

Multivariable Calculus

Reference: Michael Spivak, *Calculus on Manifolds*

Introduction

We study calculus on \mathbb{R}^n which is a field, but more importantly we study a particular *topological space*, the one on \mathbb{R}^n .

Spectral theorem for orthogonal operators holds true on \mathbb{R}^n

We have that the standard inner product on \mathbb{R}^n - the dot product - is symmetric, bilinear and positive definite. A general inner product on \mathbb{R}^n is a map from $\mathbb{R}^n \times \mathbb{R}^n$ to \mathbb{R} that takes (x, y) to $x^t A y$ where A is an $n \times n$ matrix.

Any standard norm induces a metric (space)

Be careful about the topology you consider when defining open and closed sets in \mathbb{R}^n

Ball and rectangle definitions of open sets are equivalent

Proof that closed rectangles are closed:

Equivalent to proving its complement is open. Let the rectangle be $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$. Take any point (x_1, x_2, \dots, x_n) outside this rectangle. Outside implies $\exists i \in [n] : x_i \notin [a_i, b_i]$ 2 cases then: $x > b_i \vee x < a_i$. Whichever case it be, take the average on one end and extend by 1 the other, and you're done.

For a linear transformation T , there is a scalar M such that $|T(h)| \leq M|h| \forall h \in \mathbb{R}^n$. When viewed as mappings/functions, this property implies that linear transformations form continuous functions. Suppose f is a continuously differentiable function with $f'(a) = 0$ in an open interval; then f has a continuously differentiable inverse function in a local set around point a .

Continuous differentiability: all partial derivatives are continuous. \cdot . The chain rule is applicable

Proof of IFT excluded from midsem; statement and use is important, though

Summary

\mathbb{R}^n is a vector space over $+, \cdot$ with dimension n : $(\mathbb{R}^n, +)$ and (\mathbb{R}_*^n, \cdot) are commutative.

Usual inner product on \mathbb{R}^n is the standard dot product

Any inner product must satisfy symmetry, bilinearity and positive definiteness.

Euclidean norm on \mathbb{R}^n is standard: $|x| = \sqrt{\langle x, x \rangle}$

Norm must satisfy Cauchy-Schwarz and Triangle inequality

$\langle x, y \rangle = (|x + y|^2 - |x - y|^2)/4$ (polarisation identity)

Linear transformation T is norm preserving iff it is inner product preserving

If T is norm/inner product preserving, then it is one-one and its inverse also preserves norm

\mathbb{R}^n is a metric space so standard induced topology: a set is open iff every element has an open rectangle containing it, fully inside the set.

Closed sets are complements of open, or sets containing all limit points.

Linear transformations T are ones satisfying $T(ax + by) = aT(x) + bT(y)$

$\forall T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \exists M \in \mathbb{N} : |T(h)| < M|h| \forall h \in \mathbb{R}^n$.

This implies the continuity of T as a function from \mathbb{R}^n to \mathbb{R}^m .

$\lim_{x \rightarrow a} f(x) = a$ iff $\forall \epsilon > 0 \exists \delta > 0 : \|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| < \epsilon$

Function f is continuous on open set A iff $\lim_{x \rightarrow a} f(x) = f(a) \forall a \in A$.

Function f is differentiable at $x = a$ iff there exists a linear map D satisfying:

$$\lim_{|h| \rightarrow 0} \frac{|f(a+h) - f(a) - D(h)|}{|h|} = 0$$

For differentiable functions in $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $Df(a)$ is the Jacobian matrix with

To show differentiability, show Jacobian satisfies the above equation. To show otherwise, find 2 sequences so that the limit $\lim_{|h| \rightarrow 0} |f(a+h) - f(a)|/|h|$ doesn't exist.

Inverse Function Theorem: If f is continuously differentiable in an open set containing a and $|f'(a)| \neq 0$, then $\exists V, W$ both open ($a \in V, f(a) \in W$) such that $f : V \rightarrow W$ has a continuous inverse $f^{-1} : W \rightarrow V$ which is differentiable and $\forall y \in W : f^{-1}(y) = [f'(f^{-1}(y))]^{-1} = \frac{1}{f'(f^{-1}(y))}$.

Implicit Function Theorem: If $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuously differentiable function in an open set containing (a, b) and $f(a, b) = 0$, then let $M = (D_{n+j}f^i(a, b))_{m \times m} : 1 \leq i, j \leq m$. If $|M| \neq 0$ then \exists an open set $A \in \mathbb{R}^n$ with $a \in A$ and an open set $B \in \mathbb{R}^m$ with $b \in B$ such that: $\forall x \in A \exists$ unique $g(x) \in B$ such that $f(x, g(x)) = 0$. This function g is differentiable.

Rank Theorem: If f is continuously differentiable in an open set containing a , wherein $p \leq n$. If $f(a) = 0$ and $\text{rank}(D_j f^i(a)) = p$, then there is an open set $A \in \mathbb{R}^n$ containing a and a differentiable function $h : A \rightarrow \mathbb{R}^n$ with a differentiable inverse such that $f \circ h(x_1, x_2, \dots, x_n) = (x_{n-p+1}, \dots, x_n)$.

