PH 422: Quantum Mechanics II
Tutorial Sheet 4

This tutorial sheet contains problems related to the perturbation theory for time-independent Schrödinger equation.

1. Assume that a particle of mass m, is carrying out simple harmonic motion in the x direction.

   (a) If the particle carries a charge q, and it is exposed to an electric field $\mathcal{E}_0$ in the x direction, obtain its exact eigenvalues and eigenvectors. In particular, show that the exact eigenket for the n-th level $|\psi_n\rangle$ is related to that of the unperturbed state $|\psi_n^{(0)}\rangle$, by $|\psi_n\rangle = e^{-iqE_0 p/m\omega^2\hbar}|\psi_n^{(0)}\rangle$, where p is the momentum operator.

   (b) Assuming that the electric field is small, treat the extra term in the Hamiltonian $V = -q\mathcal{E}_0 x$ as a small perturbation, to calculate:

      i. corrections to the energy eigenvalues up to second order in perturbation theory

      ii. corrections to the wave functions up to first order in perturbation theory.

   (c) Expand your exact results in leading orders in $\mathcal{E}_0$, and compare them with the results obtained in part (b).

2. Consider a hydrogen atom in its $1s$ ground state. If we apply an electric field of strength E_0 in the z direction on the hydrogen atom, make an estimate of the first non-vanishing order of correction in the energy of the $1s$ state, due to the presence of the electric field. Using this, estimate the static polarizability α of the atom, employing the formula

$$\alpha = -\left(\frac{\partial^2 \Delta E}{\partial E_0^2}\right)_{E_0=0},$$

   where ΔE is the shift in the energy of the atom, due to the presence of the external electric field. This effect is called quadratic stark effect.

3. Consider a 1D SHO with particle mass m, and frequency ω, with the Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$. To this Hamiltonian we add a perturbing term $V = \frac{1}{2}\epsilon m\omega^2 x^2$, where $\epsilon \ll 1$.

   (a) Solve this problem exactly, and expand the energies to the second order in ϵ, and wave functions to the first order in ϵ.

   (b) Calculate, using the perturbation theory, the energy corrections to the second order in ϵ, and wave functions to first order in ϵ, treating V as a perturbation. For the wave function, perform the calculations only for the ground state. Compare your results to those obtained in part (b).

4. Consider a 1D SHO with particle mass m, and frequency ω. Assume that the Hamiltonian is perturbed by a term $V = \lambda x^3$, where λ is a small number. Calculate the perturbation corrections to energy up to second order, and to the wave functions, up to the first order.

5. Consider a hydrogen atom in its first excited state with principal quantum number $n = 2$. If we apply an electric field $\mathbf{E} = E_0 \hat{k}$ (E_0 is a constant) on the atom, using the degenerate perturbation theory, calculate the first order corrections to the energy, and the modified zeroth order wave functions. What happens to the four-fold degeneracy of this level?

6. Consider a particle in a two-dimensional box with the potential

$$V_0 = \begin{cases} 0 & \text{for } 0 \le x \le a,\ 0 \le y \le a \\ \infty & \text{otherwise.} \end{cases}$$

What are the energy eigenvalues and eigenfunctions for this system? If we add a perturbation term V to this Hamiltonian, defined by

$$V = \begin{cases} \lambda xy & \text{for } 0 \le x \le a,\ 0 \le y \le a \\ 0 & \text{otherwise.} \end{cases}$$

Obtain the zeroth order energy eigenfunctions, and the first-order energy shifts to the ground and the first excited states.

7. A p-orbital electron characterized by $|n, l = 1, m = \pm 1, 0\rangle$ (ignore spin) is subjected to a potential

$$V = \lambda(x^2 - y^2), \qquad (\lambda = \text{constant}).$$

Obtain the correct zeroth-order energy eigenstates that diagonalize the perturbation. You need not evaluate the energy shifts in detail, but show that the original threefold degeneracy is now completely removed.

8. A Hamiltonian matrix for a two-level system is given by

$$H = \begin{pmatrix} E_1^0 & \lambda\Delta \\ \lambda\Delta & E_2^0 \end{pmatrix}.$$

Clearly, the energy eigenfunctions for the unperturbed problem ($\lambda = 0$) are given by

$$\phi_1^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \phi_2^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) Solve this problem exactly to find the energy eigenfunctions ψ_1 and ψ_2, and the energy eigenvalues E_1 and E_2.

(b) Assuming that $\lambda|\Delta| \ll |E_1^0 - E_2^0|$, solve the same problem using perturbation theory up to first order in the energy eigenfunctions and up to secon order in energy eigenvalues. Compare with the exact results obtained in part (a).

(c) Suppose that the two unperturbed energies are "almost degenerate," that is

$$|E_1^0 - E_2^0| \ll \lambda|\Delta|.$$

Show that the exact results of part (a) closely resemele what you would expect by applying degenerate perturbation theory to this problem with $E_1^0 = E_2^0$.

9. When a magnetic field $\mathbf{B} = B\hat{k}$, is applied to a hydrogen atom, it leads to the addition of the following perturbation term to its Hamiltonian

$$V = -\frac{eB}{2mc}(L_z + 2S_z),$$

where c is the speed of light, e is the charge of the electron, m its mass, while L_z and S_z, respectively, are the components of the z component of its orbital and spin angular momenta. Using the first-order perturbation theory, calculate the energy shifts due to this term when the hydrogen atom is in: (a) an s state, and (b) in a p state. This shift in energy levels due to an external magnetic field is called the Zeeman effect.