Combinatorics

Reference: Miklós Bóna, *A Walk Through Combinatorics*

Pigeonhole principle, extremal principle, invariance principle (mono).
Graph Theory, counting, random walks, induction, construction.

When optimally colouring vertices, traverse by vertex labelling and when colouring edges traverse an eulerian path.
When extremising any entity, try to select entries in a suitably sorted order
Getting rid of the worst creates a monotonic variant that ensures successful termination
Greedy algorithms work as minimality has to be contradicted for them to not hold

Graphs

Graph got nodes/vertices, edges joining them
Simple graph: without loops and parallel edges
On n vertices we got $2^{\binom{n}{2}}$ simple graphs possible
Sum of degrees is twice the number of edges in a simple graph
K_n is complete graph, $K_{m,n}$ is complete bipartite graph
Graph is bipartite when it can be split into 2 disjoint sets such that every edge is from one to other,
Consequently only even cycles exist in bipartite graphs

Hypercube is an n-dimensional cube, with 2^n vertices: $(x_1, x_2, \ldots, x_n)$ where $x_i \in \{0, 1\}$. Edges are between vertices differing on exactly one coordinate, $n \cdot 2^{n-1}$ in number
Petersen Graph: $J(5, 2, 0)$ has 10 vertices and 15 edges. Each vertex is some 2-subset of $\{1, 2, 3, 4, 5\}$. Edges occur between every disjoint subset-vertices only. Visually, draw a star normally inside a pentagon and join the corresponding vertices.
Every two non-adjacent vertices have exactly one common neighbour in Petersen Graph.

Paths & Circuits

A sequence of k edges from v_a to v_b is a k-walk. Edges can repeat. k can be 0 even, when $a = b$. Trail is no-edge-repeat, **path** is no-vertex-repeat. Ofc, every path is a trail. Closed trail is circuit, closed path is **cycle**.

Existence of a walk between two vertices is equivalent to existence of a path
Every odd circuit in a multigraph contains an odd cycle
A graph if bipartite iff it doesn't contain any odd cycle
Girth is the length of the shortest cycle in a graph

A multigraph is Eulerian if it has a alternating circuit spanning all edges, called the **Eulerian circuit**. A graph is Eulerian iff it is connected with all degrees even.
(Proof follows by showing the longest trail at any vertex is closed and Eulerian)
It is possible to balance a digraph with all degrees even.
A Hamiltonian path/cycle is a path/cycle that traverses all vertices without repetition. A graph with n vertices and degree of each vertex $\geq n/2$ admits a **Hamiltonian cycle**.

(Proof follows by assuming not, getting path & showing existence of ham cycle by PHP)

A digraph admits an Eulerian circuit iff every vertex has equal indegree and outdegree.
Every tournament admits a Hamiltonian path, proof by induction.
A tournament admits a Hamiltonian cycle iff it is strongly connected, by contradiction.

Automorphisms

An isomorphism is a map on vertices that equivalently maps edges.
A graph isomorphic to its complement is called self-complementary
An n-vertex graph is self-complementary iff K_n decomposes into two of its copies
Adjacency matrix of a graph is symmetric with an all-0 diagonal

An automorphism is an isomorphism of a graph onto itself (edge preserving). An automorphism can be deeped by gauging the symmetry in the graph.
A graph is **vertex-transitive** if for every pair u, v of vertices, there is an automorphism that maps u to v. We can prove a statement about vertex-transitive by showing it for any one vertex, by symmetry.

Trees

A tree is an acylic connected graph, at the very core. Following are equivalently trees:

- Existence of a unique path between any two vertices
- Minimally connected (removing any edge disconnects it)
- Maximally acyclic (adding any edge creates a cycle)
- Connected with $n - 1$ edges
- Acyclic with $n - 1$ edges

Tree construction is easiest using rooted methods.

Linear Algebra

Real symmetric matrices have real eigenvalues and are diagonalisable
Ways to find eigenvalues of a positive semi-definite matrix:

- Solve the characteristic equation (direct $|A - \lambda I| = 0$ method)
- Find nice eigenvectors by eyeballing or logic
- Power method from NumAnal lmao

A^k_{ij} gives the walks (NOT paths) of length k from i to j.
G is connected iff each entry of $(I + A)^{n-1}$ is positive
Diameter of a graph is the longest possible *distance* between two vertices, where distance is defined as length of shortest path between 2 vertices.
For a graph with diameter d, the adjacency matrices $I, A, A^2, \ldots, A^d$ are all linearly independent.
Proof is iterative - if ij is diameter then $A^r_{ij} = 1$ for only $r = d$, implying $a_d = 0$. Then to the same with $i(j - 1)$, and so on.