Prereqs: Measure Theory

Measure 0 sets must have closed rectangles spanning the set, sum of whose areas is arbitrarily small f is integrable by sup partition definition sure, but more so f 's integrability is also equivalent to: $\forall \epsilon > 0 \exists P$ such that $U(f, P) - L(f, P) < \epsilon \iff$ set of discontinuities of f is of measure 0.

The countable union of measure 0 sets is measure 0.

$\mathbb{Q} \cap [0, 1]$ is a bounded set of a measure 0.

Riemann integrability is preserved over addition and multiplication.

For any bounded function on a closed rectangle, the lower and upper integrals always exist. They simply may not be equal, and are equal iff the function is integrable.

Partitions of unity is the tool required to extrapolate the notions of differentiability and integrability to general open sets and the interfaces in \mathbb{R}^n .

In general to ensure that various open balls in \mathbb{R}^n act well with each other to permit all operations, we have "patching conditions" to enforce that.

Riemann Integration

Partition of a rectangle $[a_1, b_1] \times \dots \times [a_n, b_n]$ is (P_1, \dots, P_n) where each P_i is a partition of $[a_i, b_i]$. A partition yields subrectangles over which we have $m = \text{inf}$ and $M = \text{sup}$ defined. Volume of

rectangle = $\prod_{i=1}^n (b_i - a_i)$.

$L(f, P) = \sum m_s(f) \cdot v(s)$; $U(f, P) = \sum M_s(f) \cdot v(s)$; $L(f, P) \leq U(f, P) \forall P$

Any refinement raises the lower sum while also lowering the upper sum.

A bounded function f is integrable iff $\forall \epsilon > 0 \exists P : U(f, P) - L(f, P) < \epsilon$

Consequently, if there exists even one partition P with $U(f, P) = L(f, P)$ then the function is riemann integrable.

Measure 0: covered by arbitrarily many rectangles with negligible total volume

Content 0: covered by finitely many rectangles with negligible total volume

If A is a closed (hence compact) rectangle and f is a **bounded** real-valued function on it, then it is continuous if and only if its set of discontinuities is of measure 0.

The characteristic function of a set is integrable iff its boundary has measure (and thus content) 0. Such a set is known as Jordan measurable.

If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are closed rectangles and $f : A \times B \rightarrow \mathbb{R}$ is an integrable function then $\mathcal{L} = L \int_B f(x, y) dy$ and $\mathcal{U} = U \int_B f(x, y) dy$ are both integrable on A and

$$\int_{A \times B} f = \int_A \left(\int_B f(x, y) dy \right) dx = \int_B \left(\int_A f(x, y) dx \right) dy$$

Fubini's theorem allows us to pick whichever order is more convenient for us in evaluating a mixed (double) integral.

Change of variables theorem: Let $g : A \rightarrow B$ be a diffeomorphism (bijective bicontinuousⁿ) of open sets in \mathbb{R}^n . Let $B \rightarrow \mathbb{R}$ to be a continuous function. Then f is integrable over B iff the function $f \circ g | \det(Dg)|$ is integrable over A and thence

$$\int_B f = \int_A (f \circ g) | \det(Dg) |$$

Multilinear Algebra

$T : V^k \rightarrow \mathbb{R}$ is multilinear/**k-tensor** on V if it is linear in each variable.

$\mathcal{T}^k(V) :=$ set of all k-tensors on V , which is a vector space.

Inner product is a symmetric 2-tensor on V .

Derivative is a linear transformation from the tangent space at p to that at $f(p)$.

A one-form is a linear function $w : T_p \mathbb{R}^n \rightarrow \mathbb{R}$

$w(\langle dx_1, dx_2, \dots, dx_n \rangle) = a_1 dx_1 + a_2 dx_2 + \dots + a_n dx_n$

$w_1 \wedge w_2(v_1, v_2) = \det([w_1(v_1) \ w_1(v_2)]^T [w_2(v_1) \ w_2(v_2)]^T)$

\wedge is anti-commutative and distributes over $+$

An m-form is a multilinear alternating function $w : (T_p \mathbb{R}^n)^m \rightarrow \mathbb{R}$

The space of m-forms has a basis given by $\{dx_{i_1} \wedge \dots \wedge dx_{i_m} | 1 \leq i_1 < i_2 < \dots < i_m \leq n\}$ where $dx_1 \wedge \dots \wedge dx_n = \det(v_i^{(j)})$. Evidently its dimension is $\binom{n}{m}$.

A function w with $w(p) \in \Lambda^k(\mathbb{R}^n_p)$ is called a k-form on \mathbb{R}^n or a differential form.

If $w \in \Lambda^k(V)$, $n \in \Lambda^l(V)$ then $w \wedge n = \frac{(k+l)!}{k!l!} \text{Alt}(w \otimes n) \in \Lambda^{k+l}(V)$

Poincare's Lemma: if $A \in \mathbb{R}^n$ is an open set which is star-shaped around 0, then every closed form is exact. By default, every exact form is anyway closed.

$w = I(dw) + d(Iw)$ where I is the single function omitted operator

A singular n -cube is a continuous curve from $[0, 1]^n$ to \mathbb{R} . Its boundary $\partial I^n = \sum_{i=1}^n \sum_{\alpha=0,1} (-1)^{i+\alpha} I_{(i,\alpha)}^n$

Stokes' Theorem: If w is a $(k-1)$ form on an open set $A \in \mathbb{R}^n$ and C is a k -chain in A then $\int_C dw = \int_{\partial C} w$.