Elementary Particle Physics

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Physics is like sex:sure
it may give some practical
results,
but that's not why we do it.

RICHARD FEYNMAN

Contents

Chapter 1

Particloduction

1.1 Motivation

The most fundamental question which everyone asks is "What are the elementary things in this universe?" -which particle physics soughts to answer. For me, learning particle physics is a journey into the fundamental nature of the universe, offering profound insights into its most basic building blocks. The intellectual challenge and personal fulfillment I gain from understanding the cosmos make particle physics an immensely rewarding endeavor.

1.2 Current Picture

The Standard Model of particle physics is a comprehensive theory describing the fundamental particles and their interactions, excluding gravity. It categorizes particles into fermions and bosons.

1.2.1 Fermions

Fermions are the building blocks of matter, divided into quarks and leptons, each with three generations:

Quarks:

- First Generation: Up(u) and Down(d) quarks.
- Second Generation: Charm(c) and Strange(s) quarks.
- Third Generation: Top(t) and Bottom(b) quarks.

Leptons:

- First Generation: Electron(e) and Electron Neutrino(ν_e).

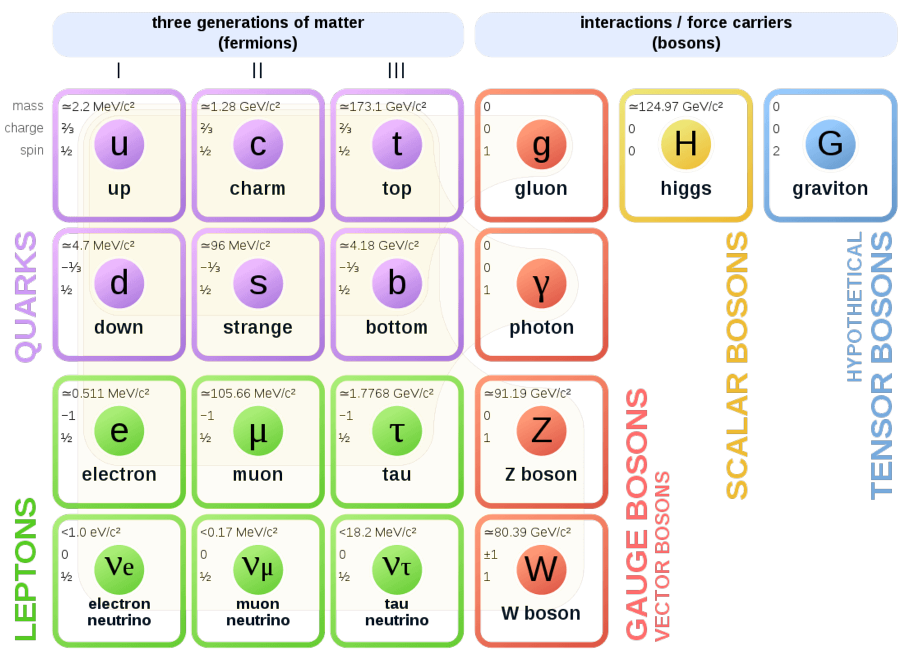
- Second Generation: Muon(μ) and Muon Neutrino(ν_μ).
- Third Generation: Tau(τ) and Tau Neutrino(ν_τ).

1.2.2 Bosons

Bosons are force carriers mediating fundamental interactions:

- Photon(γ): Mediates the electromagnetic force.
- W and Z Bosons: Mediate the weak force.
- Gluons(g): Mediate the strong force.
- Higgs Boson(H): Provides particles with mass.

Standard Model of Elementary Particles and Gravity



1.3 The Fundamental Forces

In this universe,there are four fundamental forces, or *interactions* namely gravitational, electromagnetic, strong nuclear, and weak nuclear. Each of these forces has distinct characteristics in terms of range and strength:

Gravitational Force:

It is an infinite range force that acts between objects having mass. Despite being the weakest of the four, it is significant on large scales, such as between planets, stars, and galaxies. Classical theory of gravity was given by Newton, in his famous law of universal gravitation. The relativistic generalisation is the general relativity theory by Einstein, and its quantum version is yet to be formulated. The hypothetical mediator particle is the **Graviton** (having $spin = 2$).

Electromagnetic Force:

It is the force acting between electrically charged particles. It has an infinite range and is much stronger than gravitation. The governing theory corresponding to it is electrodynamics, developed by Maxwell (*consistent with relativity*). The quantum version of it is known as QED, and mediator particle is the **Photon**.

Strong Nuclear Force:

It is a very short range force, in the order of $10^{-}15$ meters, and is the strongest of all. It acts between quarks and is instrumental in binding protons and neutrons together in the nucleus. The theory governing this force is known as quantum chromodynamics(*pioneered by Yukawa*) and mediator particles are the **Gluons**, which are total 8 in number.

Weak Nuclear Force:

Its the force responsible for radioactive decay and certain nuclear reactions. It has a very short range, and is stronger than gravitational force but weaker than electromagnetic and strong nuclear forces. It acts on both quarks and leptons. The governing theory is known as Glashow-Weinberg-Salam theory or simply *flavourdynamics*. The mediator particle are $W^{\pm}$ and Z.

Due to their diminutive size and rapid motion, particles necessitate the application of quantum mechanics and special relativity for comprehensive understanding. These foundational principles are indispensable in the realm of particle physics. Let's begin with them!

Chapter 2

Catch the Light!

In 1905, Albert Einstein developed the Special Theory of Relativity, which asserts that the laws of physics hold true in all inertial frames of reference i.e. they apply equally in any non-accelerating frame. Special relativity introduced the constancy of the speed of light for all observers and the interdependence of space and time.

2.1 Lorentz Transformations

Consider a frame of reference, S and another frame, S' moving with relative uniform speed v along a direction,say x-axis. Let (x, y, z, t) and (x', y', z', t') be the coordinates of an event in S and S', and further suppose that both coordinate system coincide at t=0. Then the relation between them is given by Lorentz transformation equations:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \end{aligned}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$
The inverse lorentz transformation can be written as:

$$\begin{aligned} x' &= \gamma(x + vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t + vx/c^2) \end{aligned}$$

2.1.1 Matrix Notations

These equations can be written using matrix notations more conveniently,using shorthand

$$x^\mu = (ct, x, y, z)$$

with μ varying from 0 to 3, i.e. $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$, commonly called as *four-vector*. Equations can be written as

$$x^{'\mu} = \Lambda^\mu_\nu x^\nu$$

where Λ is matrix:

$$\begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where ν is summed over 0 to 3[1]. Here $\mu + 1$ is the column number and $\nu + 1$ is the row number of the matrix. This notation works in all cases, the velocity of frames S and S' need not be parallel - just Λ becomes more complicated.

2.1.2 Invariant

This is the quantity which has the same value in any inertial frame of reference and is conserved under lorentz transformation. It's quite similar to the norm, generally defined in the vector space.

$$I = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

In terms of matrix notations, it can be represented as

$$\begin{aligned} I &= c^2t^2 - x^2 - y^2 - z^2 \\ I &= \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu \\ I &= g_{\mu\nu} x^\mu x^\nu && \text{(Einstein Notation)} \end{aligned}$$

where g, known as metric, is defined as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Taking this further, we define **covariant** (index down) x_μ as

$$x_\mu = g_{\mu\nu} x^\nu$$

[1] Known as Einstein's summation convention, $x^{'\mu} = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu$

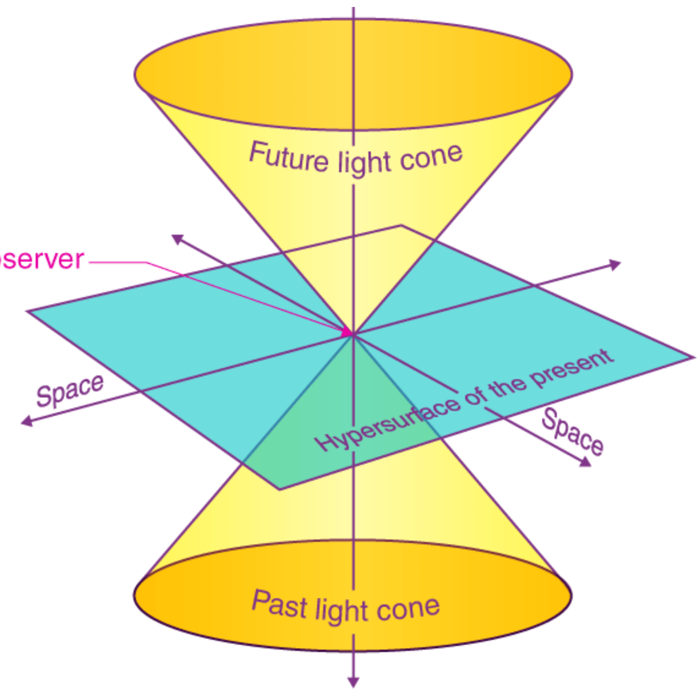


Figure 2.1: Spacetime diagram

i.e. $x^0 = ct$, $x^1 = -x$, $x^2 = -y$ and $x^3 = -z$. Therefore I becomes:

$$I = x_\mu x^\mu$$

The index up notation is known as **contravariant**. These notation generalise for all four-vectors. For any two four-vectors a^μ and b^μ, scalar product is defined as

$$a \cdot b = a^\mu b_\nu = a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3$$

Now if

$$\begin{aligned} I &> 0, && x^\mu \text{ is time-like} \\ I &< 0, && x^\mu \text{ is space-like} \\ I &= 0, && x^\mu \text{ is light-like} \end{aligned}$$

Some Math around this notation:

Consider a n-dimensional vector space with orthogonal basis B, $e_1, e_2, ...e_n$. Let B', $e'_1, e'_2, ...e'_n$ be the new basis. Then the relation between them can given as

$$\begin{pmatrix} e'_1 \\ . \\ . \\ e'_n \end{pmatrix} = \begin{pmatrix} \frac{e_1 \cdot e'_1}{|e_1|^2} & . & . & \frac{e_n \cdot e'_1}{|e_n|^2} \\ . & . & . & . \\ . & . & . & . \\ \frac{e_1 \cdot e'_n}{|e_1|^2} & . & . & \frac{e_n \cdot e'_n}{|e_n|^2} \end{pmatrix} \begin{pmatrix} e_1 \\ . \\ . \\ e_n \end{pmatrix}$$

All vectors in the space can be represented as linear combinations of the basis i.e $v = \sum_{i=1}^{n} a_i e_i$. The coordinate vectors a_i's change as B $\rightarrow$ B' accordingly-

$$\begin{pmatrix} a'_1 \\ . \\ . \\ a'_n \end{pmatrix} = \begin{pmatrix} \frac{e_1 \cdot e'_1}{|e'_1|^2} & . & . & \frac{e_n \cdot e'_1}{|e'_1|^2} \\ . & . & . & . \\ . & . & . & . \\ \frac{e_1 \cdot e'_n}{|e'_n|^2} & . & . & \frac{e_n \cdot e'_n}{|e'_n|^2} \end{pmatrix} \begin{pmatrix} a_1 \\ . \\ . \\ a_n \end{pmatrix}$$

The above matrix is the transpose of inverse of the previous one i.e. if former matrix was Λ,then the above one is $(\Lambda^{-1})^{\mathrm{T}}$. The vectors like v, which change inversely to their base change matrix, are known as **contravariant**(v^{α}). Consider the case of gradient of a scalar function in B -

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i$$

If the base is changed from B to B', by applying chain rule and above mentioned relations, the modified gradient comes out to be -

$$(\nabla f)' = \Lambda \nabla f$$

The vectors like gradient[2], which change same as their base change matrix, are known as **covariant**(v_{α}).

