Complex Analysis (Conway)

Reference: John Conway, *Functions of One Complex Variable*

Cardano in 1525: *Ars Magna* x+y=10, xy=40. Satisfying algebraic properties.
Starting with nothing, you can create something - 1 is the cardinality of the set containing nothing but the empty set.
Supremum of set - unique upper bound smaller than which there is no upper bound
lub axiom - every real subset bounded above has a supremum (may not be real)

Well-ordering principle: a total order can be found on ANY set
There is a compatible order on $\mathbb{R}$ (leading to trichotomy) Complex numbers do not have an order compatible with the field operations and axioms
$\mathbb{C}$ is defined as a pair of real numbers coupled with iota (consequently ordered)
($\mathbb{C}$,+) is a commutative group and $(\mathbb{C}*, \cdot)$ is also a comm group and distributivity holds

R-linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ are in one-one correspondence with 2x2 matrices
$\mathbb{R}$-linear = a map that is linear over real numbers (operating on complex)
Complex conjugation is a R-linear map satisfying $\bar{\bar{z}} = z$ and $z + \bar{z}$ is real
Show that $z + \bar{z}$ and $z - \bar{z}$ is indeed complex conjugation by this definition
(One way is by taking a 2x2 matrix for a general R-linear map)

Modulus function: defined as the map from $\mathbb{C}$ to $\mathbb{R}^+$ such that $z \cdot \bar{z} = |z|^2$
Deducible properties - modulus function satisfies: $|z| = |\bar{z}|$; $|zw| = |z||w|$; $|z + w| \leq |z| + |w|$ (equality when positive multiples); $|a + bi| = \sqrt{a^2 + b^2}$
Unit circle and real numbers generate complex numbers and for any complex number and there exists a unique pair of unit complex and real number whose multiplication is our complex number; proof by definition

Lemma. *Any R-linear map f(z) satisfying $z + f(z) \in \mathbb{R}$ and $z - f(z) \in i\mathbb{R}$ for all z, must be the conjugate operator*

Proof by simply observing $z = 0.5(z + f(z)) + 0.5(z - f(z))$ and comparing coeffs
We define $|z|^2 = z \cdot \bar{z}$ and owing to this we directly get $|zw| = |z||w|$
Existence of nth real root of a positive real can be proven by considering sets above and below and using *LUB axiom* (standard dic proof)
Following this very elementary theorem, we directly obtain (upon squaring) Fermat's theorem of 2 squares: $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$

Does there exist a product on $\mathbb{R}^3$ for vectors which satisfies mod multiplicity? NO.
(Numbers by Ebbinghaus and others) However there is such a product in $\mathbb{R}^4$ (quarternions). But never thereafter, lol.

Triangle inequality for complex numbers can be proven by simply squaring the mod and using the expansion $|z|^2 = z \cdot \bar{z}$ (can directly use $Re(z), Im(z) \leq |z|$)
Equality occurs iff z = kw for some non-negative k

Let w be a unit complex number, if it makes angle θ with the positive real line, then it can be expressed as $z = cos\theta + i \cdot sin\theta$
In general, $z = |z| \cdot cis\theta$ coz $|z/|z|| = 1$ where $\theta \in [0, 2\pi)$
By direct multiplication and compound angle, we obtain $z_1 \cdot z_2 = |z_1 z_2| cis(\theta_1 + \theta_2)$.
This observation enables us to solve the equation $z^n = w \; \forall \; n \in \mathbb{N}$

Stereographic projection

Represent the complex plane by x plane and draw the unit sphere. Define north pole as (0,0,1) (not on the complex plane). Then any complex number joined to the north pole will intersect the sphere at precisely one point (aside from our pole) and this 1-1 mapping forms the spherical representation of complex numbers (lines go to circles).
$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$
$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$
If z=(x,y,0) and Z=(x1,x2,x3) then we want to find the relation between them.
The line joining z and N is of the form tz + (1-t)N for $t \in \mathbb{R}$ (N=0,0,1)
So a general point on the line is (tx,ty,1-t) for some real t. The point Z is then obtained by solving the equation $t^2x^2 + t^2y^2 + (1-t)^2 = 1$:

