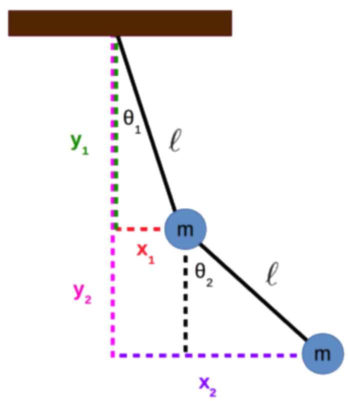
EP 222: Classical Mechanics
Tutorial Sheet 7: Solution

This tutorial sheet contains problems related to Hamiltonian formalism of classical mechanics.

1. Consider a double pendulum composed of two identical pendula of massless rods of length l, and masses m, attached along the vertical direction. Obtain the Hamiltonian of this system, and derive Hamilton's equations of motion.
   Soln:



We showed in the lectures that using the point of suspension of the upper pendulum as the origin of the coordinate system, the Lagrangian of a double pendulum consisting of equal masses m, and equal length (l) pendula is given by

$$L = ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2 + ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\ + 2mgl\cos\theta_1 + mgl\cos\theta_2.$$

Using the definition of the generalized momenta, we have

$$p_1 = \frac{\partial L}{\partial \dot{\theta}_1}$$
$$p_2 = \frac{\partial L}{\partial \dot{\theta}_2},$$

leading to

$$p_1 = 2ml^2\dot{\theta}_1 + ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_2$$
$$p_2 = ml^2\dot{\theta}_2 + ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_1.$$

We can solve for $\dot{\theta}_1$ and $\dot{\theta}_2$ in terms of p_1 and p_2, to obtain

$$\dot{\theta}_1 = \frac{p_1 - p_2\cos(\theta_1 - \theta_2)}{ml^2(1 + \sin^2(\theta_1 - \theta_2))} \quad (1)$$
$$\dot{\theta}_2 = \frac{2p_2 - p_1\cos(\theta_1 - \theta_2)}{ml^2(1 + \sin^2(\theta_1 - \theta_2))}. \quad (2)$$

Hamiltonian is defined as the Legendre transform of the Lagrangian

$$H = p_1\dot{\theta}_1 + p_2\dot{\theta}_2 - L,$$

where the generalized velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ are expressed in terms of generalized momenta p_1 and p_2, using Eqs (1) and (2) above

$$\begin{aligned}
H = {} & p_1\left(\frac{p_1 - p_2\cos(\theta_1-\theta_2)}{ml^2(1+\sin^2(\theta_1-\theta_2))}\right) + p_2\left(\frac{2p_2 - p_1\cos(\theta_1-\theta_2)}{ml^2(1+\sin^2(\theta_1-\theta_2))}\right) \\
& - ml^2\left(\frac{p_1 - p_2\cos(\theta_1-\theta_2)}{ml^2(1+\sin^2(\theta_1-\theta_2))}\right)^2 - \frac{1}{2}ml^2\left(\frac{2p_2 - p_1\cos(\theta_1-\theta_2)}{ml^2(1+\sin^2(\theta_1-\theta_2))}\right)^2 \\
& - ml^2\cos(\theta_1-\theta_2)\left(\frac{p_1 - p_2\cos(\theta_1-\theta_2)}{ml^2(1+\sin^2(\theta_1-\theta_2))}\right)\left(\frac{2p_2 - p_1\cos(\theta_1-\theta_2)}{ml^2(1+\sin^2(\theta_1-\theta_2))}\right) \\
& - 2mgl\cos\theta_1 - mgl\cos\theta_2.
\end{aligned}$$

This, after some tedious algebra, can be simplified to

$$\begin{aligned}
H = {} & \frac{1}{ml^2(1+\sin^2(\theta_1-\theta_2))}\left\{\frac{p_1^2}{2} + p_2^2 - p_1p_2\cos(\theta_1-\theta_2)\right\} \\
& - 2mgl\cos\theta_1 - mgl\cos\theta_2.
\end{aligned}$$

Question: Is the Hamiltonian same as total energy for this system, i.e., $H = T + V$?
Answer: We studied in the lectures that it is the case if the following two conditions are followed: (a) Potential energy is independent of generalized velocity, which is the case here, and (b) kinetic energy is a homogeneous function of degree 2 of the generalized velocities, which in this case means that $\frac{\partial T}{\partial \dot{\theta}_1}\dot{\theta}_1 + \frac{\partial T}{\partial \dot{\theta}_2}\dot{\theta}_2 = 2T$, which can be verified to be true here. Hence, the given Hamiltonian is the total energy of the system.