2.1.3 Tensors

The above mentioned contravariant and covariant vectors can be combined to form a multi-dimensional array called tensors. A tensor of rank (or type) [m,n] has m contravariant (upper) indices and n covariant (lower) indices:

$$T = T^{i_1 i_2 \ldots i_m}_{j_1 j_2 \ldots . j_n}$$

Generally a tensor can be produced by multiplying vectors and covectors, or covectors and covctors together without pairing them up. This results in a array formation. For example,

$$s^{\mu}_{\nu} := x^{\mu} y_{\nu} = \begin{pmatrix} x^1 \cdot y_1 & x^1 \cdot y_2 & x^1 \cdot y_3 \\ x^2 \cdot y_1 & x^2 \cdot y_2 & x^2 \cdot y_3 \\ x^3 \cdot y_1 & x^3 \cdot y_2 & x^3 \cdot y_3 \end{pmatrix}$$

[2]Every gradient is covector but not vice-versa

The tensor t can be symmetric or antisymmetric depending on swapping of indices μ and ν. If the tensor is symmetric,

$$t^{\mu\nu} = t^{\nu\mu}$$

and if it is anti-symmetric

$$t^{\mu\nu} = -t^{\nu\mu}$$

2.1.4 Tensor Product

It combines two tensors to form a new tensor. The resultant rank is the sum of ranks of original tensor. Given two tensors T and S, the tensor product $T \otimes S$ is defined such that if T is of rank (m, n) and S is of rank (p, q), then $T \otimes S$ is a tensor of rank $(m + p, n + q)$.

If T has components $T^{i_1 \ldots i_m}_{j_1 \ldots j_n}$ and S has components $S^{k_1 \ldots k_p}_{l_1 \ldots l_q}$, the components of the tensor product $T \otimes S$ are given by:

$$(T \otimes S)^{i_1 \ldots i_m k_1 \ldots k_p}_{j_1 \ldots j_n l_1 \ldots l_q} = T^{i_1 \ldots i_m}_{j_1 \ldots j_n} S^{k_1 \ldots k_p}_{l_1 \ldots l_q}$$

Properties of Tensor Product

Linearity

The tensor product is linear in both of its arguments. For tensors T, U of the same type and a tensor S:

$$(T + U) \otimes S = T \otimes S + U \otimes S$$

$$T \otimes (S + V) = T \otimes S + T \otimes V$$

Associativity

The tensor product is associative. For tensors T, S, R:

$$(T \otimes S) \otimes R = T \otimes (S \otimes R)$$

Distributivity

The tensor product distributes over tensor addition. For tensors T, S, U, V:

$$(T \otimes S) \otimes (U \otimes V) = (T \otimes U) \otimes (S \otimes V)$$

2.2 General results of Special Relativity

Relativity of Simultaneity:

If two events happen simultaneously in reference frame S but at different locations, they will not occur at the same time in reference frame S'. If $t_A = t_B$,

$$t'_A = t'_B + \frac{\gamma v(x_B - x_A)}{c^2}$$

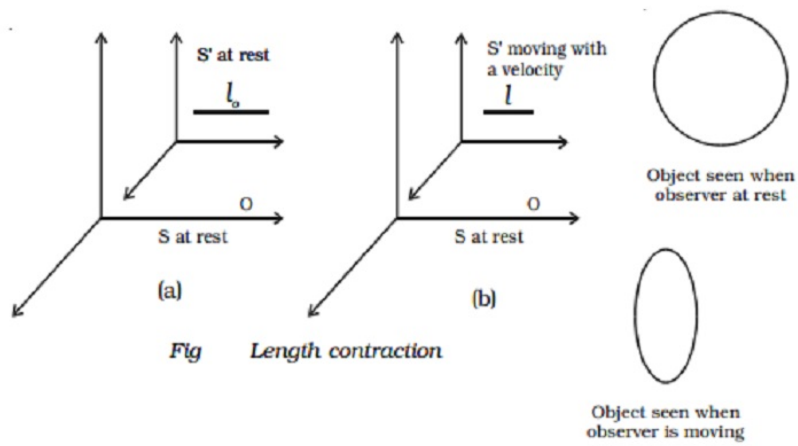


Figure 2.2: Length Contraction

Length Contraction:

Length of an object moving relative to an observer appears shorter along its direction of motion. Dimensions perpendicular to the motion remain unaffected.

$$L_{new} = \frac{L_0}{\gamma}$$

where L_0 is length measured when object is at rest.

Time Dilation

Time appears to pass slower for an observer in motion relative to a stationary observer.

$$T = \gamma T'$$

Relativistic Velocity Addition

If an object moves with velocity v relative to S' and S' moves with velocity u relative to S, then velocity of object relative to S is given by

$$V = \frac{u+v}{1+\frac{uv}{c^2}}$$

2.3 Energy and Momentum

Similar to position and time, energy and momentum can also be represented by four-vector.

$$p^\mu = (\frac{E}{c}, p_x, p_y, p_z)$$

where E and p are related by-

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

m_0 being the rest mass.
The p_μ is a contravariant vector, and follows same transformation laws as established earlier. The invariant in this case is

$$p^\mu p_\mu = (\frac{E}{c})^2 - p^2 = (m_0 c)^2$$

Proper Time

It refers to the time interval measured by a clock that is moving along with the object i.e. in the rest frame of object. Denoted by τ.

$$d\tau = \frac{dt}{\gamma}$$

Proper time is a invariant quantity.

Proper Velocity

Likewise time, let's define proper velocity as well. It is the rate of change of distance (in laboratory frame) with respect to proper time. It is also a four-vector.

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx}{dt}$$

and

$$\eta^\mu \eta_\mu = \gamma^2 (c^2 - v^2) = c^2$$

2.4 Collisions

Similar to classical collisions, following quantities are conserved in relativistic collisions-

1. Total energy: Here energy is conserved unlike mass. $E = mc^2$
2. Total momentum: Momentum is conserved.

Can be represented by equation-

$$p^\mu_A + p^\mu_B = p^\mu_C + p^\mu_D$$

Chapter 3

An Apple and a Cat

Classical Mechanics

3.1 Newton's Laws of Motion

Sir Isaac Newton's laws are fundamental to classical mechanics:

- **First Law (Law of Inertia)**: Objects remain at rest or in uniform motion unless acted upon by an external force. Mathematically, this can be expressed as:
$$\vec{F} = 0 \Rightarrow \text{constant velocity}$$
- **Second Law (Law of Acceleration)**: The acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass:
$$\vec{F} = m\vec{a}$$
where $\vec{F}$ is the net force, m is the mass of the object, and $\vec{a}$ is its acceleration.
- **Third Law (Action-Reaction Law)**: For every action, there is an equal and opposite reaction. If object A exerts a force $\vec{F}_{A \to B}$ on object B, then object B exerts a force $-\vec{F}_{A \to B}$ on object A.

3.2 Conservation Laws

Conservation laws describe fundamental properties of isolated systems:

- **Conservation of Energy**: The total energy E within an isolated system remains constant over time:
$$\frac{dE}{dt} = 0 \Rightarrow \sum_{i=1}^{n} E_i = \sum_{i=1}^{n} E'_i$$

- **Conservation of Momentum**: The total momentum $\vec{p}$ of an isolated system remains constant:

$$\frac{d\vec{p}}{dt} = 0 \Rightarrow \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} \vec{p'}_i$$

- **Conservation of Angular Momentum**: The total angular momentum $\vec{L}$ of an isolated system remains constant:

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} \vec{L}'_i$$

Phase Space: The space of positions in components of momentum is termed as phase space. For example, a harmonic oscillator in phase space executes circular motion in phase lengths of different radii corresponding to different energies.
In terms of potential energy V, force can be also written as:

$$F(\vec{r}, t) = -\nabla V(\vec{r}, t)$$

3.3 Lagrangian Mechanics

3.3.1 D'Alembert's Principle

D'Alembert's principle states that the total virtual work done by all forces acting on a system in equilibrium is zero:

$$\sum_{i=1}^{n} \vec{F}_i \cdot \delta\vec{r}_i = 0$$

Here, $\vec{F}_i$ represents the forces acting on each particle of the system, and $\delta\vec{r}_i$ denotes the virtual displacements of these particles.
Virtual work is defined as the work done by a force acting through a virtual displacement, which is a hypothetical small displacement satisfying the constraints of the system.

3.3.2 Action

Action is a quantity that describes the behavior of a system over time. It is defined as

$$A = \int_{t_0}^{t_1} \mathcal{L}(r, \dot{r})\, dt$$

where $\mathcal{L}$ is the Lagrangian, r and is the position of the object and $\dot{r}$ is velocity of the object. According to fundamental lemma of calculus of variations, δf of a function can be concentrated on an arbitrarily small interval, but not on a single point. If a continuous function f on an open interval (a, b) satisfies the equation

$$\int_a^b f(x)h(x)\,dx = 0$$

for all sufficiently smooth functions $h(x)$ on $(a, b) \Rightarrow f(x) = 0$.

3.3.3 Euler-Lagrangian Equation

As all things in nature try to remain in equilibrium, optimising their path each step, they follow principle of **least** action, according to which

$$\frac{\partial A}{\partial x_i} = 0$$

Solving by varying this equation over $\vec{r}$, we get

$$\boxed{\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0}$$

This is the Euler-Lagrangian equation.
For moving particle in a potential field, Lagrangian $\mathcal{L}$ is generally defined as:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - V(r)$$

Putting this value in the equation yields the familiar,

$$-\frac{\partial V(x)}{\partial x} = m\frac{d^2x}{dt^2}$$

This implies thar Lagrangian is a function which contains information about the entire laws governing the physics of a particular system. Also, Lagrangian in **invariant** under change of coordinate system.

3.4 Hamiltonian Mechanics

3.4.1 The Hamiltonian

Let us differentiate our beloved $\mathcal{L}$ w.r.t time t, assuming L has no explicit time dependence. Also let $p = \frac{\partial \mathcal{L}}{\partial \dot{r}}$ be the canonical momentum, then

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\frac{d\dot{x}_i}{dt}$$

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{n} \frac{d(p_i\dot{x}_i)}{dt}$$

Bringing both quantities on same side,

$$\frac{d(\sum_i p_i \dot{x_i} - \mathcal{L})}{dt} = 0$$

This quantity $\sum_i p_i \dot{x_i} - \mathcal{L} = \mathcal{H}$ is known as Hamiltonian, which is the lagrangian analogue for energy, as clearly visible as it is constant w.r.t to time(conservation law fulfiled). In general hamiltonian function $\mathcal{H}$ is written as

$$\mathcal{H} = \frac{1}{2}m\dot{r}^2 + V(r) = E$$

If the Lagrangian has an explicit dependence on time, $\mathcal{L}(x, \dot{x}, t)$, the time derivative of $\mathcal{H}$ turns out to be -

$$\frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$

3.4.2 Hamiltonian Equations

The Hamiltonian equations are the equations of motion for a system in terms of the canonical coordinates and momenta.

$$\frac{dr}{dt} = \frac{\partial \mathcal{H}}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial r}$$

The Hamiltonian equations can be derived from the Lagrangian equations by using the relation between $\mathcal{L}$ and $\mathcal{H}$. These can also be derived from the principle of least action by using the Legendre transform to convert $\mathcal{L}$ in $\mathcal{H}$.