$$-t^2|z|^2 + 1 = (1-t)^2 \Rightarrow -t^2|z|^2 = t^2 - 2t$$

Thus we obtain $t = 2/(1 + |z|^2)$.
Then the point Z is given by: $(\frac{z+\bar{z}}{1+|z|^2}, \frac{-i(z-\bar{z})}{1+|z|^2}, \frac{|z|^2-1}{1+|z|^2})$
If (x1,x2,x3) = Z then t = 1-x3 and z = (x1,x2,0)/(1-x3)
We already have a notion of norm (distance) in the standard complex plane; we have a similar one on the sphere: $d[(x_1, x_2, x_3), (x_1', x_2', x_3')] = \sqrt{2 - \sum 2x_i x_i'}$

$$\Rightarrow d = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}$$

In $\mathbb{R}$ we had a unique infinity, *i.e.* $x \to \infty$. But in the complex plane, we will take limits as $z \to \infty$ by which we mean taking $Z \to N$ in S^2 where $\mathbb{C} \equiv S^2 - \{N\}$
(same as saying $|z| \to \infty$ in $\mathbb{R}$?)
Example: $f : \mathbb{C} - \{1\} \to \mathbb{C}$ $f(z) = 1/(1-z)$. Does $\lim_{z\to\infty}$ exist?
Sometimes we will allow ∞ as a value by considering, for instance, $f : \mathbb{C} \to \mathbb{C}_\infty$ then if $\lim_{Z\to N} \tilde{f} = \infty$ we will say that the limit exists. Essentially, lines and circles in $\mathbb{C}$ are mapped to circles in S^2

Metric Spaces

Metric space = non-empty set X + function $d : X \times X \to \mathbb{R}$ satisfying positive definiteness, symmetry and the triangle inequality.
In metric space (X,d) open ball $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$
Closed is defined as the complement of open and vice-versa
A metric space (X,d) is called connected if it has no proper, non-empty subset which is both open and closed (*clopen*).
Examples:

- $(\mathbb{C}, d)$ is a metric space with $d(z, w) = |z - w|$ and the set $H = \{x + iy \in \mathbb{C} : y > 0\}$ is an open subset of $\mathbb{C}$. Why? Coz take the ball $B_\epsilon(z)$ with $\epsilon = y/2$. Note that $\forall\ z \in H, B_\epsilon(z) \subset H$ and thus QED

- Take $X = \mathbb{R} \cup B_{\frac{1}{2}}(2i)$, then the subset $\mathbb{R}$ is open coz take $x \in R$ and $\epsilon = 1/2$, then the ball doesn't intersect the complement of $\mathbb{R}$ in X, *i.e.* $B_{\frac{1}{2}}(2i)$, which is good enough (note that our universe isn't the whole complex plane)

- Evidently, $\mathbb{R}$ is closed as well in (no limit points), thus it's *clopen*. Hence, its complement in X, $B_{\frac{1}{2}}(2i)$ is also closed AND open

A metric space is called **Connected** if it has no proper, non-empty subset which is both closed and open (clopen)
Manjul Bhargava: universal quadratic forms (truant)
Door space: every subset of X is either open or closed

A subset V of a metric space (X,d) is called *Compact* if every open cover of V has a finite subcover. Thus if there are open subsets $U_\alpha, \alpha \in I$ with $\bigcup_I U_\alpha \supset V$ then there is a finite subset J of I with $\bigcup_J U_\alpha \supset V$.

Heine-Borel Property: a subset V of $\mathbb{R}^k$ with the usual metric is compact if and only if is closed and bounded (and $\mathbb{C} \equiv \mathbb{R}^2$).
(x,d) metric space, $Y \subset X$. $V \subset Y$ is compact in Y iff it is compact in X.
(X,d), (Y,e) metric spaces. $f : X \to Y$ be a continuous function. Then $f(V)$ is compact or connected whenever V is so. Ofc, open sets are pulled back to open sets under f.