2. The Lagrangian for a system can be written as

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2+y^2},$$

where a, b, c, f, g, and k are constants. What is the Hamiltonian? What quantities are conserved?
Soln: Hamiltonian will be

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L,$$

where

$$\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} \\
p_y &= \frac{\partial L}{\partial \dot{y}} \\
p_z &= \frac{\partial L}{\partial \dot{z}}.
\end{aligned}$$

Thus

$$p_x = 2a\dot{x} + c\dot{y} + fy^2\dot{z} \tag{3}$$
$$p_y = \frac{b}{x} + c\dot{x} + g \tag{4}$$
$$p_z = fy^2\dot{x} \tag{5}$$

Here, Eqs. (4) and (5) give separate expressions for $\dot{x}$ in terms of momenta, so it is better to first compute the Hamiltonian in terms of velocities, and then eliminate them to get the momenta. With this we have

$$\begin{aligned} H &= \dot{x}(2a\dot{x} + c\dot{y} + fy^2\dot{z}) + \dot{y}(\frac{b}{x} + c\dot{x} + g) + \dot{z}(fy^2\dot{x}) \\ &- a\dot{x}^2 - b\frac{\dot{y}}{x} - c\dot{x}\dot{y} - fy^2\dot{x}\dot{z} - g\dot{y} + k\sqrt{x^2+y^2} \\ &= a\dot{x}^2 + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + k\sqrt{x^2+y^2} \\ &= \dot{x}(2a\dot{x} + c\dot{y} + fy^2\dot{z}) - a\dot{x}^2 + k\sqrt{x^2+y^2} \\ &= (\frac{p_z}{fy^2})p_x - a(\frac{p_z}{fy^2})^2 + k\sqrt{x^2+y^2} \\ &= (\frac{p_z}{fy^2})(p_x - a\frac{p_z}{fy^2}) + k\sqrt{x^2+y^2} \end{aligned}$$

Above, we used Eqs. (3) and (5) to eliminate the velocities. This Hamiltonian cannot be total energy because it is easy to verify that the velocity dependent part of it is not a second degree homogeneous function of velocities. However, Hamiltonian is not an explicit function of time, therefore, it is conserved. Furthermore, it does not depend on z, i.e., z is a cyclic coordinate, therefore, p_z will also be conserved.

3. A dynamical system has the Lagrangian

$$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1 q_1^2 + k_2\dot{q}_1\dot{q}_2,$$

where a, b, k_1, and k_2 are constants. Find the equations of motion in the Hamiltonian formalism.
Soln: As before

$$H = \dot{q}_1 p_1 + \dot{q}_2 p_2 - L,$$

with

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = 2\dot{q}_1 + k_2\dot{q}_2$$
$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = \frac{2\dot{q}_2}{a + bq_1^2} + k_2\dot{q}_1$$

These can be solved to obtain $\dot{q}_1/\dot{q}_2$ in terms of p_1/p_2

$$\dot{q}_1 = \frac{\{-2p_1 + k_2(a + bq_1^2)p_2\}}{\{k_2^2(a + bq_1^2) - 4\}} \tag{6}$$
$$\dot{q}_2 = \frac{\{(a + bq_1^2)(k_2 p_1 - 2p_2)\}}{\{k_2^2(a + bq_1^2) - 4\}} \tag{7}$$

But the velocity dependent part of the Lagrangian is a homogeneous function of degree 2 in the velocities, there is a part which is totally independent of the velocity. Thus, Hamiltonian will be total energy

$$H = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_2\dot{q}_1\dot{q}_2 - k_1 q_1^2.$$