Quantum Mechanics

"God doesn't play dice" - statement by Albert Einstein clearly defines what quantum mechanics indeed is. Unlike classical mechanics, this is a complete probabilistic theory, where amateurs like me, try to find certainity.

3.5 The Wave Function

The wave function is a mathematical function that describes the quantum state of a physical system. It is a complex-valued function of position and time, denoted by $\psi(x, t)$. The wave equation, also known as the Schrödinger equation, is a partial differential equation that gives relation between ψ, x and t, just like its classical counterpart. It is given by-

$$\imath\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

3.5.1 Copenhagen Interpretation

The Copenhagen interpretation is the most widely accepted interpretation of quantum mechanics. It states that the wave function ψ is a probability amplitude, and the square of its absolute value, $|\psi|^2$, gives the probability density of finding a particle at a given point in space. It asserts that quantum systems exist in superpositions of states until measured, whereupon the wave function collapses to a single outcome. Observers play a crucial role in this process, influencing the outcome through measurement. This emphasizes the probabilistic nature of measurements and the impossibility of predicting precise outcomes before measurement.

3.5.2 Operators

Physical observables are represented by linear operators. These operators act on the wave function and linearly transform it into another wave equation. Mathematically, wave function is a vector,living in a special vector space, called the **Hilbert Space**. Hence the operators are matrices, operating on principles of linear algebra.

3.5.3 Dirac Notation

Dirac notation, also known as bra-ket notation, is a concise mathematical framework used in quantum mechanics used to represent vectors and inner products. This notation elegantly represents vectors (states) and linear operators (observables) in a vector space.

- $|\psi\rangle$: a ket, represents a state vector, defined by $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$
- $\langle\psi|$: a bra, represents a dual vector, defined by $\begin{pmatrix} a_1^* & a_2^* & \cdots & a_n \end{pmatrix}$
- $\langle\psi \,|\, \phi\rangle$: an inner product, represents the dot product of two vectors, defined by $\begin{pmatrix} a_1^* & a_2^* & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + ... + a_n^* b_n$

3.6 The Hilbert Space

Hilbert Space is a complete[1] inner product space, which is set of all square-integrable functions, on a specified interval, i.e. $\forall$ f, $\int_a^b |f(x)|^2 dx < \infty$.

[1] Any cauchy sequence of functions in Hilbert Space converges to a function in the same space.

The inner product in Hilbert space is defined as:

$$\langle f \mid g \rangle = \int_a^b f^* g \, dx$$

Hence the norm is

$$\langle f \mid f \rangle = \int_a^b |f|^2 \, dx$$

In terms of orthogonal basis $|e_i\rangle$ of Hilbert space, a function can be expressed as

$$|\psi(x)\rangle = \sum_{n=1}^{\infty} c_i |e_i\rangle$$

where $c_i = \langle e_i \mid \psi \rangle$, are the coefficients of expansion[2]. For a normalized wave function i.e. $\int |\psi|^2 \, dx = 1$

$$\sum_n |c_n|^2 = 1$$

3.6.1 Hermitian Operators

Hermitian operators are linear operators that satisfy the following property:

$$\langle \psi \mid \hat{A}\phi \rangle = \langle \hat{A}\psi \mid \phi \rangle$$

$\forall$ f and g. These operators have *real eigenvalues and orthogonal eigenvectors*. They are used to represent physical observables in quantum mechanics. In dirac notation, hermitian operations, $\langle \psi \mid \hat{A}\phi \rangle$ can be written as $\langle \psi | \hat{A} \mid \phi \rangle$. The hermitian conjugate of an operator $\hat{A}$ is $\hat{A}^\dagger$ such that

$$\langle \psi \mid \hat{A}\phi \rangle = \langle \hat{A}^\dagger \psi \mid \phi \rangle$$

For hermitian operators, $\hat{A} = \hat{A}^\dagger$.

3.6.2 Eigenvalue Problem

The eigenvalue equation for a hermitian operator is:

$$\hat{A}|\psi\rangle = a|\psi\rangle$$

where a is the eigenvalue and $|\psi\rangle$ is the eigenvector. Determinate states of any observable A are eigenfunctions of $\hat{A}$.
For any operator $\hat{A}$,

$$\langle A \rangle = \sum_n a_n |c_n|^2$$

where a_n is corresponding base eigenvalue of $\hat{A}$[3].

[2] Fourier's Trick

[3] $\hat{A}|e_n\rangle = a_n|e_n\rangle$

3.7 The Heisenberg Uncertainity Principle

The Heisenberg Uncertainty Principle states that it is impossible to determine both the position and momentum of a particle with infinite precision. Mathematically, this is expressed as:

$$\Delta A \Delta B \geq \frac{\langle [\hat{A}, \hat{B}] \,|\,|\,\rangle}{2\iota}$$

where ΔA and ΔB are the uncertainties (standard deviation) in observables A and B respectively, provided $\hat{A}$ and $\hat{B}$ are non-commutative. For example, taking position and momentum,

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

This principle implies that particles do not have definite properties until measured, highlighting the active role of the observer. These errors are not random experimental errors, but rather fundamental limits imposed by statistical nature of quantum mechanics.

3.8 Angular Momentum

3.8.1 Introduction

In quantum mechanics, the concept of angular momentum is pivotal, mirroring its classical counterpart but with significant quantum mechanical changes. The angular momentum operator, denoted by $\hat{L}$, covers the rotational symmetries and dynamics of quantum systems. It is defined in terms of the position operator $\hat{r}$ and the momentum operator $\hat{p}$ as:

$$\hat{L} = \hat{r} \times \hat{p}$$

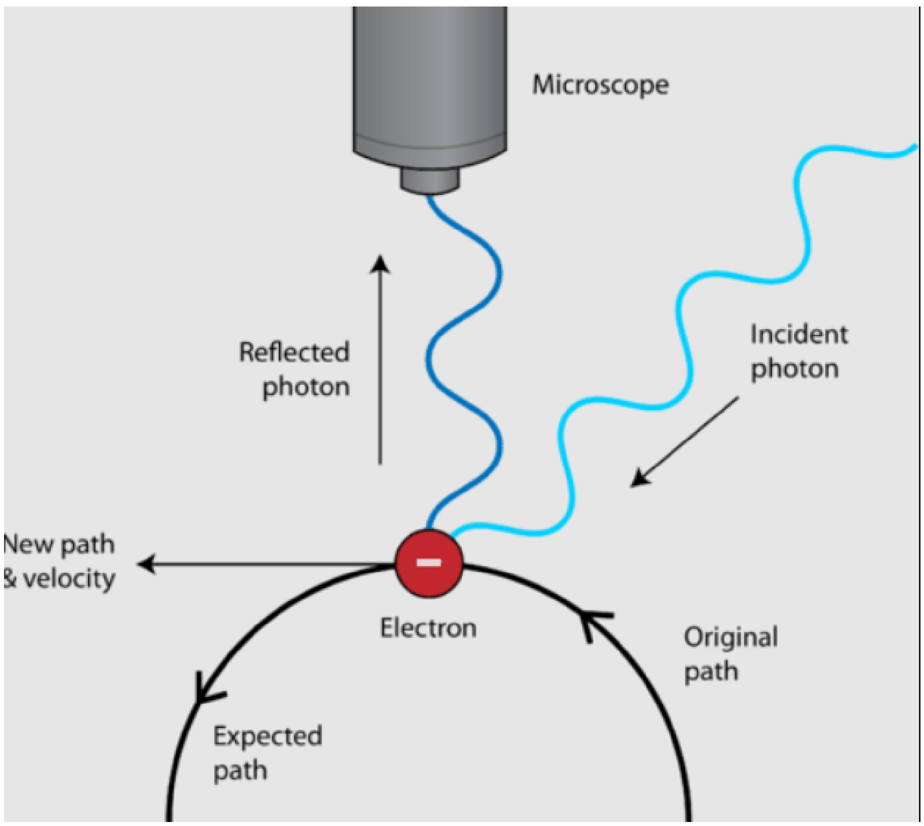


Figure 3.1: Actual Reason

3.8.2 Angular Momentum Operators

The components of the angular momentum operator in Cartesian coordinates are:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y,$$
$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z,$$
$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,$$

where the momentum operator $\hat{p}_\alpha$ is expressed as:

$$\hat{p}_\alpha = -\imath\hbar\frac{\partial}{\partial\alpha}$$

These components collectively describe the total angular momentum of a quantum system.

3.8.3 Commutation Relations

The angular momentum components are non-commutative in nature and hence their commutators are respectively:

$$\left[\hat{L}_x, \hat{L}_y\right] = \imath\hbar\hat{L}_z,$$
$$\left[\hat{L}_y, \hat{L}_z\right] = \imath\hbar\hat{L}_x,$$
$$\left[\hat{L}_z, \hat{L}_x\right] = \imath\hbar\hat{L}_y.$$

3.8.4 Eigenvalue Equations

The quantization of angular momentum is expressed through its eigenvalue equations. The square of the angular momentum operator, $\hat{L}^2$, is given by:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2.$$

Using the ladder operations, the eigenvalue equation for $\hat{L}^2$ is:

$$\hat{L}^2|\psi\rangle = \hbar^2 l(l+1)|\psi\rangle$$

where l is the orbital angular momentum quantum number, taking on integer values $l = 0, 1, 2, \ldots$. The eigenvalue equation for the z-component of the angular momentum, $\hat{L}_z$, is:

$$\hat{L}_z|\psi\rangle = \hbar m|\psi\rangle$$

where m is the magnetic quantum number, ranging from $-l$ to l in integer steps. The components of angular momentum $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ cannot be simultaneously determined due to their non-commutative nature. Specifically, the commutation relations $\left[\hat{L}_x, \hat{L}_y\right] = \imath\hbar\hat{L}_z$, $\left[\hat{L}_y, \hat{L}_z\right] = \imath\hbar\hat{L}_x$, and $\left[\hat{L}_z, \hat{L}_x\right] = \imath\hbar\hat{L}_y$ imply that measuring one component of angular momentum disturbs the others. Consequently, while we can precisely know the total angular momentum $\hat{L}^2$ and one of its components, typically $\hat{L}_z$, the other two components ($\hat{L}_x$ and $\hat{L}_y$) remain indeterminate. This uncertainty is a direct consequence of the Heisenberg uncertainty principle.