Functions & Sequences

Uniform continuity is the same as continuity alongside a uniform δ for a given ϵ
Proposition: if X is compact and $f : X \to \mathbb{C}$ is continuous then f is uniformly continuous
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. Let $p_k = \sum_{n=1}^{k} a_n$. Here $\{p_k\}$ is a sequence of complex numbers. If $\lim_{k\to\infty} p_k$ exists in $\mathbb{C}$ then we say that $\sum_{n=1}^{\infty} a_n$ is convergent. Otherwise we say that the sum is not convergent.
Examples:
The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent (group terms in powers of 2 to create infinite sum)
Series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent; the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges
The series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.
An absolutely convergent series is convergent
Proof: absolute convergence implies $q_k = \sum_{n=1}^{k} |a_k|$ is a convergent sequence. Thus given $\epsilon > 0, \exists\ N \in \mathbb{N}$ such that $|q_k - \sum |a_n|| < \epsilon\ \forall\ k \geq N$. Hence $\sum |a_n| < \epsilon\ \forall\ k \geq N$. We will show that $\{p_k\}$ is a Cauchy sequence.
We choose an $\epsilon > 0$. We need to show that $\exists M \in \mathbb{N}$ such that $|p_m - p_l| < \epsilon\ \forall\ m, l \geq M$. Assume that $m \geq l$, then $|p_m - p_l| = |\sum_{n=l+1}^{m} a_n| \leq \sum_{n=l+1}^{m} |a_n| \leq \sum_{n=l+1}^{\infty} |a_n|$. If $m, l \geq N_\epsilon$ then $|p_m - p_l| \leq \sum_{n=l+1}^{m} |a_n| \leq \epsilon$. Thus $\{p_k\}$ is Cauchy. By completeness of $\mathbb{C}, \{p_k\}$ is convergent and hence $\sum a_n$ is a convergent series.

(proving convergence without the use of Cauchy in complex numbers is near impossible as convergence is a strong result and to prove we must know the limit itself)

Let $G \subset \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be a function. We say that f is differentiable at $z \in G$ if

$$\lim_{w \to z} \frac{f(z) - f(w)}{z - w}$$

exists. The value is then called as the derivative of f at z and we denote it by $f'(z)$
A function is said to be analytic if it is continuously differentiable on G that is f' continuous on G

Series (infinite)

Weierstraß M-test:
A series is said to converge if the sequence of its partial sums converges (tail becomes negligible eventually)
Proof: choose an $x \in X$. We need to show that $\sum u_n(x)$ converges. The partial sums are $p_k(x) = \sum_{n=1}^{k} u_n(x)$. Then

$$|p_k(x) - p_l(x)| = |\sum_{n=l+1}^{k} u_n(x)| \leq \sum_{n=l+1}^{k} |u_n(x)| \leq \sum_{n=l+1}^{k} M_n \leq \sum_{n=l+1}^{\infty} M_n$$

Since $\sum M_n$ is convergent, for $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that $\sum_{n=l+1}^{\infty} M_n < \epsilon \; \forall \; l \geq N$
Hence, for $k, l \in \mathbb{N} : |p_k(x) - p_l(x)| < \epsilon$ and thus $\{p_k(x)\}$ is Cauchy and hence $\sum u_n(x)$ converges for each x.
Now define $f(x) = \sum u_n(x)$. We prove that $\sum u_n(x)$ converges uniformly to f(x).
Here $|f(x) - \sum_{n=1}^{k} u_n(x) = |\sum_{n=k+1}^{\infty} u_n(x)| \leq \sum_{n=k+1}^{\infty} |u_n(x)| \leq \sum_{n=k+1}^{\infty} M_n$
Now for $\epsilon > 0$ $\exists$ $L \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} M_n < \epsilon \; \forall \; k \geq L$
$\Rightarrow \forall x \in X, |f(x) - \sum_{n=1}^{k} u_n(x)| < \epsilon \; \forall \; k \geq L$

For $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that $|f(x) - \sum_{n=N+1}^{\infty} u_n(x)| < \epsilon \; \forall \; x \in X$
Proof: Since l is independent on x, $\sum u_n(x)$ converges to f(x) uniformly

Power Series

Theorem. *Consider the series* $\sum_{n=0}^{\infty} a_n(z-a)^n$ *where* $a_n \in \mathbb{C}$.

