perp - I) = 0 \tag{7}$$

$$I_\perp\dot{\omega}_3 - \omega_1\omega_2(I - I_\perp) = 0 \tag{8}$$

Eqn. (6) can be integrated immediately to yield

$$\omega_1 = \omega_s(\text{ a constant})$$

Next, we differentiate Eq.(7) and (8) to obtain

$$I_\perp\ddot{\omega}_2 - \omega_s\dot{\omega}_3(I_\perp - I) = 0 \tag{9}$$

$$I_\perp\ddot{\omega}_3 - \omega_s\dot{\omega}_2(I - I_\perp) = 0 \tag{10}$$

Eliminating $\dot{\omega}_3$ from Eq. (9), and $\dot{\omega}_2$ from Eq. (10), using Eqs. (8) and (7), respectively, we obtain

$$\begin{aligned} \ddot{\omega}_2 + \frac{(I - I_\perp)^2\omega_s^2}{I_\perp^2}\omega_2 &= 0 \\ \ddot{\omega}_3 + \frac{(I - I_\perp)^2\omega_s^2}{I_\perp^2}\omega_3 &= 0 \end{aligned}$$

Thus both the frequencies ω_2 and ω_3 execute simple harmonic oscillations of frequency $\gamma = \left|\frac{(I-I_\perp)\omega_s}{I_\perp}\right|$. One possible solution to these equations is

$$\omega_2(t) = A\sin\gamma t$$
$$\omega_3(t) = A\cos\gamma t.$$

This, as explained in the lectures, corresponds to a precession of the angular velocity vector $\boldsymbol{\omega}$ around ω_1, by frequency γ.

11. In the lectures we proved the following result about the angular momentum of a rigid body of mass M

$$\mathbf{L} = \mathbf{R} \times (M\mathbf{V}_{cm}) + \sum_i m_i \mathbf{r}'_i \times \dot{\mathbf{r}}'_i,$$

above $\mathbf{V}_{cm}$ is the velocity of the center of mass, $\mathbf{R}$ its location, while $\mathbf{r}'_i$ and $\dot{\mathbf{r}}'_i$ are positions and velocities, respectively of the i-th particle of the rigid body w.r.t. to its center of mass. Using this equation show that

(a)

$$L_z = I_0\omega + M(\mathbf{R} \times \mathbf{V}_{cm})_z,$$

where I_0 is the moment of inertia of the body about z-axis passing through the center of mass of the body.

Soln: It is given

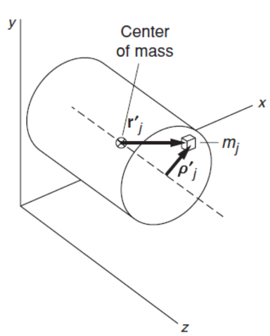
$$\mathbf{L} = \mathbf{R} \times (M\mathbf{V}_{cm}) + \sum_i m_i \mathbf{r}'_i \times \dot{\mathbf{r}}'_i,$$

therefore,

$$L_z = M(\mathbf{R} \times \mathbf{V}_{cm})_z + \sum_i m_i (\mathbf{r}'_i \times \dot{\mathbf{r}}'_i)_z,$$

if $\boldsymbol{\rho'}_i$ is the vector which is perpendicular to the axis of rotation (z axis), and connects it to the i-th particle (see the figure), then

$$(\mathbf{r}'_i \times \dot{\mathbf{r}}'_i)_z = (\boldsymbol{\rho}'_i \times \dot{\boldsymbol{\rho}}'_i)_z.$$



But $\dot{\boldsymbol{\rho}}'_i = \boldsymbol{\omega} \times \boldsymbol{\rho}'_i$, therefore,

$$(\mathbf{r}'_i \times \dot{\mathbf{r}}'_i)_z = (\boldsymbol{\rho}'_i \times \dot{\boldsymbol{\rho}}'_i)_z = \rho_i'^2\omega$$

leading to

$$\begin{aligned} L_z &= M(\mathbf{R} \times \mathbf{V}_{cm})_z + (\sum_i m_i \rho_i'^2)\omega \\ &= M(\mathbf{R} \times \mathbf{V}_{cm})_z + I_0\omega \end{aligned}$$

where $I_0 = \sum_i m_i \rho_i'^2$ is the moment of inertia about the z axis passing through the CM of the body.