3.9 Spin Angular Momentum

Apart from orbital angular momentum, elementary particles possess intrinsic angular momentum as well, known as Spin Angular Momentum. Strange enough, despite being point particles, these particles exhibit rotational momentum intrinsically. The spin angular momentum operator is denoted by $\hat{S}$ and its components are given by:

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x$$
$$\hat{S}_y = \frac{\hbar}{2}\sigma_y$$
$$\hat{S}_z = \frac{\hbar}{2}\sigma_z$$

where σ_x, σ_y, and σ_z are the Pauli spin matrices given by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvectors for the $\hat{S}_z$ matrix are:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Similarly, the eigenvectors for the $\hat{S}_x$ and $\hat{S}_y$ matrices can be calculated:

$$\text{Eigenvectors of } \hat{S}_x : \quad |+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

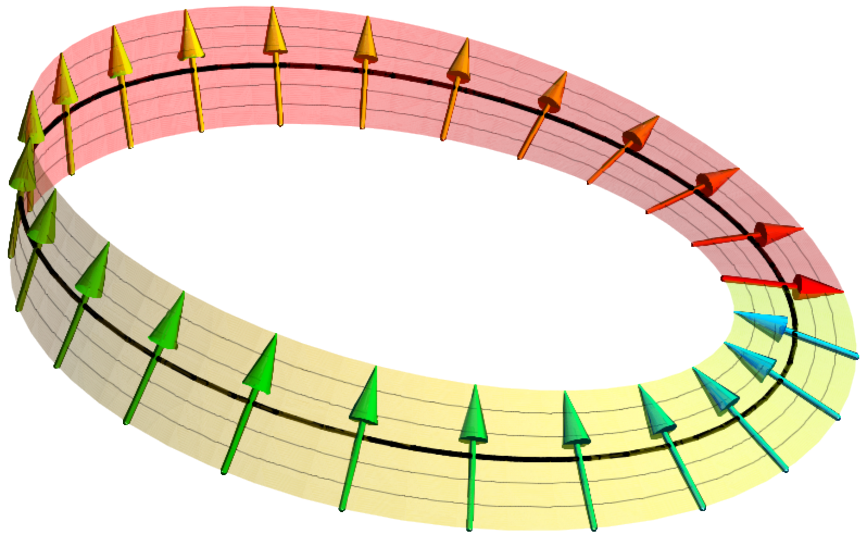


Figure 3.2: Visualisation for a Spinor

$$\text{Eigenvectors of } \hat{S}_y : \quad |+_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \iota\end{pmatrix}$$
$$|-_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\iota\end{pmatrix}$$

These eigenvectors are also referred to as spinors and are normalized to ensure that their total probability is equal to 1, a requirement in quantum mechanics:

$$\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$$

These spinors occupy an intermediate position between scalars and vectors. It rotates to same configuration after 4π angle, unlike vectors (2π). When the coordinate axes are rotated, this spinor changes as -

$$\chi' = \mathbb{U}(\theta)\chi$$

where $\mathbb{U}(\theta)$ is the rotation matrix. The rotation matrix, $\mathbb{U}(\theta) = e^{\frac{-\iota\theta\cdot\sigma}{2}}$ where θ points along the axis of rotation and σ corresponds to pauli matrices.

Stern-Gerlach Experiment

The Stern-Gerlach experiment is a classic demonstration of spin angular momentum. In this experiment, silver atoms are passed through a magnetic field gradient, causing the atoms to deflect in different directions depending on their spin orientation. The experiment shows that the spin of an atom can take on

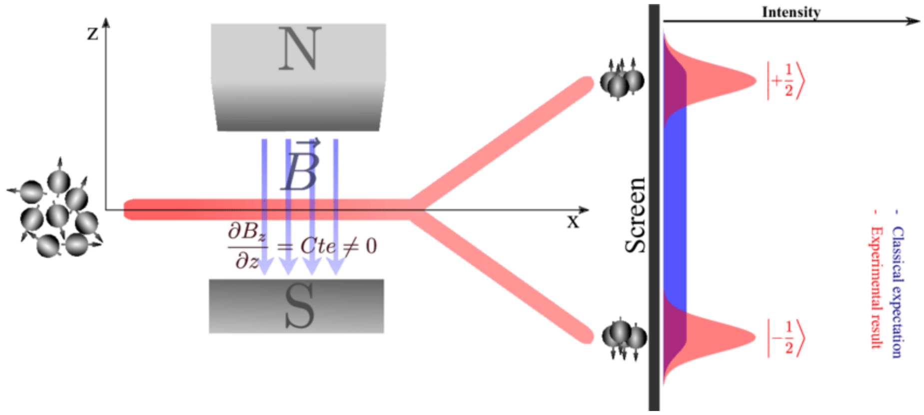


Figure 3.3: Stern-Gerlach Experiment

only two values, corresponding to the two possible orientations of the spin angular momentum vector. This is a direct manifestation of the quantization of spin angular momentum. In summary, the deflection of particles in the experiment, into discrete paths confirms the existence of spin and its quantized nature.

Chapter 4

Tango with Invariance

Symmetry is an operation you can perform on a system that leaves it **invariant**, carrying it into a configuration indistinguishable from the original. For example, if you rotate a sphere by 360 degrees, it will look exactly the same as it did before the rotation. This is an example of a symmetry operation. Properties of set of all symmetry operations, in general -

1. *Closure:* If $\mathbb{R}_i$ and $\mathbb{R}_j$ are in the set, then the product $\mathbb{R}_i\mathbb{R}_j$ is also in the same set. In other words, $\exists\, \mathbb{R}_k \in$ set such that $\mathbb{R}_k = \mathbb{R}_i\mathbb{R}_j$.
2. *Associativity:* $\mathbb{R}_i(\mathbb{R}_j\mathbb{R}_k) = (\mathbb{R}_i\mathbb{R}_j)\mathbb{R}_k$.
3. *Identity:* There exists an identity element $\mathbb{I}$ such that $\mathbb{R}_i\mathbb{I} = \mathbb{I}\mathbb{R}_i = \mathbb{R}_i\, \forall\, \mathbb{R}_i$ in the set.
4. *Inverse:* For every $\mathbb{R}_i$ in the set, there exists an inverse $\mathbb{R}_i^{-1}$ such that $\mathbb{R}_i\mathbb{R}_i^{-1} = \mathbb{R}_i^{-1}\mathbb{R}_i = \mathbb{I}$.

A set of elements that satisfies the above properties is called a **group**. The set of symmetry operations of a system is called the **symmetry group** of the system. This symmetry group provides a mathematical framework for describing the symmetries of the system.
Groups can be represented using matrices, particle physicists' favourite tool! In the context of symmetries, matrices are used to represent transformations such as rotations, reflections, and other symmetry operations.

4.1 Types of Groups

- **Finite or Infinite:** A finite group has a limited number of elements, while an infinite group has an unbounded number of elements.
- **Discrete or Continuous:** A discrete group consists of isolated elements, whereas a continuous group, also known as a Lie group, has elements that form a continuum, often parameterized by continuous variables.

- **Abelian Group:** A group in which the order of the elements does not matter. Mathematically, the elements *commute*, $R_iR_j = R_jR_i$.

The important matrix for our use is the Unitary Matrix, which satisfies the property $\mathbb{U}^{-1} = \mathbb{U}^{\dagger}$ i.e. Hermitian. Unitary matrices preserve the inner product in complex vector spaces. When the determinant of a unitary matrix $\mathbb{U}$ is restricted to be 1, the group formed by such matrices is called the Special Unitary group, denoted as SU(n), where n is the dimension of the matrices.

SU(n)'s that we need

- **SU(2):** It explains the spin of particles and the weak interactions in the Standard Model.
- **SU(3):** It explains the theory of strong interactions, known as Quantum Chromodynamics (QCD). It describes the symmetries of quarks and gluons.

The group of real unitary matrices is called orthogonal group, O(n). Again if determinant is restricted to 1, it is SO(n).

4.1.1 Homomorphic Groups

A group $\mathbb{G}$ is said to be homomorphic to a group H if there exists a function ϕ from $\mathbb{G} \to \mathbb{H}$ such that $\forall$ a,b $\in \mathbb{G}$,

$$\phi(ab) = \phi(a)\phi(b)$$

- **Kernel:** It is the set of elements in $\mathbb{G}$ that maps to the identity element in H, $\phi(g) = I_{\mathbb{H}} \, \forall g \in \mathbb{G}$ where $I_{\mathbb{H}}$ is identity element in $\mathbb{H}$
- **Image:** It is the set of elements in $\mathbb{H}$ having preimage in $\mathbb{G}$, or ϕG.

4.1.2 Isomorphism

A homomorphism $\mathbb{G} \to \mathbb{H}$ is said to be isomorphism if ϕ is both injective and surjective $\Rightarrow$ Bijective. Apart from the properties of homomorphism, existence of inverse is unique to isomorphism. If $\phi : \mathbb{G} \to \mathbb{G}$ is an isomorphism, then there exists $\phi^{-1} : \mathbb{H} \to \mathbb{G}$ such that $\phi^{-1}(\phi(g) = g) \, \forall g \in G$.

4.2 Noether's Theorem

For every differentiable symmetry of the action of a physical system corresponds to a conservation law. Consider a physical system described by a Lagrangian L and action S. As we know,

$$S = \int L(q_i, \dot{q}_i, t) \, dt$$

Symmetry		Conversation Law
Translation in time	$\longleftrightarrow$	Energy
Translation in space	$\longleftrightarrow$	Momentum
Rotation	$\longleftrightarrow$	Angular momentum
Gauge transformation	$\longleftrightarrow$	Charge

Figure 4.1: Symmetry and corresponding conservation law

A symmetric transformation (continuous) can be represented by:

$$q_i \to q_i + \epsilon\eta_i(q, \dot{q}, t)$$

where $\epsilon \to 0$ and η_i is the transformation generator. Noether's theorem states that if the action S is invariant under a continuous symmetry transformation, then there exists a conserved quantity. If $\delta S = 0$, then for the above changes, there exists conserved current,J^ν , known as **Noether's Current** such that

$$\frac{\partial J^\nu}{\partial x^\nu} = 0$$

4.3 Isospin

Isospin or isobaric spin, is a artificial analogue to spin to describe the similarities between protons and neutrons under the strong nuclear force. Despite their different electric charges, protons and neutrons are nearly identical in mass and interact similarly with other hadrons, suggesting an underlying symmetry. It is denoted by **I**. This is mathematically represented using the formalism of spinors, mainly SU(2) group.

4.3.1 Isospin Spinor and Its Components

Introduced by Heisenberg, the proton and neutron are treated as two states of a nucleon doublet. The isospin state of a nucleon can be represented as a two-component spinor:

$$\chi = \begin{pmatrix} \chi_p \\ \chi_n \end{pmatrix}$$

where χ_p represents the proton state and χ_n represents the neutron state. $\chi_p =$ 1 and $\chi_n = 0$ in case of proton and vice-versa.

4.3.2 Isospin Operators

The isospin operators, analogous to the Pauli matrices used in spin-½ systems, are given by:

$$I_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These operators act on the χ and obey the commutation relations of the $SU(2)$(Lie) algebra.

