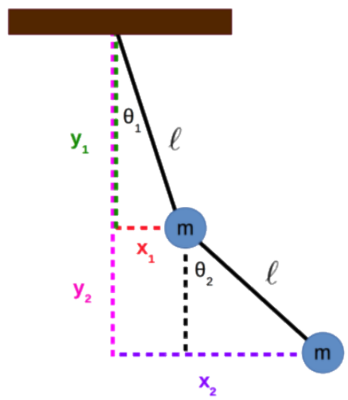
EP 222: Classical Mechanics
Tutorial Sheet 6: Solution

This tutorial sheet contains problems related to small oscillations of coupled harmonic oscillators, their eigenfrequencies, and normal modes.

1. Consider a double pendulum composed of two identical pendula of massless rods of length l, and masses m, attached along the vertical direction. Obtain the frequencies of the normal modes and the normal coordinates for small oscillations of this system.
   Soln:



Using the point of suspension of the upper pendulum as the origin of the coordinate system, the kinetic energy and the potential energies of the system can be written in terms of two generalized coordinates θ_1 and θ_2 as (refer to classnotes for the derivation)

$$T = \frac{1}{2}(2m)l^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 + ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$
$$V = -2mgl\cos\theta_1 - mgl\cos\theta_2$$

The equilibrium positions of both the pendula correspond to $\theta_1^{(0)} = \theta_2^{(0)} = 0^\circ$, so the displacements from the equilibrium positions are $\eta_1 = \theta_1$; $\eta_2 = \theta_2$, so that in terms of them

$$T = ml^2\dot{\eta}_1^2 + \frac{1}{2}ml^2\dot{\eta}_2^2 + ml^2\cos(\eta_1 - \eta_2)\dot{\eta}_1\dot{\eta}_2$$
$$V = -2mgl\cos\eta_1 - mgl\cos\eta_2.$$

For small oscillations, i.e., small values of η_1 and η_2, up to quadratic terms we have

$$\cos\eta_1 \approx 1 - \frac{\eta_1^2}{2}$$
$$\cos\eta_2 \approx 1 - \frac{\eta_2^2}{2}$$
$$\cos(\eta_1 - \eta_2) \approx 1,$$

so we have

$$T \approx ml^2\dot{\eta}_1^2 + \frac{1}{2}ml^2\dot{\eta}_2^2 + ml^2\dot{\eta}_1\dot{\eta}_2$$
$$V \approx -3mgl + mgl\eta_1^2 + \frac{1}{2}mgl\eta_2^2.$$

Ignoring the constant term in the potential energy, we have the matrix representation of the T and V operators

$$T = \begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix}$$
$$V = \begin{pmatrix} 2mgl & \\ & mgl \end{pmatrix},$$

with the Lagrangian being $L = \frac{1}{2}\dot{\eta}^T T\dot{\eta} - \frac{1}{2}\eta^T V\eta$, where $\dot{\eta} = \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix}$ and $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$. Frequencies of the normal modes are obtained by solving the characteristic polynomial

$$\det(V - \omega^2 T) = 0$$
$$\implies \begin{vmatrix} 2ml(g - l\omega^2) & -ml^2\omega^2 \\ -ml^2\omega^2 & ml(g - l\omega^2) \end{vmatrix} = 0$$
$$\implies 2(g - \omega^2 l)^2 - l^2\omega^4 = 0$$
$$\implies (g\sqrt{2} - \omega^2 l\sqrt{2} - \omega^2 l)(g\sqrt{2} - \omega^2 l\sqrt{2} + \omega^2 l) = 0$$
$$\implies \omega^4 l^2 - 4\omega^2 gl + 2g^2 = 0$$
$$\implies \omega_{1,2}^2 = \frac{4gl \pm \sqrt{16g^2l^2 - 8g^2l^2}}{2l^2}$$
$$\implies \omega_{1,2}^2 = \frac{g}{l}(2 \pm \sqrt{2})$$
$$\omega_{1,2} = \sqrt{\frac{g}{l}}\left(2 \pm \sqrt{2}\right)^{1/2}.$$