Define $R \in [0, \infty]$ by $1/R = lim\ sup\ |a_n|^{\frac{1}{n}}$ (note the closed bracket on infinity)

- If $|z - a| < R$ then $\sum a_n(z-a)^n$ converges absolutely
- If $|z - a| > R$ then $\sum a_n(z-a)^n$ diverges
- On the disc $\overline{B_r(a)}$ for $r < R$, $\sum a_n(z-a)^n$ converges uniformly
  (open or closed ball immaterial)

Note that once we prove this theorem, R's uniqueness will follow directly from the first 2 points above. We call it the *radius of convergence* of the series $\sum a_n(z-a)^n$.

Let $\{x_n\}$ be a sequence of real numbers. We define:
$lim\ inf\ x_n = \lim_k\{inf\{x_k, x_{k+1}, \dots\}\}$; $lim\ sup\ x_n = \lim_k\{sup\{x_k, x_{k+1}, \dots\}\}$
Examples: for a constant sequence both limsup and liminf are both defined and same as the constant. For the sequence $x_n = 1/n$, sup of kth tail $= 1/k$ while inf of kth tail $= 0$. Thus, liminf $=$ limsup $= 0$.
Note that for any convergent sequence limsup and liminf are both defined and are equal
When we say a function is continuous/differentiable on a set, either the set itself is open or we're considering a small open set containing the given set

Theorem. *Let $\sum a_n(z-a)^n$ be a power series. Define the ratio $\alpha = \lim |a_n/a_{n+1}|$, provided the limit exists. Then $\alpha = R$.*

Observe that this is an easier way to compute the radius of convergence but is not applicable as ubiquitously as the og definition, as the limit may not exist. Former is called root test, latter is terms as ratio test
In order to prove the theorem all we must be do is prove the satisfiability of the properties of RoC (absolute convergence inside the ball, divergence outside it). By its uniqueness, we'll obtain that α is indeed RoC itself.

Proof. We have proven in the og proof already that if $|z-a| < \alpha$ then $\sum a_n(z-a)^n$ converges absolutely. Thus $\alpha < R$. Now assume that $|z-a| > \alpha$. Choose an r such that $|z-a| > r > \alpha$. Then $\exists\ N \in \mathbb{N}$ such that $r > |a_n/a_{n+1}|\ \forall\ n \geq N$ or $|a_{n+1} > |a_n|\ \forall\ n \geq N$.

Define $B = a_N r^N$. Then $|a_{N+1}r^{N+1}| = |a_{N+1}r||r^N| > |a_N r^N| = B$. Similarly, $|a_n r^n| > B\ \forall\ n \geq N$. Further, $|a_n(z-a)^n| = |a_n r^n||\frac{z-a}{r}|^n > B|\frac{z-a}{r}|^n > B > 0\ \forall\ n \geq N$. Hence $\sum a_n(z-a)^n$ cannot converge. Thus, $\alpha \geq R$, which implies $\alpha = R$. □

Holomorphism and Analyticity

We say that f is analytic on G if f' is defined on G and is continuous on G. If f and g are analytic, all operational compositions of them are also analytic, including f(g) wherever it makes sense. This implies that all standard derivative formulae continue to hold true.

Theorem. *Let $\sum a_n(z-a)^n$ have radius of convergence $= R$. Then $\sum n \cdot a_n(z-a)^{n-1}$, or in general $\forall\ k \geq 1 : \sum_k^\infty n(n-1)(n-2)\dots(n-k)a_n(z-a)^{n-k}$ converges and has RoC $= R$.*