4.3.3 Total Isospin and Projection

The total isospin I and its third component I_3 are used to classify particles, just like spin. For the nucleons:

$$I = \frac{1}{2}, \quad I_3 = \begin{cases} +\frac{1}{2} & \text{for the proton} \\ -\frac{1}{2} & \text{for the neutron} \end{cases}$$

Strong Interactions are *invariant* under rotations in isospin space, and following the Noether theorem, isospin is the conserved quantity. I is assingned to a multiplet and I_3 is assigned to each member of the multiplet. The multiplicity is given by

$$\text{multiplicity } = 2I + 1$$

4.3.4 The Nishijima-Gell Mann Relation (for hadrons composed of u,d and s quarks)

The third component of isospin is related to charge, Q of the concerned particle and multiplet member with highest charge gets highest value $I_3 = I$. The charge of a particle can be related to its isospin and hypercharge (associated with $SU(3)$) through the Nishijima-Gell Mann relation:

$$Q = I_3 + \frac{Y}{2}$$

where Q is the electric charge, I_3 is the third component of isospin, and Y is the hypercharge, given by $Y = A + S$, A being baryon number and S is strangeness. For consistency, all leptons and mediators are assigned I = 0, as it doesn't affects them.

4.3.5 Isospin Multiplets

Particles can be grouped into multiplets based on their isospin values:

Isospin Doublets

The nucleon (proton and neutron) forms an isospin doublet:

$$\begin{pmatrix} p \\ n \end{pmatrix}$$

Isospin Triplets

Pions (π^+, π^0, π^-) form an isospin triplet with $I = 1$:

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$$

In this case, π^+ has $I_3 = +1$, π^0 has $I_3 = 0$, and π^- has $I_3 = -1$.

4.3.6 Isosinglets

An isosinglet refers to a particle or a state that has an isospin quantum number I=0. This means that the particle is invariant under isospin transformations, much like a scalar is invariant under rotations in ordinary three-dimensional space. Examples include λ baryon and deutron nucleus.

4.3.7 Symmetric and Antisymmetric States

The particle states can be further classified as either symmetric or antisymmetric under exchange:

Symmetric States

For two nucleons, the symmetric combination is:

$$\chi_{\text{symmetric}} = \frac{1}{\sqrt{2}}(\chi_A \chi_B + \chi_B \chi_A)$$

Antisymmetric States

The antisymmetric combination for two nucleons is:

$$\chi_{\text{antisymmetric}} = \frac{1}{\sqrt{2}}(\chi_A \chi_B - \chi_B \chi_A)$$

4.4 Gell-Mann's Eightfold Way

Gell-Mann extended this concept to the larger group $SU(3)$. The Eightfold Way is a classification scheme for hadrons that organizes hadrons into multiplets based on their properties under the SU(3) flavor symmetry, which includes the up, down, and strange quarks.

4.4.1 Baryon Octet

The baryon octet includes particles like the proton, neutron, Λ, and Σ baryons, arranged in a hexagonal pattern reflecting their isospin (as **s** in the image) and hypercharge values (as **q**).

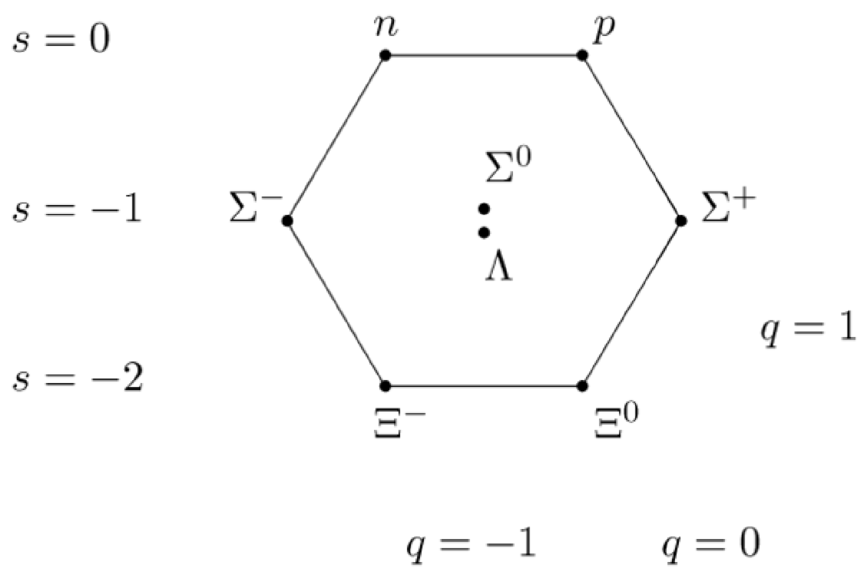


Figure 4.2: Baryon Octet

4.4.2 Meson Octet

The meson octet includes particles like the pions and kaons, also arranged in a hexagonal pattern.

4.4.3 Baryon Decuplet

The baryon decuplet includes 10 baryons under SU(3), including particles like Δ, Σ, Ξ, and Ω.

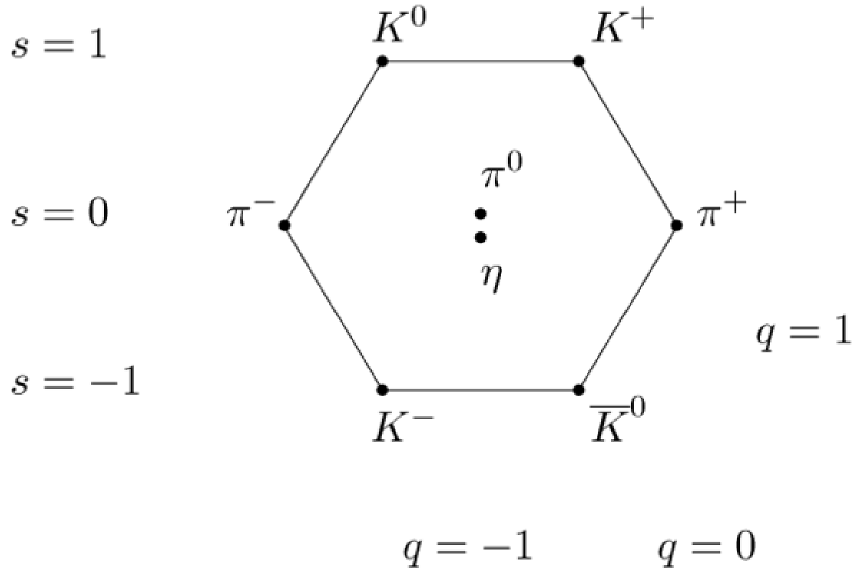


Figure 4.3: Meson Octet

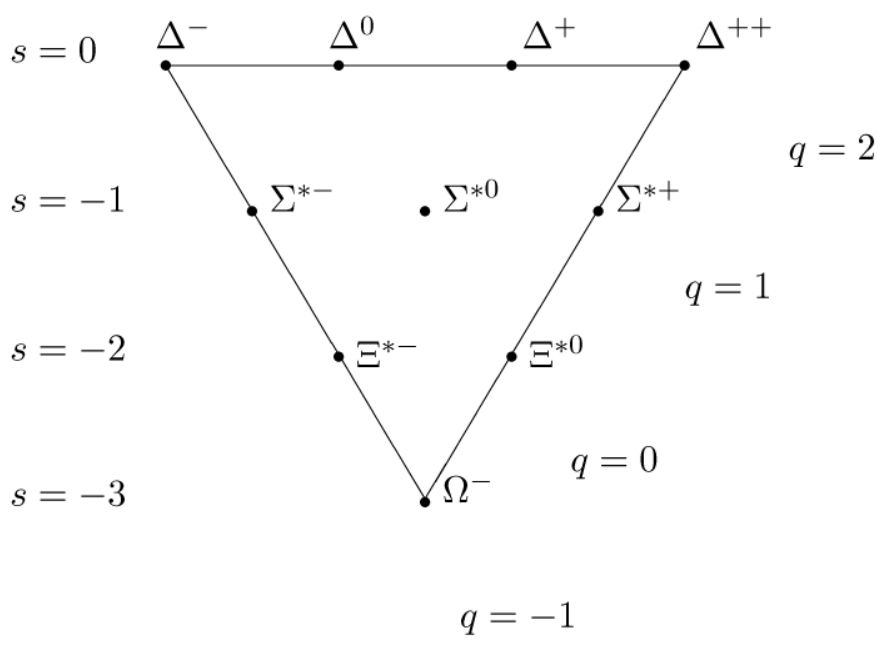


Figure 4.4: Baryon Decuplet

Chapter 5

The Eccentric Elegance of Symmetry

5.1 Parity

Parity is a fundamental symmetry operation that involves the inversion of spatial coordinates, i.e. r $\rightarrow$ -r. If a system has a wave function $\psi(r)$, this operation transforms it into $\psi(-r)$. Parity can be even or odd depending on ψ. For mass-containing particles, like muons and electrons; notion of parity is quite simple. It solely depends on frame of reference of the observer. For example, if moving particle is rotating clockwise for a observer in frame with $v_{rel} = v_{particle}$, then it would be rotating anticlockwise for a observer in frame moving with velocity greater than particle, i.e. $v_{rel} < 0$.
In case of massless particles, like neutrinos (almost zero rest mass), things become complicated as they move at speed equals to that of light, which is an absolute quantity in relativity. Here particle can be only in one defined rotation or in other words, the parity/helicity is not interconvertible.

5.1.1 Helicity

Helicity is defined as the projection of a particle's spin onto its direction of motion. For a particle with momentum $\vec{p}$ and spin $\vec{S}$, helicity is given as:

$$h = \frac{\vec{p} \cdot \vec{S}}{|\vec{p}|^2}$$

On simpler terms, helicity can be +1 or -1, representing **right handed** and **left handed** spins respectively. Helicity is invariant under Lorentz transformations but not under parity transformations for massless particles as explained above. Strong and Electromagnetic Interactions conserve parity, implying that physical processes and their mirror images are indistinguishable. But conservation of parity gets violated in weak interactions, this was famously demonstrated in the

1950s through experiments on beta decay, where it was observed that neutrinos are produced predominantly with left-handed helicity.

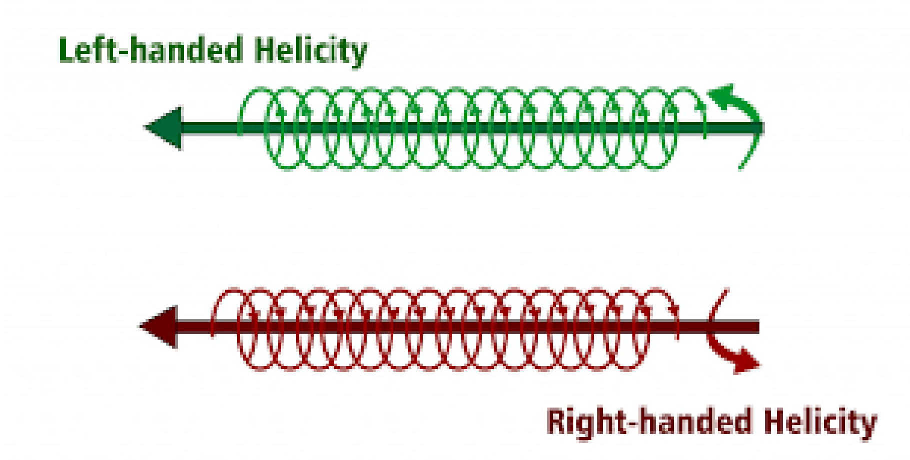


Figure 5.1: Parity States

5.1.2 Parity Operator

It is defined as the operator that reverses the sign of all spatial coordinates. It is represented by the symbol $\mathcal{P}$ and acts on $\psi(r)$ as:

$$\mathcal{P}\psi(r) = \psi(-r)$$

Applying P again we get

$$\mathcal{P}^2\psi(r) = \psi(r)$$

implies $\mathcal{P}^2 = I$ or eigenvalues of parity operator are ± 1. A system in an eigenstate of the parity operator satisfies:

$$\mathcal{P}\psi(r) = \pm\psi(r)$$

Classification

Vectors and scalars can be categorized based on their transformation properties under parity.

