Chapter 1 : Lie Groups and Algebra

Reference: Jacques Faraut, *Analysis on Lie Groups*

1.1 Fundamentals

1.1.1 Basic Lemmae

A topological group is a group coupled with a topological space. More precisely, it is a **group and topology** G such that the maps $(x, y) \to xy$ and $x \to x^{-1}$ are continuous.
For example, $(\mathbb{R}, +), (\mathbb{R}^*, \cdot), (GL_n(\mathbb{R}), \cdot)$

For any open $V \subset G$, $\forall g \in G : gV = \{gv \mid v \in V\}$ is open.
If H is an open subgroup of G, it is also closed.
The connected component G_0 of G containing e is a normal subgroup of G
If V is a connected open subset of G containing e then $G_0 = \bigcup_{n=1}^{\infty} V^n$

Separable polynomials are dense in $\mathbb{C}[x]$ because their complement (polynomials with repeated roots) is a closed, measure-zero set. This fails over $\mathbb{R}$ due to unavoidable complex conjugate roots.

1.1.2 Notable Subgroups

$GL(n, \mathbb{R})$ is open in $\mathbb{R}^{n^2}$; $SL(n, \mathbb{R})$ is closed but not compact
Proof follows easily using the fact that determinant is a continuous map
Orthogonal group $O(n)$ is a normal subgroup of $GL(n, \mathbb{R})$
It's closed and bounded, hence compact by Heine-Borel. Same with $U(n, \mathbb{R}), SO(n, \mathbb{R})$

Matrix B associated with bilinear form $b : b(x, y) = \langle Bx, y\rangle = y^T Bx$
We define $O(b) = \{g \in GL(n, \mathbb{R}) : b(gx, gy) = b(x, y)\ \forall x, y \in \mathbb{R}^n\}$
$SO(p, q) = I_{pq}$: usual identity for north p then minus identity for south q
$SP(n, \mathbb{R}) = ([0\ \ -I_n][I_n\ 0]$: symplectic matrix

Every invertible matrix in $GL(n, \mathbb{C})$ is similar to an upper triangular matrix
$GL(n, \mathbb{C})$ is path connected as there's a path from $A = CUC^{-1}$ to I via CDC^{-1}
$U(n)$ and $O(n)$ and their respective special subgroups are all connected
$SO(n)/SO(n-1) \cong S^{n-1}$ and thus $SO(n)$ is connected

1.1.3 Polar Decomposition

$\forall g \in GL(n, \mathbb{R})\ \exists! k \in O(n), p \in P_n : g = kp$. Orthogonal $\times$ positive semi-definite.
The map $(k, p) \to kp$ from $O(n) \times P_n$ to $GL(n, \mathbb{R})$ is a homeomorphism

For $g \in GL(n, \mathbb{R})$ we have $g^T g = hDh^{-1} : h \in O(n)$. Then $p = h\sqrt{D}h^{-1}, k = gp^{-1}$.
$\forall g \in GL(n, \mathbb{R})\ \exists k_1, k_2 \in O(n) : g = k_1 d k_2$ where d is a +ve diagonal matrix

1.2 Exponential Map

$e : \mathcal{M}_n(\mathbb{C}) \to GL(n, \mathbb{C})$ such that $e^A = I + A/1 + A^2/2! + A^3/3! + \ldots = \sum_{n\in\mathbb{N}_0} A^n/n!$
$e^{A+B} = e^A \cdot e^B \iff [A, B] = AB - BA = 0$
$\forall g \in GL(n, \mathbb{R}) : ge^X g^{-1} = e^{gXg^{-1}}$ and $det(e^X) = e^{tr(X)}$
For diagonalisable $X : X = kDk^{-1}$ then $e^X = ke^D k^{-1}$ (raising the diagonal)

For any $X \in \mathcal{M}_n(\mathbb{R}), \exists S$ diagonalisable and N nilpotent such that $X = S + N, SN = NS$
Exponential map is neither injective nor surjective over $\mathbb{R}-$matrices
Exp map forms a bijective map from symmetric matrices to PSD matrices

$exp(tX) \cdot exp(tY) = exp(t(X + Y) + \frac{1}{2}t^2[X, Y] + O(t^3))$
$exp(tX) \cdot exp(tY) \cdot exp(-tX) \cdot exp(-tY) = exp(t^2[X, Y] + O(t^3))$

$\lim_{k\to\infty}(e^{X/k}e^{Y/k})^k = e^{X+Y}$
$\lim_{k\to\infty}(e^{X/k}e^{Y/k}e^{-X/k}e^{-Y/k})^{k^2} = e^{[X,Y]}$

