# EP 222: Classical Mechanics
# Tutorial Sheet 8

This tutorial sheet contains problems related to canonical transformations, Poisson brackets etc.

1. One of the attempts at combining two sets of Hamilton's equations into one tries to take $q$ and $p$ as forming a complex quantity. Show directly from Hamilton's equations of motion that for a system of one degree of freedom the transformation

$$Q = q + ip, \qquad P = Q^*$$

is not canonical if the Hamiltonian is left unaltered. Can you find another set of coordinates $Q'$ and $P'$ that are related to $Q$, $P$ by a change of scale only, and that are canonical?

2. Show that the transformation for a system of one degree of freedom,

$$\begin{aligned} Q &= q\cos\alpha - p\sin\alpha \\ P &= q\sin\alpha + p\cos\alpha, \end{aligned}$$

satisfies the symplectic condition for any value of the parameter $\alpha$. Find a generating function for the transformation. What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$? Does your generating function work for both the cases?

3. Show directly that the transformation

$$Q = \log\left(\frac{1}{q}\sin p\right), \qquad P = q\cot p$$

is canonical.

4. Show directly that for a system of one degree of freedom the transformation

$$Q = \arctan\frac{\alpha q}{p}, \qquad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical, where $\alpha$ is an arbitrary constant of suitable dimensions.

5. The transformation between two sets of coordinates are

$$\begin{aligned} Q &= \log(1 + q^{1/2}\cos p), \\ P &= 2(1 + q^{1/2}\cos p)q^{1/2}\sin p. \end{aligned}$$

   (a) Show directly from these transformation equations that $Q$, $P$ are canonical variables if $q$ and $p$ are.

   (b) Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2\tan p.$$

6. Prove directly that the transformation

$$\begin{aligned} Q_1 &= q_1, \qquad & P_1 &= p_1 - 2p_2, \\ Q_2 &= p_2, \qquad & P_2 &= -2q_1 - q_2 \end{aligned}$$

is canonical and find a generating function.

7. (a) Using the fundamental Poisson brackets find the values of $\alpha$ and $\beta$ for which the equations

$$Q = q^\alpha \cos \beta p, \qquad P = q^\alpha \sin \beta p$$

represent a canonical transformation.

(b) For what values of $\alpha$ and $\beta$ do these equations represent an extended canonical transformation? Find a generating function of the $F_3$ form for the transformation.

8. Show by the use of Poisson brackets that for a one-dimensional harmonic oscillator, there is a constant of motion $u$ defined as

$$u(q, p, t) = \ln(p + im\omega q) - i\omega t, \qquad \omega = \sqrt{\frac{k}{m}}.$$

9. A system of two degrees of freedom is described by the Hamiltonian

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2,$$

where $a$ and $b$ are constants. Show that

$$F_1 = \frac{p_1 - a q_1}{q_2} \quad \text{and} \quad F_2 = q_1 q_2$$

are constants of the motion.