- **Scalar**: A quantity with spin = 0 that is invariant under parity transformations, $a \xrightarrow{P} a$.
- **Pseudoscalar**: A quantity with spin = 1 that changes sign under parity transformations, $a \xrightarrow{P} -a$.

- **Vector**: A quantity with spin = 1 that changes sign under parity transformations, $\vec{V} \xrightarrow{P} -\vec{V}$. Also known as polar or true vector.
- **Pseudovector**: A quantity with spin = 1 that is invariant under parity transformations, $\vec{V} \xrightarrow{P} \vec{V}$. Also known as axial or false vector.

5.1.3 Composite Systems

The parity of a composite system in ground state can be determined by the product of the parities of its constituents. Let say a system has constituents with parity eigenvalues $\mathcal{P}_1$ and $\mathcal{P}_2$, the net parity eigenvalue is given by

$$\mathcal{P} = \mathcal{P}_1 \mathcal{P}_2$$

This gives intution that parity operator could be of unitary form (i.e $e^{-\iota \hat{P}}$).
In the context of particle physics, fermions and their antiparticles exhibit opposite intrinsic parity, whereas bosons and their antiparticles share the same intrinsic parity. By convention, quarks are assigned an intrinsic parity of +1, while antiquarks are assigned an intrinsic parity of -1. The photon, being a vector particle, possesses an intrinsic parity of -1.

5.2 Charge Conjugation

Charge conjugation, denoted by the operator $\mathcal{C}$, transforms a particle into its corresponding antiparticle. For a particle/field $\psi(x)$, the charge conjugated particle/field $\psi^c(x)$ is given by:

$$\psi^c(x) = \mathcal{C}\psi(x)\mathcal{C}^{-1}$$

In short, it involves changing the sign of all internal quantum numbers (such as charge, baryon number, lepton number). If $\psi(x)$ is an eigenstate of $\mathcal{C}$, we have:

$$\mathcal{C}\psi(x) = \lambda_C \psi(x)$$

where λ_C is the eigenvalue. For bosons, $\lambda_C = \pm 1$, corresponding to even (symmetric) and odd (antisymmetric) states under charge conjugation.

5.2.1 Composite Systems

Charge conjugation is a multiplicative, same as parity. For a composite system consisting of two particles with charge conjugation eigenvalues C_1 and C_2, the net charge conjugation eigenvalue C is given by:

$$C = C_1 \cdot C_2$$

This implies that charge conjugation operator could be of unitary form similar to parity(i.e $e^{-\iota \hat{C}}$).

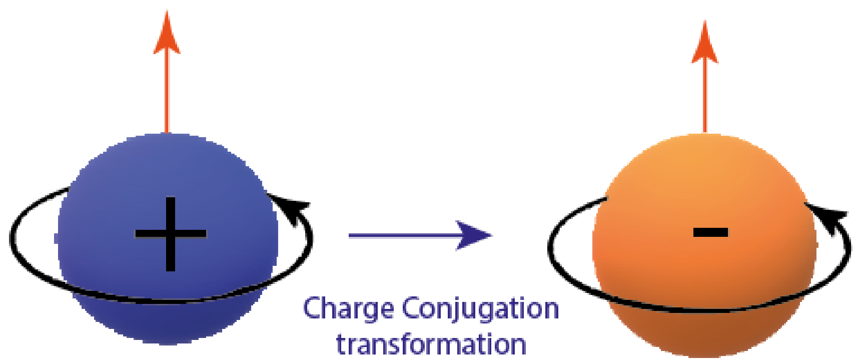


Figure 5.2: Parity States

5.2.2 Limited Usage of $\mathcal{C}$

Most particles are not eigenstates of the charge conjugation operator. This is because charge conjugation changes particles into antiparticles, and for most particles, the resultant state is distinct from the original state. For example, an electron is not an eigenstate of $\mathcal{C}$ because $\mathcal{C}$ transforms it into a positron, which is a different particle. A special condition where particles can be eigenstates of charge conjugation occurs in neutral mesons and photon. Neutral mesons, such as the neutral pion π^0, are their own antiparticles and can be eigenstates of $\mathcal{C}$ with eigenvalues ± 1
Charge conjugation symmetry is conserved in electromagnetic and strong interactions but **violated** in *weak interactions.*

5.2.3 G-Parity

G-parity is a extension of charge conjugation, combining charge conjugation with isospin symmetry. If rotated about axis-2 in isospin space, the G-parity operator $\mathcal{G}$ is defined as:

$$\mathcal{G} = \mathcal{C}e^{i\pi I_2}$$

where I_2 is the second component of the isospin operator. G-parity is useful in classifying the properties of hadrons. All mesons with strangeness/beauty/charm/truth

$= 0$ are eigen states of $\mathcal{G}$, eigenvalues given by:

$$\mathcal{G}\psi = (-1)^I \mathcal{C}\psi$$

5.3 $\mathcal{CP}$ Violations

Weak Interactions are not invariant neither under parity nor charge conjugation. But the combined operation of $\mathcal{CP}$ works quite well. $\mathcal{CP}$ symmetry implies that the laws of physics should remain unchanged if a particle is replaced by its antiparticle and its spatial coordinates are inverted. Mathematically, the $\mathcal{CP}$ acts on particle field ψ as:

$$\mathcal{CP}\psi = \lambda_{CP}\psi$$

where λ_{CP} is the eigenvalue of $\mathcal{CP}$ operator.

5.3.1 K^0 Dilemma

Neutral kaons (K^0 and $\bar{K^0}$) provide a classic example of CP symmetry and its violation. These particles can oscillate between each other due to weak interactions. The CP operator in this case comes out to be:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of this matrix are ± 1. The result of this operation is that the neutral kaons are in a superimposed state, one which has long lifetime (with $\lambda_{CP} = 1$) and short life time($\lambda_{CP} = -1$). The $\mathcal{CP}$ eigenstates are:

$$|K_L\rangle = \frac{|K^0\rangle + |\bar{K^0}\rangle}{\sqrt{2}}$$
$$|K_S\rangle = \frac{|K^0\rangle - |\bar{K^0}\rangle}{\sqrt{2}}$$

In this system, CP violation is observed in the decay processes of the long-lived kaon (K_L). The K_L decay into 3π system and K_S into 2π system. When the long-lived particle was studied experimentally, discreprencies were encountered, that is, K_L is not a perfect eigenstate of CP but has a small mixture of K_S as well. Mathematically,

$$|K'_L\rangle = \frac{|K_L\rangle + \epsilon|K_S\rangle}{\sqrt{1+\epsilon^2}}$$

where ϵ is the CP violation parameter. The value of ϵ was measured to be 2.24×10^{-3}.

The primary reason for CP violation is the presence of complex phases in the parameters of the Standard Model, particularly in the Cabibbo-Kobayashi-Maskawa (CKM) matrix that describes quark mixing. The CKM matrix includes complex phases that lead to differences in the behavior of particles and

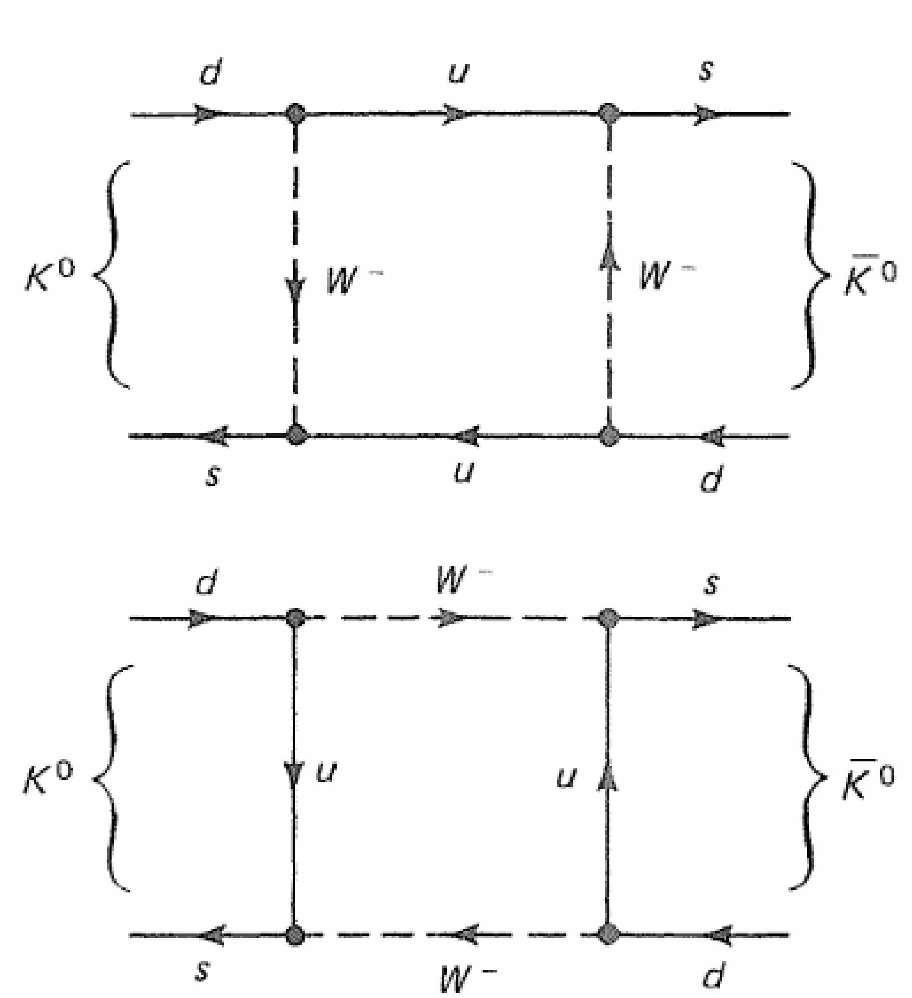


Figure 5.3: Interconversions of K^0

antiparticles.. Understanding CP violation is crucial for explaining the matter-antimatter asymmetry in the universe.

5.4 Time Reversal

Time reversal symmetry ($\mathcal{T}$) refers to the invariance of physical laws under the reversal of the direction of time. If a physical process is described by a Hamiltonian $\mathcal{H}$, time reversal symmetry implies that:

$$\mathcal{H} = \mathcal{T}\mathcal{H}\mathcal{T}^{-}1$$

where $\mathcal{T}$ is the time reversal operator.
For electromagnetic and strong interactions, time reversal is true always, but again comes **weak interaction** to spoil the party!
This shows that time reversal is not a perfect symmetry in nature, but something else is, and by intution it is $\mathcal{TCP}$.

