Linear Algebra

Reference: Sheldon Axler, *Linear Algebra Done Right*

Basics

A list of n elements is an ordered $n-$tuple of finite length.
$\mathbb{F}^n := \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{F}\ \forall j \in [n]\}$ - the collection of all $n-$tuples
A **field** is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all the field axioms

A **vector space** is a set V over a field F satisfying the vector axioms
Set of all functions from set S to field $\mathbb{F}$ i.e. $\mathbb{F}^S$ is a vector space
A subset of V is a subspace iff it is also a vector space, essentially 0 and closure

$U_1 + U_2 + \ldots + U_n = \{u_1 + u_2 + \ldots + u_n \mid u_i \in U_i\ \forall i \in [n]\}$
$U_1 + U_2 + \ldots + U_n$ is the smallest subspace of V containing all the U_is
$U_1 \oplus U_2 \oplus \ldots \oplus U_n$ is a direct sum iff 0 can be uniquely written as sum of 0s
For $U, W \leq V$, $U + W$ is direct iff $U \cap W = \{0\}$

F.D. Vector Spaces

For all $a_i \in \mathbb{F}, v_i \in V$, $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$ is a linear combination of $v_1, v_2, \ldots, v_n$
The set of all linear combinations of a finite set of vectors constitutes its **span**, which is also the smallest subspace of V containing 'em all. Finite span $\implies$ FDVS
A list of vectors is linearly independent if only their trivial combination becomes null
In any linearly dependent list, there is a vector which an LC of all previous vectors
Length of any linearly independent list is always less than that of a spanning list

A **basis** is a list of linearly independent vectors spanning V - unique expression possible
Every spanning list can be reduced to and every LI list can be extended to a basis
Every subspace of V is part of a direct sum equal to V

Linear Transformations

Matrix T that is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ has dimension $m \times n$
To find the matrix of T, fill in the columns with the images of the basis vectors
To find T from its matrix, multiply matrix by the variables' column vector

A **linear map** $T : V \to W$ satisfies $T(u + v) = T(u) + T(v)$ and $T(\alpha v) = \alpha T(v)$.
Null space: $\ker T = \{v \in V : T(v) = 0\}$ (read as "kernel")
Range: $\operatorname{im} T = \{T(v) : v \in V\}$ (analogous to the range of a function)

Fundamental theorem of linear maps:

$$\dim V = \dim \ker T + \dim \operatorname{im} T$$

Two vector spaces V, W are **isomorphic** if $\exists$ bijective linear map $V \to W$.
Matrices represent linear maps relative to bases so their composition is matrix multiplication.
Invertibility of a map is equivalent to the inveribility of its matrix.

Algebra

A **quotient space** V/U is a vector space of all **cosets** of a subspace $U \subseteq V$.
A **coset** of U is the set $v + U = \{v + u : u \in U\}$.
The dimension of the quotient space is: $\dim(V/U) = \dim(V) - \dim(U)$

The **First Isomorphism Theorem** for linear maps $T : V \to W$ states that the quotient space $V/\ker(T)$ is isomorphic to the image of T, $\mathrm{im}(T)$.

$$V/\ker(T) \cong \mathrm{im}(T)$$

This theorem shows that modulo the kernel, our space is structurally identical to the range.

Eigenspaces

An **eigenvalue** of T is $\lambda \in \mathbb{F}$ s.t. $\exists v \neq 0$ with $Tv = \lambda v$. Such v is an **eigenvector**.
The set of eigenvectors for λ plus 0 forms the **eigenspace** $E_\lambda = \ker(T - \lambda I)$.

T has a basis of eigenvectors $\iff$ T is **diagonalisable**.
On $\mathbb{C}$, every operator has at least one eigenvalue $\implies$ every operator is triangularizable.

An eigenvalue λ of an operator T has two multiplicities:

- **Geometric Multiplicity**: $\dim(E_\lambda)$, the dimension of its eigenspace.
- **Arithmetic Multiplicity**: The number of times λ is a root of the characteristic polynomial.

The geometric multiplicity is always less than or equal to the arithmetic multiplicity.

Cayley-Hamilton Theorem: every square matrix satisfies its own characteristic equation. If $p(t)$ is the characteristic polynomial of a matrix A, then $p(A) = 0$.
The theorem is useful for computing the matrix inverse or powers of a matrix.

Inner Products

An **inner product** on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ that is linear in the first slot, conjugate symmetric, and positive definite.
Associated **norm** induced by the inner product: $\|v\| = \sqrt{\langle v, v \rangle}$.
Vectors u, v are **orthogonal** iff $\langle u, v \rangle = 0$.

Orthonormal basis: basis $\{e_i\}$ with $\langle e_i, e_j \rangle = \delta_{ij}$ (kronecker delta).
Every finite-dimensional inner product space admits one (Gram–Schmidt).
Projection: For subspace U, $v = u + w$ with $u \in U$, $w \perp U$.
Then u is the orthogonal projection of v onto U.

Operators

For $T : V \to V$ on an inner product space:
Adjoint T^* satisfies $\langle Tv, w\rangle = \langle v, T^*w\rangle$ and T is **self-adjoint** if $T = T^*$.
Self-adjoint $\implies$ all eigenvalues real; eigenspaces of distinct eigenvalues are orthogonal.

T is **normal** if $TT^* = T^*T$. Normal operators are diagonalisable by an orthonormal basis.
Isometries: preserve norm, i.e. $\|Tv\| = \|v\|$. Unitaries/orthogonals are examples.
Positive operators: $\langle Tv, v\rangle \geq 0$ for all v. Square roots exist.

Spectral Theorem

Spectral theorem (complex): A linear operator T on a finite-dimensional complex inner product space is normal $\iff$ there exists an orthonormal basis of eigenvectors.

Spectral theorem (real): A self-adjoint operator on a real inner product space admits an orthonormal basis of eigenvectors, all eigenvalues real.

Consequences:
- Every self-adjoint matrix is diagonalisable with real eigenvalues.
- Every normal matrix is unitarily diagonalisable.
- Decompositions: Polar decomposition, Singular Value Decomposition (SVD).