Proof. The general version follows directly from induction on the initial result, so we just need to prove the convergence in RoC = R of $\sum na_n(z-a)^{n-1}$. Employing the root test to this end, we ought to compute $\limsup |na_n|^{1/(n-1)}$. Observe first that $\{n^{\frac{1}{n-1}}\}$ is convergent to 1.
Further we can separate the convergent quantities from a product in a lim sup, so we get $\limsup |na_n|^{1/(n-1} = \lim n^{1/(n-1)} \cdot \limsup |a_n|^{1/(n-1)} = \limsup |a_n|^{1/(n-1)}$. Let $1/R' = \limsup |a_n|^{1/(n-1)}$. Then $\sum a_n(z-a)^n$ has RoC = R'. Now $a_n(z-a)^n = a_0 + (z-a)\sum a_n(z-a)^{n-1}$ and we know that whenever $|z-a| < R'$, the latter sum converges absolutely, which tells us that so does our original series. This gives us $R' \leq R$.

Now take $z \neq a$ (coz at equality it's anyway just a_0) and then $\sum a_n(z-a)^{n-1} = (\sum a_n(z-a)^n - a_0)/(z-a)$. For $|z-a| < R$ our latter series here converges absolutely, which implies so does the former. This gives us $R \leq R'$ due to absolute convergence, and thus we have $R' = R$. □

Let f(z) be a power series having radius of convergence $R > 0$. Then f is differentiable in $B_R(a)$ and $f'(z) = \sum na_n(z-a)^{n-1}$ on $B_R(a)$. Further f is k-times differentiable on $B_R(a)$ for $k > 1$ and $f(k)'$ can be obtained by standard calculus differentiation, still on $B_R(a)$ The power series representation of any function has coefficients determined by Taylor's polynomial wala method
Power series are computed based solely off the centre, no other point e.g. the power series $\sum z^n$ converges absolutely on $B_1(0)$ via the ratio test easily, and we also know it's equal to 1/(1-z) for z inside the ball. However the function is surprisingly differentiable over the entire complex plane bar z=1. Thus this further ensures the ball can never be larger than 1 in radius, when centered about z=0. Otherwise the radius can be arbitrarily large. In the nbhd of z=1/2 the ball would be half-unit big.

Let G be an open set and let $f : G \to \mathbb{C}$ be an analytic function. For $a \in G$ let $r > 0$ be such that $B_r(a) \subset G$. There exists a power series representation $f(z) = \sum a_n(z-a)^n$ in $B_r(a)$.

Theorem. *Let G be open and connected and let $f : G \to \mathbb{C}$ be analytic with $f'(z) = 0\ \forall z \in G$. Then f is constant.*

Remark: if G is not connected, let G1 be a connected component of G and let F2 = G - G1. Define f(z) to be 5 on G1 and 2024 on G2. Clearly f'(z) = 0 everywhere. We gonna use complex version of LMVT to prove this thing.

Proof. Let $z \in G$ and let $\alpha = f(z)$. Let $A = \{w \in G : f(w) = \alpha\}$, we want to show that $A = G$. We first show that A is closed in G, *i.e.* any limit points of any sequence in A should lie in A.
Let $\{z_n\}$ be a sequence of points in A and assume that z_n is convergent in G, so $\lim f(z_n) = f(\lim z_n)$. LHS $= \alpha$. $\lim z_n \in A$ as $f(\lim z_n) = \alpha$. Thus A is closed in G.
We now show that A is open in G. Since G is open in $\mathbb{C}$ and if $w \in A$, then $\exists r > 0$ with $B_r(w) \subset G$. We will prove that $B_r(w) \subset A$. That will prove that A is open.
If $z \in B_r(w)$ then there is a straight line segment joining w and z, given by $tw + (1-t)z, 0 < t < 1$. Define $g : (-\epsilon, 1+\epsilon) \to \mathbb{C}$ by $g(t) = f(tw + (1-t)z)$ for a suitable $\epsilon > 0$. This is a differentiable function, in fact $g'(t) = f'(tw + (1-t)z) \cdot (w-z)$. Then $g'(t) = 0\ \forall\ t \in (-\epsilon, 1+\epsilon)$ and have $g(t) = f(tw + (1-t)z)$ is constant on $(-\epsilon, 1+\epsilon)$. Therefore $f(z) = g(0) = g(1) = f(w) \Rightarrow z \in A$. Since $z \in B_r(w)$ was chosen at random, we get $B_r(w) \subset A$. Hence A is open. Since G is connected, A = G. (how? coz G is open itself so A being clopen and inside G implies A and $A^c \cap G$ are separated) □

Definition: a **region** in $\mathbb{C}$ is a non-empty, open, connected subset.
Let G be a region and $f : G \to \mathbb{C}$ be analytic. Let $z = x + iy \in G$ then $f'(z) = \lim_{w \to z}[f(z) - f(w)]/(z-w)$. We compute this limit in two directions, once on a lie parallel to the x-axis, the real axis, and once along a line parallel to the imaginary axis.