5.5 $\mathcal{TCP}$ Theorem

The $\mathcal{TCP}$ theorem from inherited from Quantum Field Theory, states that combined operation of time reversal, charge conjugation and parity (in any order) is an exact symmetry of any interaction. The physical laws should remain unchanged if a particle is replaced by its antiparticle, its spatial coordinates are inverted, and the direction of time is reversed. If the theorem is correct, every particle must have precisely the same mass and lifetime as its antiparticle. Experiments till date prove it correct within range of errors, and any departure from this theorem will unleash absolute madness in the Standard Model.

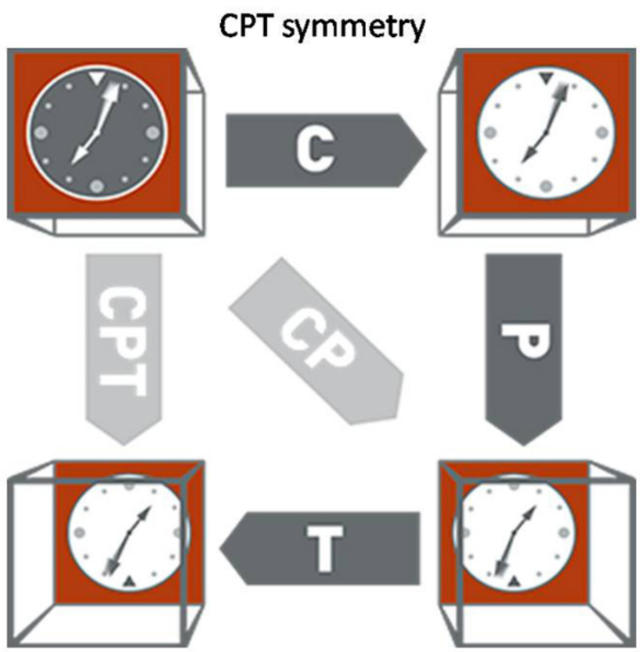


Figure 5.4: Neat Visualisation

Chapter 6

The Ultimate Theory

Quantum Field Theory is the framework which unites quantum mechanics with special relativity. It explains the interactions of elementary particles and fields they interact through.

6.1 Classical vs Quantum Fields

Classical fields describe the distribution of physical quantities in space and time, connected through differential equations.
Quantum fields are discrete, quantized; where they become operators that can create and annihilate particles.

6.1.1 Lagrangian and Hamiltonian Formulations

The dynamics of fields are described by the Lagrangian density $\mathcal{L}$. The action S is the integral of the Lagrangian density over spacetime:

$$S = \int \mathcal{L}\, dx^4$$

From the Lagrangian, the equations of motion can be derived using the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

where ϕ is the field. The Hamiltonian density $\mathcal{H}$ is defined as the Legendre transform of the Lagrangian density:

$$\mathcal{H} = \sum_i \pi_i \partial_0 \phi_i - \mathcal{L}$$

where π_i is the conjugate momentum of the field ϕ_i.

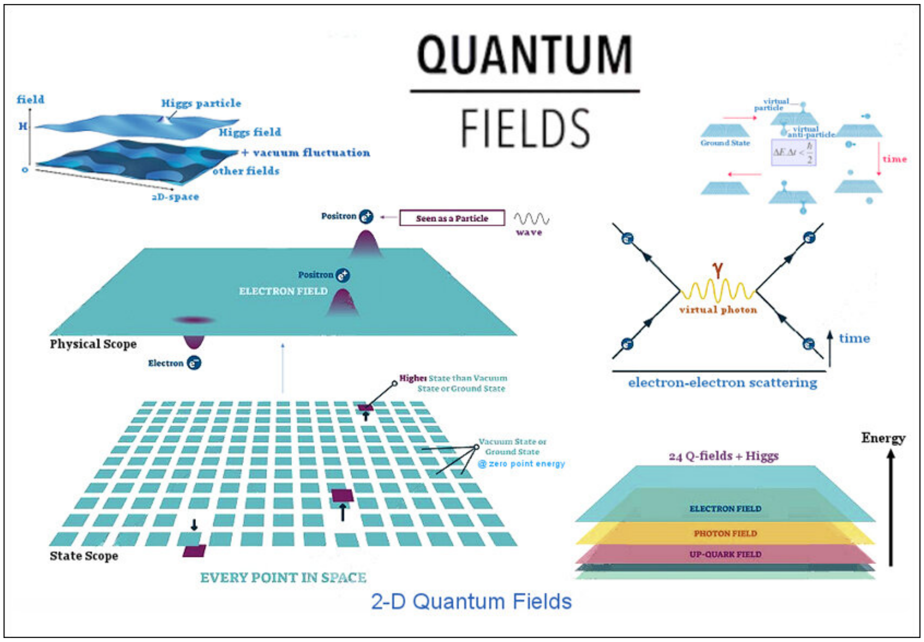


Figure 6.1: Field Theory in a glance

6.2 Path Integral Formulation

The amplitude to propagate from a point q_i to a point q_f in time T is governed by the unitary operator $e^{-\iota HT}$, where H is the Hamiltonian. Precisely writing the idea, if $|q\rangle$ represents the state of particle at q, the amplitude is $\langle q_f \,|\, e^{-\iota HT} |q_t\rangle$. The path integral formulation is a way to calculate this amplitude. The amplitude is given by the sum overall possible paths connecting the initial and final points.

$$\langle f|e^{-iHt/\hbar}|i\rangle = \int \mathcal{D}\phi \, e^{iS[\phi]/\hbar} \tag{6.1}$$

where $\mathcal{D}\phi$ is the path integral measure, and $S(\phi)$ is the action. After performing a mathematical trick, derived rigorously, known as Wick rotation in euclidean time ($t \to -\iota t$) the path integral converts to Euclidean path integral $\mathcal{Z}$ given by:

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\int_0^T \mathcal{H}\, dt} \tag{6.2}$$

6.3 Vacuum in QFT

In classical physics, the vacuum is considered to be completely empty, devoid of any particles or activity. However, QFT assumes that the vacuum is instead teeming with activity due to quantum fluctuations.

Zero Point Energy

The uncertainityprinciple implies that even in the lowest energy state, there is always some inherent uncertainty and, consequently, some residual energy. This residual energy is known as zero-point energy. The quantum fluctuations are result of this zero-point energy only. These fluctuations manifest as temporary, spontaneous creation and annihilation of particle-antiparticle pairs. The transient particles are termed as virtual particles. In the absence of external particles, Feynman diagrams can include closed loops, known as vacuum bubbles, which represent the virtual particles fluctuating in and out of existence.

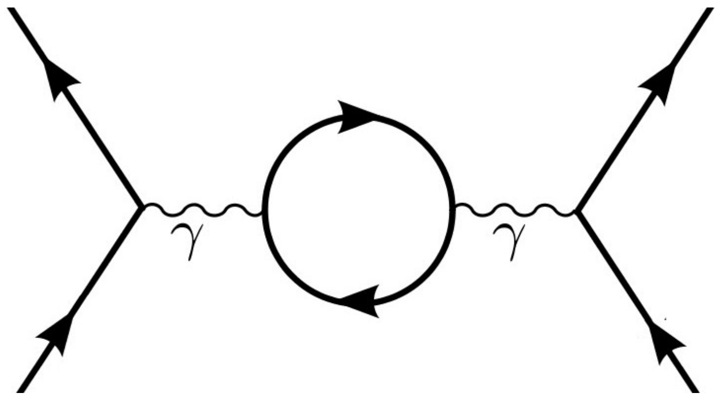


Figure 6.2: Virtual Bubble

Physical Implications include **Casimir Effect**, **Lamb Shift** and **Hawking Radiation**.

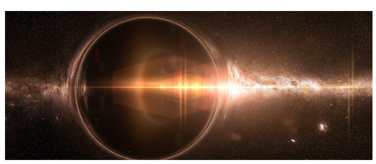


Figure 6.3: Hawking Radiation

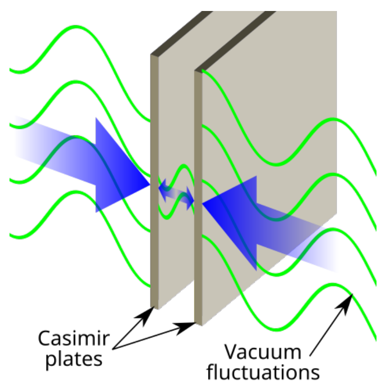


Figure 6.4: The Casimir Effect

6.4 Feynman Calculus

6.4.1 Differential Cross Section

The differential cross section, $\frac{d\sigma}{d\Omega}$, is defined as the ratio of the total cross section σ that is scattered into a given solid angle $d\Omega$[1]. Imagine a beam of particles with uniform luminosity L, then $dN = LD\sigma$ is the number of particles per unit time passing through area $d\sigma$. This **dN** is known as event rate and differential cross section $D(\theta)$ is given by :

$$\frac{d\sigma}{d\Omega} = \frac{dN}{Ld\Omega}$$

6.4.2 Fermi's Golden Rule

It formulates that a transition rate is given by the product of the phase space and the absolute square of amplitude. Without proving, I mention the mathematical result from pertubation theory:

$$W_{i \to f} = \frac{2\pi |\langle f \mid \hat{H}' | i \rangle|^2 \rho(E_f)}{\hbar} \tag{6.3}$$

where $W_{i \to f}$ is transition rate, $\hat{H}'$ is perturbing Hamiltonian and ρ is density of states.
After applying the above mathematical treatment on decay and scattering of particles, it turns out that the decay rate τ and in turn $D(\theta)$ are directly proportional to absolute square of amplitude, $|\mathcal{M}|^2$.

$$\tau \propto |\mathcal{M}|^2$$

For calculating $\mathcal{M}$, we need to use **Feynman's Rules**.

6.4.3 Feynman's Rules

These rules provide a systematic way to compute scattering amplitudes using diagrams, derived from the path integral formulation.

1. **Notation**:
   - Label the incoming and outgoing four-momenta $p_1, p_2, \ldots, p_n$.
   - Label the internal momenta $q_1, q_2, \ldots$.
   - Put an arrow beside each line to indicate the 'positive' direction (forward in time for external lines, arbitrary for internal lines).
2. **Vertex Factor**: For each vertex, write down a factor $-ig$, where g is the coupling constant that specifies the strength of the interaction between particles.

[1]Given by $d\Omega = sin(\theta) d\theta d\phi$ where θ is the polar angle and ϕ is the azimuthal angle in spherical coordinates.

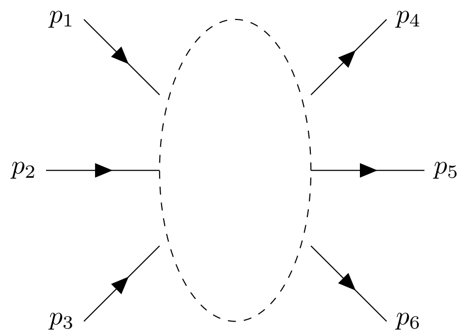


Figure 6.5: Feynman Diagram for a Toy Theory

3. **Propagators**: For each internal line, write a factor:

$$\frac{i}{q_j^2 - m_j^2 c^2}$$

where q_j is the four-momentum of the line and m_j is the mass of the particle the line describes[2].