With this

$$\begin{aligned}
H &= \frac{\left\{-2p_1 + k_2(a + bq_1^2)p_2\right\}^2}{\left\{k_2^2(a + bq_1^2) - 4\right\}^2} + \frac{1}{(a + bq_1^2)} \frac{\left\{(a + bq_1^2)(k_2 p_1 - 2p_2)\right\}^2}{\left\{k_2^2(a + bq_1^2) - 4\right\}^2} \\
&+ k_2 \frac{\left\{-2p_1 + k_2(a + bq_1^2)p_2\right\}}{\left\{k_2^2(a + bq_1^2) - 4\right\}} \times \frac{\left\{(a + bq_1^2)(k_2 p_1 - 2p_2)\right\}}{\left\{k_2^2(a + bq_1^2) - 4\right\}} - k_1 q_1^2 \\
&= \frac{p_1^2}{\left\{4 - k_2^2(a + bq_1^2)\right\}} + \frac{(a + bq_1^2)p_2^2}{\left\{4 - k_2^2(a + bq_1^2)\right\}} - \frac{k_2(a + bq_1^2)p_1 p_2}{\left\{4 - k_2^2(a + bq_1^2)\right\}} - k_1 q_1^2.
\end{aligned}$$

Hamilton's equations of motion are

$$\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\dot{q}_1 &= \frac{\partial H}{\partial p_1} = \frac{2p_1 - k_2(a + bq_1^2)p_2}{\left\{4 - k_2^2(a + bq_1^2)\right\}} \\
\dot{q}_2 &= \frac{\partial H}{\partial p_2} = \frac{(a + bq_1^2)(2p_2 - k_2 p_1)}{\left\{4 - k_2^2(a + bq_1^2)\right\}}
\end{aligned}$$

These equations are the same as Eqs. (6) and (7) above. The other two Hamilton's equations are

$$\begin{aligned}
\dot{p}_1 &= -\frac{\partial H}{\partial q_1} \\
&= -\frac{2bk_2^2 q_1 p_1^2}{\left\{4 - k_2^2(a + bq_1^2)\right\}^2} - \frac{2bq_1 p_2^2}{\left\{4 - k_2^2(a + bq_1^2)\right\}} \\
&- \frac{2bk_2^2 q_1(a + bq_1^2)p_2^2}{\left\{4 - k_2^2(a + bq_1^2)\right\}^2} + \frac{2k_2 bq_1 p_1 p_2}{\left\{4 - k_2^2(a + bq_1^2)\right\}} \\
&+ \frac{2k_2^3 b(a + bq_1^2)q_1 p_1 p_2}{\left\{4 - k_2^2(a + bq_1^2)\right\}^2} + 2k_1 q_1,
\end{aligned}$$

and, because q_2 is a cyclic coordinate, we have

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = 0.$$

4. A Hamiltonian of one degree of freedom has the form

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{ba}{2}q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2},$$

where a, b, α, and k are constants.

(a) Find a Lagrangian corresponding to this Hamiltonian
Soln: Here we have the reverse problem, compared to earlier ones. We have to obtain the Lagrangian from the Hamiltonian, using the formula

$$L = p\dot{q} - H, \tag{8}$$

where p will be eliminated using the Hamilton's equation

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{a} - bqe^{-\alpha t} \\ \implies p &= a(\dot{q} + bqe^{-\alpha t}) \end{aligned} \tag{9}$$

Using Eq. (9) in (8), we obtain the Lagrangian in terms of q and $\dot{q}$

$$\begin{aligned} L &= \dot{q}a(\dot{q} + bqe^{-\alpha t}) - \frac{a^2(\dot{q} + bqe^{-\alpha t})^2}{2a} + baq(\dot{q} + bqe^{-\alpha t})e^{-\alpha t} \\ &- \frac{ba}{2}q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{kq^2}{2} \\ &= \frac{a\dot{q}^2}{2} - \frac{kq^2}{2} + baq\dot{q}e^{-\alpha t} - \frac{ab\alpha}{2}q^2 e^{-\alpha t} \\ &= \frac{a\dot{q}^2}{2} - \frac{kq^2}{2} + \frac{d}{dt}\left(\frac{1}{2}abq^2 e^{-\alpha t}\right), \end{aligned}$$

so that

$$L = L_0 + \frac{dF}{dt},$$

with $L_0 = \frac{a\dot{q}^2}{2} - \frac{kq^2}{2}$ and $F(q,t) = \left(\frac{1}{2}abq^2 e^{-\alpha t}\right)$. Note that L_0 is the Lagriangian for a one-dimensional simple Harmonic oscillator of mass a, and force constant k.