For obtaining the normal coordinates, we need to solve the secular equation

$$(V - \omega_j^2 T)a_j = 0$$
$$\implies \begin{pmatrix} 2ml(g - l\omega_j^2) & -ml^2\omega_j^2 \\ -ml^2\omega_j^2 & ml(g - l\omega_j^2) \end{pmatrix}\begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} = 0$$
$$\implies 2(g - l\omega_j^2)a_{1j} - l\omega_j^2 a_{2j} = 0$$
$$-l\omega_j^2 a_{1j} + (g - l\omega_j^2)a_{2j} = 0$$

For $j = 1$, i.e. $\omega_1^2 = \frac{g}{l}(2 + \sqrt{2})$, the equations are

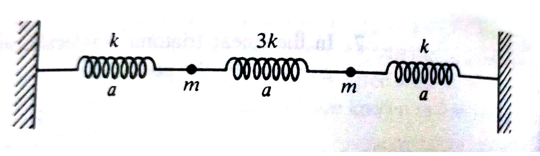
$$2(-1 - \sqrt{2})ga_{11} - (2 + \sqrt{2})ga_{21} = 0$$
$$-(2 + \sqrt{2})ga_{11} + (-1 - \sqrt{2})ga_{21} = 0$$
$$\implies a_{11} = -\frac{a_{21}}{\sqrt{2}}$$

In this mode, at a given point in time the two pendula will oscillate in opposite directions, with the displacement of the upper pendulum will be less than that of the lower one. For $j = 2$, with $\omega_2^2 = \frac{g}{l}(2 - \sqrt{2})$, we will obtain

$$\begin{aligned} 2(-1 + \sqrt{2})ga_{11} - (2 - \sqrt{2})ga_{21} &= 0 \\ -(2 - \sqrt{2})ga_{11} + (-1 + \sqrt{2})ga_{21} &= 0 \\ \implies a_{11} &= \frac{a_{21}}{\sqrt{2}} \end{aligned}$$

In this mode, at a given point in time the two pendula will oscillate in the same direction, and the displacement of the upper pendulum will again be less than that of the lower one.

2. Two particles move in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to a, and the force constants and masses are shown. Find their eigenfrequencies, and normal modes.



Soln: This problem has only two generalized coordinates x_1 and x_2, the coordinates of the two masses starting from left. Let X_1 be the coordinate of the left end of the left most spring, and X_2 be the coordinate of the right end of the right-most spring (note X_1 and X_2) are fixed. Then it is obvious that

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \\ V &= \frac{1}{2}k(x_1 - X_1 - a)^2 + \frac{1}{2}(3k)(x_2 - x_1 - a)^2 + \frac{1}{2}k(X_2 - x_2 - a)^2 \end{aligned}$$

If η_1 and η_2 are deviations from the equilibrium position of the two masses, then clearly

$$\begin{aligned} x_1 &= X_1 + a + \eta_1 \\ x_2 &= X_1 + 2a + \eta_2 = X_2 - a + \eta_2, \end{aligned}$$

we have

$$\begin{aligned} T &= \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2 \\ \implies T &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \end{aligned}$$

and

$$
\begin{aligned}
V &= \frac{1}{2}k\eta_1^2 + \frac{1}{2}(3k)(\eta_2 - \eta_1)^2 + \frac{1}{2}k\eta_2^2 \\
&= \frac{1}{2}\left(4k\eta_1^2 - 6k\eta_1\eta_2 + 4k\eta_2^2\right) \\
\implies V &= \begin{pmatrix} 4k & -3k \\ -3k & 4k \end{pmatrix}
\end{aligned}
$$

Frequencies of the normal mode are obtained by solving

$$
\begin{aligned}
&\det(V - \omega^2 T) = 0 \\
\implies &\begin{vmatrix} 4k - \omega^2 m & -3k \\ -3k & 4k - \omega^2 m \end{vmatrix} = 0 \\
\implies &(4k - \omega^2 m)^2 - 9k^2 = 0 \\
\implies &\omega^2 m = 4k \pm 3k \\
\implies &\omega_{1,2} = \sqrt{\frac{k}{m}} \text{ or } \sqrt{\frac{7k}{m}}
\end{aligned}
$$

The secular equation for normal coordinates is

$$
\begin{pmatrix} 4k - \omega_j^2 m & -3k \\ -3k & 4k - \omega_j^2 m \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} = 0
$$