4. **Conservation of Energy and Momentum**: For each vertex, write a delta function of the form:

$$(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$$

where the k's are the three four-momenta coming into the vertex. If the arrow leads outward, then k is minus the four-momentum of that line. This factor ensures conservation of energy and momentum at each vertex.

5. **Integration over Internal Momenta**: For each internal line, write down a factor:

$$\frac{1}{(2\pi)^4}\, d^4 q_j$$

and integrate over all internal momenta.

6. **Cancel the Delta Function**: The result will include a delta function:

$$(2\pi)^4 \delta^4(p_1 + p_2 + \cdots - p_n)$$

reflecting overall conservation of energy and momentum. Erase this factor and multiply by i. The result is $\mathcal{M}$.

[2]A virtual particle doesn't lies on the mass shell.

6.4.4 Renormalisation

While calculating higher order diagrams, one often encounters integrals that diverge to infinity. Renormalization is the process by which we absorb these infinities into redefined ("renormalized") parameters of the theory, such as masses and coupling constants. A high momentum cutoff is introduced and the mass and coupling factor change are considered:

$$m_{physical} = m + \delta m \qquad\qquad g_{physical} = g + \delta g$$

Renormalization reflects the fact that the bare parameters in the Lagrangian are not the physical, measurable quantities. Instead, the physical quantities are those that remain after the infinities are canceled.

6.5 The Klein - Gordon Equation

The Klein - Gordon equation is a relativistic wave equation, derived for particles with spin = 0 or scalars. Considering free particles -
The energy - mass relation is given as : $E^2 = p^2c^2 + m^2c^4$ or in a more elegant way

$$p^\mu p_\mu - m^2c^2 = 0$$

Replacing p_μ with its operator $\imath\hbar\partial_\mu$ [3] -

$$-\hbar^2\partial^\mu\partial_\mu\psi = m^2c^2\psi \text{ or}$$
$$-\frac{\partial^2\psi}{c^2\partial t^2} + \nabla^2\psi = (\frac{mc}{\hbar})^2\psi$$

This is the Klein - Gordon equation. This is a second order equation in time and is incompatible with Born's Statastical Interpretation.

6.6 The Dirac Equation

The Dirac equation is a relativistic wave equation, derived for particles with spin = $\frac{1}{2}$ or fermions.

$$p^\mu p_\mu - m^2c^2 = (\beta^\kappa p_\kappa + mc)(\gamma^\nu p_\nu - mc)$$

where β^κ and γ^ν are coefficients to be determined. Comparing the LHS and RHS terms -

$$\beta^\kappa = \gamma^\kappa p^\mu p_\mu = \gamma^\kappa\gamma^\nu p_\kappa p^\nu$$

[3] $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

In long form, this is

$$\begin{aligned}(p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2 = {} & (\gamma^0)^2(p^0)^2+(\gamma^1)^2(p^1)^2+(\gamma^2)^2(p^2)^2 \\ & +(\gamma^3)^2(p^3)^2+(\gamma^0\gamma^1+\gamma^1\gamma^0)p^0p^1 \\ & +(\gamma^0\gamma^2+\gamma^2\gamma^0)p^0p^2+(\gamma^0\gamma^3+\gamma^3\gamma^0)p^0p^3 \\ & +(\gamma^1\gamma^2+\gamma^2\gamma^1)p^1p^2+(\gamma^1\gamma^3+\gamma^3\gamma^1)p^1p^3 \\ & +(\gamma^2\gamma^3+\gamma^3\gamma^2)p^2p^3 \end{aligned} \tag{6.4}$$

Solving by assuming γ as a number, there is no solution whatsoever. Therefore γ's are matrices, and the anticommutator is given by -
$\gamma^\mu, \gamma^\nu = 2g^{\mu\nu}$
where $g^{\mu\nu}$ is the Minkowaski Metric. The smallest matrices which work are of size 4×4.

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where I is the 2×2 identity matrix and σ^i are the Pauli matrices. We obtain after putting values -

$$\gamma^\mu p_\mu = mc$$

Replacing the eigenvalues with respective operators, we get, drumrolls and behold

$$\boxed{\iota\hbar\partial_\mu\psi = mc\psi}$$

This is the Dirac equation. The Dirac equation is a first order equation in time and is compatible with Born's Statastical Interpretation. ψ is now a four-element column matrix, and is a spinor known as **bi-spinor** or **Dirac's spinor**.
The Dirac equation predicts the existence of **anti-particle**, and in general, yields four solutions. Two solutions correspond to matter and other two correspond to anti-matter, depending on **spin up** or **spin down**.

Chapter 7

Richard P. Feynman's Own Child

Quantum Electrodynamics (QED) is the quantum field theory that describes the interaction of charged particles with the electromagnetic field. It provides a comprehensive framework for understanding phenomena such as scattering, decay processes, and the interactions of photons with matter.

7.1 Maxwell Equations

In classical electrodynamics, Maxwell's formulation gives four fundamental equations -

$$\begin{aligned} \nabla \cdot \vec{E} &= 4\pi\rho & \qquad \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} &= 0 & \qquad \nabla \times \vec{B} &= \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}J \end{aligned}$$

where ρ is the charge density and J is the current density.
In relativity, $\vec{E}$ and $\vec{B}$ form a antisymmetric tensor, field strength tensor $F^{\mu\nu}$ given as-

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The four Maxwell's equations can be written as, in terms of $F^{\mu\nu}$ -

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

Also field strength can be written in terms of vector potential $\mathcal{A}$ as

$$F^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu$$

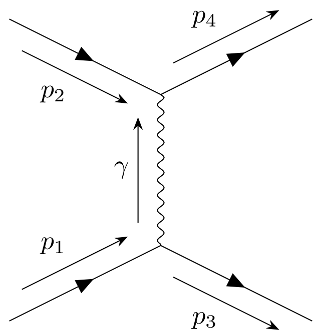


Figure 7.1: Feynman Diagram for Moller Scattering

The Maxwell Equation reduces to-

$$\partial_\mu \partial^\mu \mathcal{A}^\nu - \partial^\nu \partial_\mu \mathcal{A}^\mu = \frac{4\pi}{c} J^\nu$$

This potential $\mathcal{A}$ is not uniquely defined, i.e. if $\mathcal{A} \to \mathcal{A}' = A + \partial_\mu \Lambda$ then -

$$\partial^\mu \mathcal{A}'^\nu - \partial^\nu \mathcal{A}'^\mu = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu$$

Such a change of potentials, which has no effect on the fields, is called **gauge transformation**. This renders

$$\partial_\mu \mathcal{A}^\mu = 0$$

This is the *Lorentz Condition*.
The Maxwell equation further simplifies to

$$\partial^\mu \partial \nu \mathcal{A}^\nu = \frac{4\pi}{c} J^\nu$$

[1] For a free photon, $J^\nu = 0$, hence it satisfies $\partial^\mu \partial \nu \mathcal{A}^\nu = 0$, which is Klein-Gordon Equation for massless particle. Its solution turns out to be -

$$\boxed{\mathcal{A}^\mu = a e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \epsilon^\mu(p)}$$

where a is a constant and $\epsilon^\mu(p)$ is a four-vector.

7.2 Feynman Rules for QED

1. **Notation**:

    - Label the incoming and outgoing four-momenta $p_1, p_2, \ldots, p_n$.

[1] $\partial^\mu \partial \nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is known as d'Alembertian.

- Label the internal momenta $q_1, q_2, \ldots$.
- Put an arrow beside each line to indicate the 'positive' direction (forward in time for external lines, arbitrary for internal lines).

2. **Vertex Factor**: For each vertex, write down a factor $ig_e\gamma^\mu$, where g_e is the coupling constant that is related to the charge of electron as:

$$g_e = e\sqrt{\frac{4\pi}{\hbar c}}$$

3. **Propagators**: Each internal line contributes a factor as:

$$\text{Electrons and Positrons:} \ = \iota\frac{(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$$
$$\text{Photons:} \ = \frac{-\iota g_{\mu\nu}}{q^2}$$

4. **Conservation of Energy and Momentum**: For each vertex, write a delta function of the form:

$$(2\pi)^4\delta^4(k_1 + k_2 + k_3)$$

where the k's are the three four-momenta coming into the vertex. If the arrow leads outward, then k is minus the four-momentum of that line. This factor ensures conservation of energy and momentum at each vertex.

5. **Integration over Internal Momenta**: For each internal line, write down a factor:

$$\frac{1}{(2\pi)^4}\,d^4q_j$$

and integrate over all internal momenta.

6. **Cancel the Delta Function**: The result will include a delta function:

$$(2\pi)^4\delta^4(p_1 + p_2 + \cdots - p_n)$$

reflecting overall conservation of energy and momentum. Erase this factor and multiply by i. The result is $\mathcal{M}$.

7. **Antisymmetrization:** Include a minus sign between diagrams that differ only in the interchange of two incoming electrons or positrons.

It is important to track each fermion line backward through the diagram, to assemble the integral in correct order of multiplication.

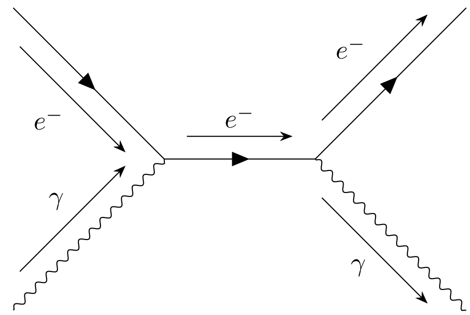


Figure 7.2: Feynman Diagram for Compton Scattering

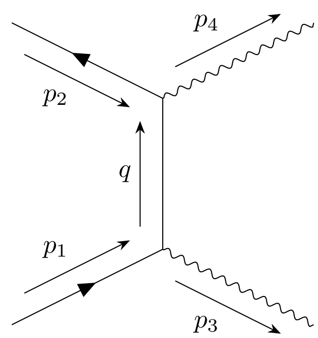


Figure 7.3: Feynman Diagram for Annihilation

7.3 The Lagrangian

The whole QED can be descirbed using a Lagrangian, which is given by:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{7.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative and ψ is the Dirac field for the particle.

Using the feynman rules over each vertex, decay rates can be calculated for a process, with help of amplitude and phase space. Example processes could be **Moller scattering, compton effect** and **pair annihilation**.

7.4 Vacuum Polarisation

Vacuum polarization refers to the process in which a photon propagates through the vacuum and interacts with virtual electron-positron pairs, the virtual bubble. This effect modifies the photon propagator and changes the coupling constant. After performing the customary renormalisation step to sweep infinities, there remains a finite correction term in the coupling constant, g, which depends on q^2. In other words, the effective charge of any particle, depends on the momentum transferred in the collision or the relative distance between interacting particles. It is like screening of the charge. These changes become quite visible in the fine structure constant and can be written as-

$$\alpha(q^2) = \alpha(0)(1 + \frac{\alpha(0)}{3\pi} f(\frac{-q^2}{m^2c^2}))$$

where f is an approximation function from renormalisation. This correction is very small and hence its effect are seldom visible in the classical world.

References

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- **Quantum Field Theory in a Nutshell** by Anthony Zee
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