(b) Is it possible to find an equivalent Lagrangian that is not explicitly dependent on time?
Soln: Above we showed that the original Lagrangian L differs from a time independent Lagrangian L_0 by a total time derivative. Which means that L and L_0 are equivalent.

(c) If you are able to solve part (b), what is the Hamiltonian corresponding the new Lagrangian, and what is the relationship between the two Hamiltonians?
Soln: It is obvious that the Hamiltonian H_0 corresponding to L_0 will also be that for 1D SHO

$$H_0 = \frac{P^2}{2a} + \frac{1}{2}kQ^2,$$

where new canonical variables are $P = \dot{q}$ and $Q = q$, so that the original Hamiltonian is

$$H = H_0 - bqpe^{-\alpha t} + \frac{ba}{2}q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}).$$

On using the fact that $p = a(\dot{q} + bqe^{-\alpha t}) = a(P + bQe^{-\alpha t})$, we obtain

$$\begin{aligned} H &= H_0 - abQ(P + bQe^{-\alpha t})e^{-\alpha t} + \frac{ba}{2}Q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) \\ &= H_0 - abQPe^{-\alpha t} - \frac{1}{2}ab^2 Q^2 e^{-2\alpha t} + \frac{ba\alpha}{2}Q^2 e^{-\alpha t} \end{aligned}$$

5. (a) The Lagrangian for a system of one degree of freedom can be written as

$$L = \frac{m}{2}\left(\dot{q}^2 \sin^2 \omega t + \dot{q}q\omega \sin 2\omega t + q^2\omega^2\right).$$

What is the corresponding Hamiltonian? Is it conserved?
Soln: We have

$$\begin{aligned} p &= \frac{\partial L}{\partial \dot{q}} = m\dot{q}\sin^2 \omega t + \frac{1}{2}mq\omega \sin 2\omega t \\ \implies \dot{q} &= \frac{(p - \frac{1}{2}mq\omega \sin 2\omega t)}{m \sin^2 \omega t} \end{aligned}$$

So that

$$\begin{aligned} H &= p\dot{q} - L \\ &= \frac{p(p - \frac{1}{2}mq\omega \sin 2\omega t)}{m\sin^2 \omega t} - \frac{m}{2}\frac{(p - \frac{1}{2}mq\omega \sin 2\omega t)^2}{m^2 \sin^4 \omega t} \\ &\quad - \frac{m}{2}q\omega \frac{(p - \frac{1}{2}mq\omega \sin 2\omega t)}{m\sin^2 \omega t}\sin 2\omega t - \frac{1}{2}m\omega^2 q^2 \end{aligned}$$

which leads to a tedious time-dependent expression

$$\begin{aligned} H &= \frac{p^2}{2m}\left(\frac{1}{\sin^2 \omega t} - \frac{1}{2\sin^4 \omega t}\right) \\ &\quad - \frac{1}{2}pq\omega \sin 2\omega t\left(\frac{1}{\sin^2 \omega t} - \frac{1}{2\sin^4 \omega t}\right) \\ &= \frac{1}{2}m\omega^2 q^2 \sin^2 2\omega t\left(\frac{1}{2\sin^2 \omega t} - \frac{1}{4\sin^4 \omega t} - 1\right), \end{aligned}$$

which is not conserved because of its explicit time dependence.

(b) Introduce a new coordinate defined by

$$Q = q\sin \omega t.$$

Find the Lagrangian in terms of the new coordinate and the corresponding Hamiltonian. Is H conserved?

Soln: We make the substitutions in the Lagrangian

$$q = \frac{Q}{\sin \omega t}$$
$$\dot{q} = \frac{\dot{Q} - \omega Q \cot \omega t}{\sin \omega t},$$

and after some tedious algebra we obtain the Lagrangian in terms of new variables

$$L = \frac{1}{2} m \dot{Q}^2 + \frac{1}{2} m \omega^2 Q^2.$$

Clearly, the Hamiltonian in new coordinates (with $P = \frac{\partial L}{\partial \dot{Q}} = m\dot{Q}$) will be

$$H = \frac{P^2}{2m} - \frac{1}{2} m \omega^2 Q^2,$$

which depends on canonical variables P and Q, both of which are explicitly time dependent. Therefore, Hamiltonian will not be conserved.