(b) kinetic energy of the rigid body also splits in similar two terms, and can be written as

$$K = \frac{1}{2} I_0 \omega^2 + \frac{1}{2} M V_{cm}^2$$

Soln: Kinetic energy of a rigid body which is both rotating and translating can be written as

$$K = \frac{1}{2} \sum_i m_i v_i^2. \tag{11}$$

But, if $\mathbf{V}_{cm}$ is the velocity of the center of mass of the rigid body, then we can write

$$\mathbf{v}_i = \mathbf{V}_{cm} + \dot{\mathbf{r}}_i'. \tag{12}$$

But, from the figure above it is obvious that

$$\mathbf{r}_i' = \mathbf{a_i} + \boldsymbol{\rho}_i',$$

where $\mathbf{a}_i$ is a constant vector for the i-th particle, which connects the origin to that point on z axis where vector $\boldsymbol{\rho}_i$ begins. Therefore,

$$\dot{\mathbf{r}}_i' = \dot{\boldsymbol{\rho}}_i'$$

Putting this in Eq. (12), we obtain

$$\mathbf{v}_i = \mathbf{V}_{cm} + \dot{\boldsymbol{\rho}}_i'. \tag{13}$$

Using Eq. (13) in (11), we obtain

$$\begin{aligned} K &= \frac{1}{2} \sum_i m_i (\mathbf{V}_{cm} + \dot{\boldsymbol{\rho}}_i')^2 \\ &= \frac{1}{2} \sum_i m_i \dot{\rho}_i'^2 + \frac{1}{2} \sum_i m_i V_{cm}^2 + \frac{1}{2} (\sum_i m_i \dot{\boldsymbol{\rho}}_i') \cdot \mathbf{V}_{cm} \end{aligned}$$

Using the fact that $\dot{\rho}_i' = \rho_i'\omega$, and $\sum_i m_i \dot{\boldsymbol{\rho}}_i' = 0$, we obtain

$$\begin{aligned} K &= \frac{1}{2} (\sum_i m_i \rho_i'^2)\omega^2 + \frac{1}{2} (\sum_i m_i) V_{cm}^2 \\ &= \frac{1}{2} I_0 \omega^2 + \frac{1}{2} M V_{cm}^2 \end{aligned}$$

(c) work-energy theorem holds for the rotational motion

$$\int_{\theta_a}^{\theta_b} \tau_0 d\theta = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_b^2,$$

where τ_0 is the torque acting on the rigid body, while subscript a and b denote initial and final quantities, respectively.

Soln: If a torque τ is acting on a rigid body of moment of inertia I_0, then its angular acceleration α can be computed from

$$\tau = I_0 \alpha$$

If in time dt, the body rotates by angle $d\theta$, then by multiplying the equation above by $d\theta = \omega dt$, on both sides, we obtain

$$\begin{aligned} \tau d\theta &= I_0 \frac{d\omega}{dt} \omega dt = I_0 \omega \frac{d\omega}{dt} dt = \frac{d}{dt} \left(\frac{1}{2} I_0 \omega^2 \right) dt \\ \implies \int_a^b \tau d\theta &= \frac{1}{2} I_0 \int_a^b \frac{d}{dt} (\omega^2) dt = \frac{1}{2} I_0 \omega_b^2 - \frac{1}{2} I_0 \omega_a^2, \end{aligned}$$

12. Prove the following results about the rotational kinetic energy $K_{rot} = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i'^2$ of a general rigid body

(a)

$$K_{rot} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}$$

Soln: We have

$$K_{rot} = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i'^2$$

But

$$\dot{\mathbf{r}}_i' = \boldsymbol{\omega} \times \mathbf{r}_i',$$

therefore

$$\begin{aligned} K_{rot} &= \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i' \cdot (\boldsymbol{\omega} \times \mathbf{r}_i') \\ &= \frac{1}{2} \boldsymbol{\omega} \cdot \sum_i (\mathbf{r}_i' \times m_i \dot{\mathbf{r}}_i') \\ &= \frac{1}{2} \boldsymbol{\omega} \cdot \sum_i (\mathbf{r}_i' \times \mathbf{p}_i') \\ &= \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} \end{aligned}$$

(b)

$$K_{rot} = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3},$$

where L_i and I_i are the angular momentum component, and moment of inertia, respectively, with respect to the i-th principal axis.

Soln: If we align our coordinate axes with the principal axes of the rigid body, the inertia tensor becomes diagonal, and we obtain

$$\mathbf{L} = I_1\omega_1\hat{\mathbf{i}} + I_2\omega_2\hat{\mathbf{j}} + I_3\omega_3\hat{\mathbf{k}} = L_1\hat{\mathbf{i}} + L_2\hat{\mathbf{j}} + L_3\hat{\mathbf{k}}, \tag{14}$$

with this

$$\begin{aligned}\boldsymbol{\omega} &= \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}} + \omega_3\hat{\mathbf{k}} \\ &= \frac{L_1}{I_1}\hat{\mathbf{i}} + \frac{L_2}{I_2}\hat{\mathbf{j}} + \frac{L_3}{I_3}\hat{\mathbf{k}}\end{aligned} \tag{15}$$

Using Eqs (14) and (15), in the expression of part (a), we have

$$\begin{aligned}K_{rot} &= \frac{1}{2}\left(\frac{L_1}{I_1}\hat{\mathbf{i}} + \frac{L_2}{I_2}\hat{\mathbf{j}} + \frac{L_3}{I_3}\hat{\mathbf{k}}\right) \cdot \left(L_1\hat{\mathbf{i}} + L_2\hat{\mathbf{j}} + L_3\hat{\mathbf{k}},\right) \\ &= \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}\end{aligned}$$