Laplacian matrix of a graph = degree diagonal - adjacency matrix

Laplacian is always positive semi-definite so all its non-0 eigenvalues are positive reals
Tree number of G, $T(G)$, is defined as the number of spanning trees in G
$T(G) = \lambda_1\lambda_2 \dots \lambda_{n-1}/n$ where λ_i are eigenvalues of Laplacian ($\lambda_n = 0$)
All principal minors of Laplacian have the same determinant, due to MTT
$T(G) =$ this said determinant of principal minor of order $n-1$, by Matrix Tree Theorem
In a regular matrix eigenvalues of Laplacian follow from those of Adj since degree same
$T(K_n) = n^{n-2}$ (Cayley's Theorem), $T(K_{m,n}) = m^{n-1} \cdot n^{m-1}$

A doubly stochastic matrix is a square matrix of nonnegative real numbers where each row and each column sums to 1. This means it is both left stochastic (each column sums to 1) and right stochastic (each row sums to 1).

- Convex Polytope: The set of all doubly stochastic matrices forms a convex polytope known as the Birkhoff polytope
- Birkhoff–von Neumann Theorem: This theorem states that any doubly stochastic matrix can be represented as a convex combination of permutation matrices
- Product: The product of two doubly stochastic matrices is also doubly stochastic
- Stationary Distribution: For an irreducible aperiodic finite Markov chain, the stationary distribution is uniform if and only if its transition matrix is doubly stochastic

Perfect Matchings

Let H be a simple graph on $2m$ vertices ($m \geq 2$) and at least $m^2 + 1$ edges. Then H contains at least m triangles.
König - Egervary: size of matching equals size of cover if and only if the matching is maximum and cover is minimum. Else $\#M \leq \#J + \#K$.
Consequently, there exists a matching of size min-cover.
Hall's Marriage: X can be matched into Y iff $|A| \leq |N(A)|$ holds $\forall\ A \in X$
Usually used by assuming contradiction and considering degrees of extreme vertices
There exists a matching of size at least $|E|/\Delta(G)$ for any bipartite graph

An M-alternating path is one which alternates in edges of M. If both its ends are unsaturated by M then it's an M-augmenting path. It does NOT have to include EVERY M-edge.
Let G be any simple graph, and let M be a matching in G. Then M is maximum if and only if G has no M-augmenting paths.

Questions

Maximum matching

Let G be a bipartite graph, and let uv be an edge of G. Prove that at least one of u and v have the following property. "All maximum matchings of G contain an edge adjacent to this vertex". Note that this is a stronger requirement than just requiring that each maximum matching contain an edge adjacent to u or v.

The note is easy to prove - if there's a matching without containing both u and v then it can be extended using the edge uv, thus every maximum matching must saturate either u or v.
Now let if possible there exist maximum matchings M and N such that v isn't covered in M and u in N. Then we can consider $M \oplus N$, the set of edges in exactly one of M and N. This is clearly non-empty since $M \neq N$. The connected components of $M \oplus N$ can only be alternating paths or cycles since they're both matchings.
The alternating paths must also be even since there cannot be an augmenting path for M or N. Since u and v are in different sets and both have degree 1 in $M \oplus N$, they cannot be parts of an even alternating path or an even cycle. So they must be in different alternating paths.
However in this case, we can simply augment N by adding edge uv to the alternating path containing v. This contradicts the maximality of N, thus solving the problem.

Matrix Tree Theorem

Find the number of spanning trees of $K_{m,n}$.

We can compute the determinant by both methods: either compute the product of eigenvalues or the principal minor. To compute using the principal minor, add all rows to the first row to get a row of first m 0s and next n-1 1s. Use this row to clean up all -1s in the following m-1 rows. This gives an easy determinant along the diagonal from the right vertex which comes out to $m^{n-1}n^{m-1}$ - one m lost to minor and one n lost to row summing.

Binomial Coefficients

$\binom{n}{k}$ can be interpreted as the number of ways of choosing k out of n, or the number of k-subsets of [n], or the coefficient of x^k in $(1+x)^n$. All interpretations are ofc equivalent.