$B(I, 1) = \{g \in M_n(\mathbb{R}) : ||g - I|| < 1\} \subset GL_n(\mathbb{R})$
We define logarithm as $\log(g) = -\sum_{k=1}^{\infty}(-1)^k(g - I)^k/k$

1.3 Lie Groups

For topological group G, we define its 1-parameter subgroup as a continuous homomorphism $\gamma : \mathbb{R} \to G$. Thus $\gamma(\mathbb{R})$ is a subgroup of G.
1-parameter subgroup γ of $GL_n(\mathbb{R})$ is a smooth map and $\exists!\ A \in GL_n(\mathbb{R}) : \gamma(t) = e^{tA}$

Linear Lie Group G is a closed subgroup of $GL_n(\mathbb{R})$
Lie Algebra of G is defined as $Lie(G) = \mathcal{G} = \{x \in \mathcal{M}_n(\mathbb{R}) \mid e^{tX} \in G\ \forall t \in \mathbb{R}\}$
$M \subset \mathbb{R}^n$ is a submanifold of dimension k if $\forall x \in M, \exists$ a neighbourhood U of x in M and an open set V in $\mathbb{R}^k$ diffeomorphic to U

Lie Manifold Theorem: If G is a lie group and its lie algebra is $\mathcal{G}$, then G is a submanifold of $GL_n(\mathbb{R})$ of dimension $\dim \mathcal{G}$
There exist diffeomorphic neighbourhoods of O in lie algebra $\mathcal{G}$ and I in lie group G under the exponentiation map which is locally diffeomorphic

1.4 Haar Measure

Locally compact group G. Radon measure $\mu \geq 0$ is left-invariant if

$$\int_G f(gx)\mu(dx) = \int_G f(x)\mu(dx)\ \forall g \in G, f \in \mathcal{C}_C(G) \iff \mu(gE) = \mu(E)\ \forall g \in G, E \subset G$$

There exists a non-trivial left-invariant measure on G unique upto scaling: left **Haar**
$f \mapsto \int_G f(gxg^{-1})\mu(dx)$ defines a left Haar measure $\forall g \in G$ so by uniqueness upto scaling we have the existence of $\Delta(g) \in \mathbb{R}^+ : \int_G f(gxg^{-1})\mu(dx) = \Delta(g) \int_G f(x)\mu(dx)$

Modular function Δ is multiplicative hence a continuous group morphism: $G \to \mathbb{R}_+^*$
Thus we also have for any borel set $E \subset G : \mu(gE) = \Delta(g)\mu(E)$

If $\Delta(g) = 1 \ \forall g \in G$ then G **unimodular**. $\{1\}$ is the only compact subgroup of $\mathbb{R}_+^*$.
All commutative, compact and discrete groups are unimodular.
$\Delta(x^{-1})\mu(dx)$ forms a right Haar measure on G and $\int_G f(x^{-1}\mu(dx) = \int_G f(x)\Delta(x^{-1})\mu(dx)$

For open $G \subset R^m$ we have maps $L(g) : x \mapsto gx, R(g) : x \mapsto xg \ (g \in G)$
Left Haar measure $\mu(dx) = h(x)\lambda(dx)$ for Lebesgue measure λ
$h(gx) \cdot |J(L(g))| = h(x) \implies h(x) = h(e) \cdot |J(L(x))|^{-1}$

Group $ax + b$ of affine linear transformations can be identified with $g = ([a \ b][0 \ 1])$
Then $L(g)x = (au, av + b), R(g)x = (au, bu + v)$ and $\Delta(g) = 1/|a|$
For $x \in \mathcal{M}_n(\mathbb{R}) : L(g)x = (gx^1, \ldots, gx^n) \implies \det L(g) = \det R(g) = (\det g)^n$
Thus the measure $|\det x|^{-n}\Pi dx_{ij}$ is invariant and $GL_n(R)$ is unimodular

1.5 Group Representations

Topological group G normed vector space V over $\mathbb{R}$ or $\mathbb{C}$; $L(V)$ bounded operator space
Representation $\pi : G \to L(V)$ of G is a multiplicative map $g \mapsto \pi(g)$ such that the map $g \mapsto \pi(g)v$ is continuous $\forall v \in V$
Subspace $W \subset V$ is invariant if $\forall g \in G : \pi(g)W = W$. Then $\pi_0(g) = \pi(g)|_W$
π is **irreducible** if the only invariant closed subspaces are $\{0\}$ and V, *e.g.* 1D reps

For (π_1, V_1) and (π_2, V_2) if there's a continuous linear map $A : V_1 \to V_2$ such that $A\pi_1(g) = \pi_2(g)A \ \forall g \in G$ then A is an **intertwining operator**