For $j = 1$ $(\omega_1 = \sqrt{\frac{k}{m}})$

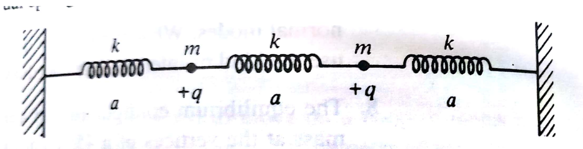
$$
\begin{aligned}
\begin{pmatrix} 3k & -3k \\ -3k & 3k \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} &= 0 \\
\implies a_{1j} &= a_{2j}
\end{aligned}
$$

For $j = 2$ $(\omega_2 = \sqrt{\frac{7k}{m}})$

$$
\begin{aligned}
\begin{pmatrix} -3k & -3k \\ -3k & -3k \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} &= 0 \\
\implies a_{1j} &= -a_{2j}
\end{aligned}
$$

which means both the masses are displaced by the same amount, but in the opposite directions. As a result, the middle spring is now stretched and compressed, leading to a larger frequency.

3. Two mass points of equal mass m are connected to each other and to fixed points by three equal springs of force constant k, as shown in the diagram. The equilibrium length of each sprint is a. Each mass point has charge $+q$, and they repel each other according to Coulomb law. Set up the secular equation for the eigenfrequencies.



Soln: This problem also has two generalized coordinates x_1 and x_2, the coordinates of the two masses starting from left. So kinetic energy operator will be same as in the previous problem.

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2 \\
\implies T &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}
\end{aligned}
$$

As far as potential energy is concerned, besides the contribution from the three springs, we will have one extra contribution due to the Coulomb repulsion between the two charges (we have used cgs units for electrostatic potential energy)

$$
\begin{aligned}
V &= \frac{1}{2}k\eta_1^2 + \frac{1}{2}k(\eta_2 - \eta_1)^2 + \frac{1}{2}k\eta_2^2 + \frac{q^2}{|x_1 - x_2|} \\
&= \frac{1}{2}\left(2k\eta_1^2 - 2k\eta_1\eta_2 + 2k\eta_2^2\right) + \frac{q^2}{|\eta_2 - \eta_1 - a|}
\end{aligned}
$$

Because $|\eta_2 - \eta_1|/a \ll 1$, we obtain by expanding the last term in the powers of $|\eta_2 - \eta_1|/a$, and retaining up to quadratic terms

$$
\begin{aligned}
V &= \frac{1}{2}\left(2k\eta_1^2 - 2k\eta_1\eta_2 + 2k\eta_2^2\right) + \frac{q^2}{a}\left(1 - \frac{|\eta_2 - \eta_1|}{a}\right)^{-1} \\
&= \frac{1}{2}\left(2k\eta_1^2 - 2k\eta_1\eta_2 + 2k\eta_2^2\right) + \frac{q^2}{a} + \frac{q^2|\eta_2 - \eta_1|}{a^2} + \frac{q^2(\eta_2 - \eta_1)^2}{a^3} + \cdots
\end{aligned}
$$

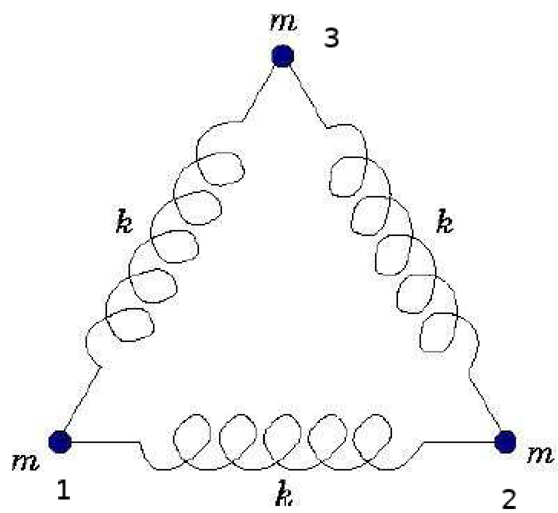
Noting that constant terms and the terms which are linear in displacement coordinates do not contribute to the potential energy matrix, we have

$$
\begin{aligned}
V &= \frac{1}{2}\left\{(2k + \frac{2q^2}{a^3})\eta_1^2 - 2(k + \frac{2q^2}{a^3})\eta_1\eta_2 + (2k + \frac{2q^2}{a^3})\eta_2^2\right\} \\
\implies V &= \begin{pmatrix} 2(k + \frac{q^2}{a^3}) & -(k + \frac{2q^2}{a^3}) \\ -(k + \frac{2q^2}{a^3}) & 2(k + \frac{q^2}{a^3}) \end{pmatrix}.
\end{aligned}
$$

