PH 422: Quantum Mechanics II
Tutorial Sheet 5

This tutorial sheet contains problems related to the time-dependent perturbation theory.

1. Suppose that the matrix element $\langle i|V|n\rangle$ which occurs in the Fermi's Golden Rule, is zero for a given quantum mechanical system. This means that the transition rate $\Gamma_{n\to i} = 0$, in the first order of perturbation theory. Compute the corresponding transition rate in the second order of time-dependent perturbation theory.

2. Consider a one-dimensional harmonic oscillator of mass m, angular frequency ω_0, and charge q. We know that for this system $H_0|n\rangle = (n+\frac{1}{2})\hbar\omega_0|n\rangle$. Assume that it is subject to the following perturbation

$$V(t) = \begin{cases} -qEx & \text{for } 0 \leq t \leq \tau \\ 0 & \text{for } t < 0 \text{ and } t > \tau, \end{cases}$$

   where $\mathcal{E}$ is the electric field. If $P_{n\to i}$ is the transition probability from the initial level n to the final level i, then

   (a) compute $P_{0\to 1}$ as a function of τ

   (b) Show that in the first order of perturbation theory $P_{0\to 2} = 0$

   (c) Compute $P_{0\to 2}$ in the second order of perturbation theory.

3. Consider two spin 1/2's, $\mathbf{S}_1$ and $\mathbf{S}_2$, interacting with each other through Hamiltonian $H(t) = a(t)\mathbf{S}_1 \cdot \mathbf{S}_2$; where $a(t)$ is a function of time which satisfies $\lim_{|t|\to\infty} a(t) = 0$, and takes on non-negligible values (of the order of a_0) only inside an interval τ, symmetrically placed about $t = 0$.

   (a) At $t = -\infty$, the system is in the state $|+,-\rangle$. Calculate, without approximations, the state of the system at $t = +\infty$. Show that probability $P(+- \to -+)$ of finding, at $t = +\infty$, the system in the state $|-,+\rangle$, depends only on the integral $\int_{-\infty}^{+\infty} a(t)dt$.

   (b) Calculate the same probability using the first-order perturbation theory, and compare your results with those obtained in the preceding part.

4. The unperturbed Hamiltonian of a two-level system is represented by

$$H_0 = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix}.$$

   This system is perturbed by a time-dependent term

$$V(t) = \begin{pmatrix} 0 & \lambda\cos\omega t \\ \lambda\cos\omega t & 0 \end{pmatrix},$$

   where λ is real.

(a) At $t = 0$, the system is known to be in the first state, represented by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using time-dependent perturbation theory, and assuming that $|E_1^0 - E_2^0| \gg \hbar\omega$, derive an expression for the probability that the system be found in the second state represented by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

as a function of t ($t > 0$).

(b) Why is this procedure not valid when $|E_1^0 - E_2^0| \approx \hbar\omega$?

5. Consider a three-level system where the unperturbed states are of the form $|j, m\rangle$, with $j = 1$, and $m = 0, \pm 1$. We define the ordered orthonormal basis as $|\psi_1\rangle = |1, -1\rangle$, $|\psi_2\rangle = |1, 0\rangle$, and $|\psi_3\rangle = |1, 1\rangle$,with $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, where $|\psi_i\rangle$ are eigenfunctions of the time-independent Hamiltonian H_0

$$\begin{aligned} H_0|\psi_1\rangle &= (E_0 - \hbar\omega_0^{'})|\psi_1\rangle \\ H_0|\psi_2\rangle &= E_0|\psi_2\rangle \\ H_0|\psi_3\rangle &= (E_0 + \hbar\omega_0)|\psi_1\rangle. \end{aligned}$$

The degeneracy of the $j = 1$ states has been broken by applying external magnetic and electric fields. Next, a radiofrequency field rotating at the angular velocity ω in the xOy plane is applied, leading to the time-dependent perturbation

$$V(t) = \frac{\omega_1}{2}\left(J_+ e^{-i\omega t} + J_- e^{i\omega t}\right),$$

where ω_1 is a constant.

(a) Assuming that

$$|\psi(t)\rangle = \sum_{i=1}^{3} a_i(t) e^{-iE_i t/\hbar}|\psi_i\rangle,$$

write down the differential equations satisfied by $a_i(t)$.

(b) Assume that $|\psi(t = 0)\rangle = |\psi_1\rangle$. Show that if we want to calculate $a_3(t)$ by time-dependent perturbation theory, the calculation must be pursued to second order.

(c) Compute $a_3(t)$ up to second order of perturbation theory. For fixed t, how does the probability $P_{1\to 3}(t) = |a_3(t)|^2$ vary with respect to ω? Show that a resonance appears, not only for $\omega = \omega_0$ and $\omega = \omega_0^{'}$, but also for $\omega = (\omega_0 + \omega_0^{'})/2$.

6. Photoelectric effect is the ejection of an electron from a system, because of its interaction with an incident radiation field. Calculate the cross-section for photoelectric effect for a hydrogen atom in its ground state, by taking the initial electronic state to be the $1s$ wave function of the hydrogen atom, and the final state to be a box normalized plane wave $\frac{1}{L^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}}$. The radiation field is represented by a plane wave of frequency ω, wave vector $\frac{\omega}{c}\hat{\mathbf{n}}$, polarized in the direction $\hat{\mathbf{e}}$.

7. A hydrogen atom in its ground state is placed between the plates of a capacitor, which applies the following time-dependent electric field

$$\mathbf{E} = \begin{cases} 0 & \text{for } t < 0 \\ \mathcal{E}_0 e^{-t/\tau}\hat{k} & \text{for } t > 0 \end{cases}$$

Using first-order time-dependent perturbation theory, calculate the transition probability $P_{1s\to 2p_0}(t \gg \tau)$, where $2p_0$ state corresponds to the $2p$ state with $m = 0$.