Let $w = p + iq$. Then we have $f'(z) = \lim_{p \to x}[f(x+iy) - f(p+iy)]/(x-p) = \lim_{q \to y}[f(x + iy) - f(z+iq)]/i(y-q)$. Let $u, v : G \to \mathbb{R}$ be defined by $f(w) = u(w) + i \cdot v(w)$. Equivalently, $u(w) = Re(f(w)), v(w) = Im(f(w))$.
$\lim_{p \to x}[f(x+iy) - f(p+iy)]/(x-p) = \lim_{p \to x}[u(x+iy) + i(v(x+iy)) - u(p+iy) - i(v(p+iy))]/(x-p) = du/dp(x+iy) + idv/dq(x+iy)$ Similarly, the above limit =

Cauchy-Riemann equations.
A function $u : \mathbb{R}^2 \to \mathbb{R}$ is said to be Harmonic iff it is twice continuously differentiable and satisfies the Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

If u and v are both harmonic functions then the function $f(z) = u + iv$ is analytic.

e^z is not one-one as after argument repeats on 2π, the value is the same. We define *exp* on an open strip of width under-equal 2π (open needed coz of analyticity issues). The image of S under e^z is understandably not the entire $\mathbb{C}$ plane: to start with we don't have the origin and a ray from it (the e^{iy_o} argument one)

$log(u + iv) = log(re^{i\theta}), y_o < \theta < y_o + 2\pi : log(r) + i\theta$. Compute image of $log(u + iv)$ without using the polar decomposition.
Find derivative of $f(z) = log(z)$

exp and log

Theorem. *A branch f of logarithm is analytic and $f'(z) = 1/z$*

Principal branch of logarithm
If $f : G \to \mathbb{C}$ is analytic then u and v, the real and imaginary parts of f, satisfy $u_x = v_y$ and $u_y = -v_x$ (CR equations)

Theorem. *Let $f : G \to \mathbb{C}$ be a function such that $u = Re(f)$ and $v = Im(f)$ have continuous partial derivatives wrt x and y. Assume further that u and v satisfy the CR equations, then f is analytic at each $z \in G$*

Proof. We wish to compute $\lim_{w\to 0}[f(z + w) - f(z)]/w$. Write z = x + iy, w = s + it, f = u + iv. Then, $f(z+w)-f(z) = u(x+s, y+t)+iv(x+s, y+t)-u(x, y)-iv(x, y) = [u(x+s, y+t)-u(x, y)]+i[v(x+s, y+t)-v(x, y)]$. Now $u(x+s, y+t)-u(x, y) = u(x+s, y+t)-u(x+s, y)+u(x+s, y)-u(x, y)$. By mean value theorem then, for some t_1 we have the first term equal to $t \cdot y_u(x + s, y + t_1)$, similarly second term is $s \cdot u_x(x + s_1, y)$. Same for v, we have: $v(x + s, y + t) - v(x, y) = tv_y(x + s, y + t_2) + sv_x(x + s_2, y)$.
Now consider $\phi(s, t) = tu_y(x + s, y + t_1) + su_x(x + s_1, y) - tu_y(x, y) - su_x(x, y)$ and $\psi(s, t) = tv_y(x + s_1, y + t_2) + sv_x(x + s_2, y) - tv_y(x, y) - sv_x(x, y)$.
Observe that the limit now becomes $\lim \phi(s, t)/[s + it] =$

□

Möbius Transformations

A **Möbius transformation** (or linear fractional transformation) is a function $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form

$$S(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are complex constants satisfying $ad - bc \neq 0$.
The set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the **extended complex plane**, or the Riemann Sphere. The function S is defined on this sphere as follows:

- If $c \neq 0$, then $S(-d/c) = \infty$ and $S(\infty) = a/c$.
- If $c = 0$, then $S(z) = \frac{a}{d}z + \frac{b}{d}$ is a linear map, and we define $S(\infty) = \infty$.