So we note that the presence of charges on the two masses have effectively modified the force constants of the springs. The problem can be easily solved by setting up the secular equation $\det(V - \omega^2 T) = 0$.

4. A plane triatomic molecule consists of equal masses m at vertices of an equilateral triangle of sides a. Assume the molecule is held together by forces that are harmonic for small oscillations and that the force constants are identical and equal to k. Allow motion only in the plane of the molecule.

   (a) Set up the secular equation for the eigenfrequencies of the system.
   Soln:



This problem clearly as six degrees of freedom, e.g., (x, y) coordinates of each of the three particles. If we number the masses as shown, and place the origin of the coordinate system at the leftmost mass, then their equilibrium coordinates are $(0, 0)$, $(a, 0)$, $(a/2, a\sqrt{3}/2)$. For the motion confined to the xy plane, their coordinates at a given instance of time can be written as (η_1, η_2), $(a+\eta_3, \eta_4)$, and $(a/2 + \eta_5, a\sqrt{3}/2 + \eta_6)$, and . Clearly, the kinetic energy is

$$T = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2 + \dot{\eta}_4^2 + \dot{\eta}_5^2 + \dot{\eta}_6^2)$$

$$\Longrightarrow T = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{pmatrix}$$

The potential energy can be written as

$$V = \frac{1}{2}k\left((r_{12} - a)^2 + (r_{13} - a)^2 + (r_{23} - a)^2\right),$$

where

$$
\begin{aligned}
r_{12} &= \sqrt{(\eta_3 + a - \eta_1)^2 + (\eta_4 - \eta_2)^2} \\
r_{23} &= \sqrt{(\eta_5 - \eta_3 - a/2)^2 + (\eta_6 + a\sqrt{3}/2 - \eta_4)^2} \\
r_{13} &= \sqrt{(\eta_5 + a/2 - \eta_1)^2 + (\eta_6 + a\sqrt{3}/2 - \eta_2)^2}
\end{aligned}
$$

These expressions for the distances are complicated, and involve square roots. Best is to expand them geometrically, to the first order in η_i coordinates. That gives us

$$
\begin{aligned}
r_{12} &\approx a + \eta_3 - n_1 \\
r_{23} &\approx a - \frac{1}{2}(\eta_5 - \eta_3) + \frac{\sqrt{3}}{2}(\eta_6 - \eta_4) \\
r_{13} &\approx a + \frac{1}{2}(\eta_5 - \eta_1) + \frac{\sqrt{3}}{2}(\eta_6 - \eta_2)
\end{aligned}
$$

These expressions will lead to potential energy which is correct to quadratic terms in η_is. With this

$$V = \frac{1}{2}k\{(\eta_3 - \eta_1)^2 + (\frac{1}{2}(\eta_5 - \eta_1) + \frac{\sqrt{3}}{2}(\eta_6 - \eta_2))^2 + (\frac{\sqrt{3}}{2}(\eta_6 - \eta_4) - \frac{1}{2}(\eta_5 - \eta_3))^2\}.$$

Which leads to

$$
V = k \begin{pmatrix}
5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 \\
\sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 \\
-1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\
0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\
-1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 \\
-\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2
\end{pmatrix}
$$

(b) Identify the zero frequency modes of this system.

Soln: Calculations for this case are tedious, but solving $\det(V - \omega^2 T) = 0$, leads to three zero frequency modes corresponding to: (a) rigid motion in the x direction, (b) rigid motion in the y direction, and (c) rigid rotation of the system about its center of mass (centroid). In all three cases the springs are left unstretched/uncompressed leading to zero frequencies. These modes are shown below