Principle of Inclusion-Exclusion: for $f, g : 2^{[n]} \to \mathbb{R}$ we have $f(S) = \sum_{T \subset S} g(T)$ if and only if $g(S) = \sum_{T \subset S} (-1)^{|S-T|} f(T)$
Alt version: $f(S) = \sum_{T \supset S} g(T) \iff g(S) = \sum_{T \supset S} (-1)^{|T-S|} f(T)$

Let M be a $|\mathcal{F}| \times |\mathcal{F}|$ matrix for any closed subset $\mathcal{F}$ of $2^{[n]}$ under inclusion, such that $M(S,T) = 1$ if $T \subset S$ and 0 otherwise. Then $M^{-1}(S,T) = (-1)^{|S-T|} M(S,T)$.

Combinatorial identities can be proven by algebraic bash, combinatorial interpretation, binomial theorem encoding, PIE or in extreme cases, sign inverting involutions.

Stirling Numbers

$(x)_k = k! \cdot {}^xC_k = (x)(x-1)(x-2)\ldots(x-k+1)$ notation
$[n\ k]$: Stirling numbers of the first kind (n cycle k)
Defined as the number of permutations in S_n having k number of disjoint cycles
Recursively defined as $[n+1\ k] = [n\ k-1] + n \cdot [n\ k]$
$\sum_0^n x^k [n\ k] = (x)(x+1)(x+2)\ldots(x+n-1)\ \ \forall n \in \mathbb{N}$

$\{n\ k\}$: Stirling numbers of the second kind (n subset k)
Defined as the number of partitions of $[n]$ having k parts
Recursively defined as $\{n+1\ k\} = \{n\ k-1\} + k \cdot \{n\ k\}$
Using PIE we also get a closed form for second stirling: maps from $[n]$ to $[k]$ are partition times $k!$, so $\{n\ k\} = 1/k! \cdot \sum_0^k (-1)^j\ {}^kC_j(k-j)^n$
$x^n = \sum_0^n \{n\ k\}(x)_k \ \ \forall n \in \mathbb{N}$

FPS OGF

A formal power series is an infinite expression of the form $a_0 + a_1x + a_2x^2 + \ldots$ and the sequence of coefficients is denoted by $\{a_n\}_0^\infty$ where $a_i \in \mathbb{C}$
The set of all formal power series is denoted by $\mathbb{C}[[x]]$, similar to $\mathbb{F}[x]$ for polynomials
For $f = a_0 + a_1x + a_2x^2 + \ldots$ its multiplicative inverse exists **iff** $a_0 = 0$ and is unique
Rational function FPS: you first guess using long division then prove using induction
Given $\{a_n\}$ its FPS is obtained by multiplying the recursion with all x^i and adding up all equations then self-substituting back f. Then split into partial fractions and expand.
$1/1-x = \sum x^n,\ -log(1-x) = \sum x^n/n,\ e^x = \sum x^n/n!$

Degree of an FPS is infinite, order is the smallest powered nonzero term
Set F of FPS is summable iff $\forall i : [x^i]f_j = 0$ for all but finitely many j
Integer partitions can be characterised by OGFs: $g = (1+x)(1+x^2)\ldots$ gives number of distinct parts of n and $f = \prod 1/(1-x^i)$ gives the number of partitions of n
The constant term of fps $g(x)$ in fog should be 0 for it to be well-defined
f has a compositional inverse if and only if $[x^0]f = 0$ and $[x^1]f \neq 0$ and it's unique

OGF of $\{a_n\}$ is its formal power series. OGF of $p(n) \cdot \{a_n\}$ is given by $p(xD) \cdot f$
If f is the OGF of $\{a_n\}$ then $f/(1-x)$ is of $\{\sum_1^n a_k\}$
Showing that a given FPS is the OGF of a sequence is usually easy in the reverse direction: show that the coefficients of the FPS satisfy the recursion satiated by the sequence entries.
Differentiating and multiplying by x is an easy way to gain an equation.
Try the first few terms to obtain a combinatorial OGF then exponentiate.

To find the OGF: proceed direct combinatorially, or algebraically as above, or by creating recursion then solving.
If we have one structure on one interval and so on then the total number is obtained by product rule. For undefined number of intervals, we have $h = 1/1 - f$
If we have the same structure on each interval then another structure over all the whole set treating intervals are elements, we have the composition rule.

EGF Structures

EGF of $\{a_n\}$ is its exponential power series: $\sum a_n x^n/n!$
EGF of $f \cdot g$ is given by $\{\sum_0^n\ {}^nC_k a_k b_{n-k}\}_0^\infty$
To solve recursions using EGF you multiply throughout by $x^i/i!$ then add. Rest same.