The condition $ad - bc \neq 0$ ensures that the transformation is not a constant function.

Group Structure. The set of all Möbius transformations forms a group under the operation of function composition. This group is often called the Möbius group.
Fundamental Decomposition. Any Möbius transformation can be written as a composition of simpler transformations:

1. **Translation:** $T(z) = z + b$.
2. **Dilation and Rotation (Homothety):** $H(z) = az$ (for $a \neq 0$). This is a dilation by $|a|$ and a rotation by $\arg(a)$.
3. **Inversion:** $J(z) = 1/z$.

Proof Sketch. If $c = 0$, the transformation is already of the form $az + b$, which is a homothety followed by a translation. If $c \neq 0$, we can use polynomial division:

$$S(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

This shows that $S(z)$ is the composition $S = T_2 \circ H_2 \circ J \circ T_1 \circ H_1$, where: $H_1(z) = cz$, $T_1(z) = z + d$, $J(z) = 1/z$, $H_2(z) = \frac{bc-ad}{c}z$, and $T_2(z) = z + \frac{a}{c}$.

The Circline Property: The most important geometric property of a Möbius transformation is that it maps circles and lines to other circles and lines. We use the term **circline** to refer to a set that is either a circle or a line.

Möbius transformations map circlines to circlines.
Based on the decomposition, we only need to check that each elementary transformation type preserves the set of circlines. Translations and homotheties clearly map lines to lines and circles to circles.
The crucial case is inversion, $z \mapsto 1/z$. The general equation for a circline in the complex plane is $A(x^2 + y^2) + Bx + Cy + D = 0$, where $z = x + iy$. In terms of z and $\bar{z}$, this becomes $Az\bar{z} + Ez + \bar{E}\bar{z} + D = 0$ for some complex E. Let $w = 1/z$. Substituting $z = 1/w$ gives

$$A\frac{1}{w\bar{w}} + E\frac{1}{w} + \bar{E}\frac{1}{\bar{w}} + D = 0$$

Multiplying by $w\bar{w}$ yields $A + E\bar{w} + \bar{E}w + Dw\bar{w} = 0$. This is another equation of a circline in the variable w. A line corresponds to the case $A = 0$, and a circle to $A \neq 0$. Inversion can map a line to a circle and vice-versa if the circline passes through the origin.

The Cross-Ratio

A Möbius transformation is uniquely determined by its action on any three distinct points in $\mathbb{C}_\infty$. This is formalized using the cross-ratio.

Cross-Ratio. For four distinct points z, z_1, z_2, z_3 in $\mathbb{C}_\infty$, their cross-ratio is defined by the transformation that maps $z_1 \to 1$, $z_2 \to 0$, and $z_3 \to \infty$. This transformation is

$$T(z) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

The value $T(z)$ is denoted (z, z_1, z_2, z_3).

Invariance of the Cross-Ratio. If S is any Möbius transformation, then the cross-ratio is invariant under S. That is, for any four distinct points z_1, z_2, z_3, z_4,

$$(z_1, z_2, z_3, z_4) = (S(z_1), S(z_2), S(z_3), S(z_4))$$

This property is the primary tool for constructing a specific Möbius transformation that sends a given triple of points to another triple.

Summary

Complex derivative can be interpreted in 3 ways:

1. Convert to stereographic projection and let $Z \to N$
2. Evaluate the limit $f[(z) - f(z_o)]/[z - z_o]$ for all sequences $\{z_n\} \to z_o$ (can be understood as $|z - z_o| \to 0$ or simply any open ball containing a z_n tail)
3. Find a complex number $f'(z_o)$ such that we have a linear map $f(z) = f(z_o)+f'(z_o)(z-z_o)+\phi(z)$ where $\phi(z)$ goes to 0 faster than z to z_o.

Limits work the same as before - to show existence and to evaluate, use the open ball or epsilon-delta definitions, and to show DNE just find 2 sequences that yield a different value of the limit.
To CHECK for differentiability, either evaluate the limits above and check for their existence or simply use the Cauchy Riemann equations.

A complex function must always be defined on an *open set* U and if it is differentiable at ALL points in U then f is said to be *holomorphic* (on U).
Holomorphic complex function on $\mathbb{C}$ is known as *entire*.
All standard derivative rules and theorems apply on holomorphic functions too and consequently polynomials, exponentials, logarithms, trigons, etc. are holomorphic.

$\mathbb{C}$ is $\mathbb{R}^2$ with a multiplication $\cdot$ and thus each complex function induces a function in $\mathbb{R}^2 \to \mathbb{R}^2$.
We already have the notion of differentiability in $\mathbb{C}$, so we can extend that to $\mathbb{R}^2$ by means of the *Cauchy-Riemann equations.*
If $f(z) = u(x, y) + i \cdot v(x, y)$ then it is complex differentiable at x_o iff at x_o:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We also need ALL partial derivatives to be continuous at x_o. This is a necessary and sufficient condition for differentiability.
If $\phi(s, t)$ is a continuous function on a real rectangle to complex plane, and g is defined as $g(t) = \int_a^b \phi(s, t)ds$ then g is continuous, and if its partial derivative $\frac{\partial g}{\partial t}$ exists and *is continuous*, then g is continuously differentiable and:

$$\frac{\partial g}{\partial t} = \int_a^b \frac{\partial \phi}{\partial t} ds$$

A function is said to be harmonic if all partial derivatives of u of order 2 exist, are continuous and $u_{xx} + u_{yy} = 0$.
If ϕ satisfies $\phi'(x) + u_y(x, y_o) = 0$ then v is a harmonic conjugate to u, implying $u_x = v_y$.

To show equality of two integrals, we invoke of the definition of the integral being a complex number which is ϵ-close to any polygonal sum with norm under δ, then use the inequalities after making polygonal sums equal
Variation of a curve is calculated as $\int |\gamma'(t)|dt$

$$\int_a^b f d\gamma = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(\tau_k) \cdot (\gamma(t_k) - \gamma(t_{k-1}))$$

Rectifiable paths are limit points of piecewise smooth curves in an appropriate topology
For all rectifiable curves we have a piecewise smooth path epsilon-close to it in terms of the integral

f is integrable by sup partition definition sure, but more so we go 2 equivalent conditions: f is integrable $\iff \forall \epsilon > 0\ \exists P$ such that $U(f, P) - L(f, P) < \epsilon \iff$ set of discontinuities of f is of measure 0.
If f is analytic in $B(a, R)$ then $f(z) = \sum a_n(z - a)^n$ for $|z - a| < R$ where $a_n = \frac{f^{(n)}(a)}{n!}$ and this series has radius of convergence $\geq R$.

Variations and Integrability

Jordan Curve theorem - a closed curve partitions the complex plane into two parts, of which one is bounded and one isn't (sounds fkin easy but is a pain to prove)
history of mathematics (UK); math overflow/stackexchange; wolfram, wikipedia.
Complex integration will essentially count the number of single-order singularities bound inside a closed complex curve

The line integral of any continuous function on the trace of a curve can be acomputed using any rectifiable path in the equivalence class
Non-bounded functions in $\mathbb{C}$ are of bounded variation and the length of a curve is one such function, which gives us the following result:

$$|\int_\gamma f| \leq \int_\gamma |f||dz| \leq V(\gamma)$$

We first prove that if $|z| = 1$ then $|f(z)| = 1$. For we have

$$|f(z)| = |\frac{z-a}{1-\bar{a}z}| = |\frac{1}{\bar{z}}\frac{z-a}{1-\bar{a}z}| = |\frac{z-a}{\bar{z}-\bar{a}}| = 1$$

Now apply maximum principle to get $|f(z)| < 1$ for $|z| < 1$.