Chapter 1 : Measure Theory

Reference: Gerald Folland, *Real Analysis*

1.1 Quiz 1

1.1.1 Prelims

Issues with Riemann integration:

- $f_n(x) = 1$ iff $x \in \{r_1, \ldots, r_n\}$ where $r_i \in \mathbb{Q}$ is riemann integrable for all $n \in \mathbb{N}$ but not for $\lim\ n \to \infty$
- $\lim_{n\to\infty} \int_a^b f_n(x)dx \neq \int_a^b \lim_{n\to\infty} f_n(x)dx$ for $f_n(x) = nxe^{-nx^2}$ in $[0, 1]$

There doesn't exist a (measure) function satisfying finiteness, translation invariance and countable additivity. So our measures satisfy **non-negativity** and **disjoint countable additivity**.
Counter-example found in **Vitali set function**: on $\mathbb{R}$ define $x \sim y$ iff $x - y \in \mathbb{Q}$. Then the set of equivalence classes is uncountable. Then there exists a choice function taking each class to one of its elements. We define vitali set as $V := \{\{f(a)\} \in (0, 1]|a \in EC\}$. Then $\mu(V)$ can never exist as $\sum \mu(V) \leq 3 \implies \mu(V) = 0$ but $1 \leq \sum \mu(V) = 0$.

1.1.2 Algebras

For Ω non-empty, we have the following notions for subsets of $P(\Omega)$: $(\phi, \Omega \in S)$

- S semi-algebra if closed under intersection and complement is disjoint union
- S algebra if closed under finite intersection and complementation
- S sigma-algebra is closed under countable intersection and complementation

Finite unions of intervals forms an algebra but not a sigma-algebra
Intersection of semi-algebras need not be semi-algebra but arbitrary intersection of (sigma) algebras is a (sigma) algebra
Every sigma algebra is an algebra, so we have $\mathcal{A}(S) \subset \mathcal{F}(S)$

Let Ω be a metric space. Then the minimal sigma-algebra of all open sets of Ω is known as its **Borel sigma-algebra**.
An **additive set function** over a collection of subsets of Ω satisfies $\mu(\phi) = 0$ and finite disjoint additivity. Equivalently, sigma additivity.
A (sigma) additive set function over a semi-algebra S can be uniquely extended to a (sigma) additive set function over algebra $\mathcal{A}(S)$.

If $S \subset P(\Omega)$ is a semi-algebra, then $A \in \mathcal{A}(S)$ iff A is a finite disjoint union of elements of the semi-algebra.
For $E_i \in S$ we have if G is the set of all their finite disjoint unions, then it is $= \mathcal{A}(S)$.

1.1.3 Continuity

For $\{E_i\}_{n=1}^{\infty} \subset P(\Omega) : E_n \uparrow E$ if $E_n \subset E_{n+1}$ and $E = \bigcup E_n$ (similarly down)
Continuity from below: $E_n \uparrow E_n \implies \mu(E_n) \to \mu(E)$
μ is continuous if it's continuous from both sides (so $\mu(\Omega) < \infty$)
Length function $(a, b]$ is σ-additive

If μ is σ-additive then it's continuous at all $E \in \mathcal{A}$
If μ is continuous from below at all $E \in \mathcal{A}$ then it's σ-additive
If μ is continuous from above at ϕ and $\mu(\Omega) < \infty$ then μ is σ-additive

1.1.4 Measurability

$\mathcal{F}$ is a σ-algebra on Ω with measure μ: $(\Omega, \mathcal{F}, \mu)$ is a **measure space**
Almost everywhere $\iff$ true everywhere except on E where $\mu(E) = 0$
Completion of $(\Omega, \mathcal{F}, \mu)$ is $(\Omega, \mathcal{F}', \mu')$ where $\mathcal{F}'$ is complete, *i.e.* it contains all subsets of measure zero sets, and μ' restricted to $\mathcal{F}$ is μ
Measure is sub-additive: $\mu(\bigcup A_i) \leq \sum \mu(A_i)\ \forall \{A_i\} \subset \mathcal{F}$ (countable sum)

The completion of $(\Omega, \mathcal{F}, \mu)$ is given by $(\Omega, \mathcal{F}_\mu, \bar{\mu})$, where $\bar{\mu}$ is an extension of μ and $\mathcal{F}_\mu = \{A \cup N \mid A, H \in \mathcal{F}; N \subset H : \mu(H) = 0\}$. $\bar{\mu}(A \cup N) = \mu(A)$.
$\mathcal{F}_\mu$ is a σ-algebra and $\bar{\mu}$ is unique. Furthermore $\mathcal{F}_\mu$ is the **minimal completion**

1.1.5 Monotonicity & Extensions

$M \subset P(\Omega)$ is a monotone class if $\{A_i\} \in M$ and $A_i \uparrow A$ or $A_i \downarrow A$ imply $A \in M$
Every σ-algebra is a monotone class and if a monotone class is an algebra then it is σ
Arbitrary intersection of monotone classes is a monotone class

For an algebra $\mathcal{A} \in P(\Omega)$, the monotone class $M(\mathcal{A}) = \mathcal{F}(\mathcal{A})$, coz $M(\mathcal{A})$ is algebra
Caratheodary Theorem: Algebra $\mathcal{A} \in P(\Omega)$. Let $v : \mathcal{A} \to [0, \infty]$ be σ-finite measure. Then there is a **unique extension** measure $\pi : \mathcal{F}(A) \to [0, \infty]$ such that $\pi|_{\mathcal{A}} = v$.

1.2 Quiz 2

1.2.1 Outer Measurability

$\phi \in C \subset P(\Omega)$. Function $\mu' : C \to [0, \infty)$ is an outer measure if $\mu'(\phi) = 0$ and it is **countably subadditive**: $\mu'(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu'(E_i)\ \forall \{E_i\} \in P(C)$.
Let μ be a measure on algebra $\mathcal{A}$ then we have an outer measure π:

$$\pi(A) = \inf \left(\sum_{i=1}^{\infty} \mu(E_i) \mid A \subset \bigcup_{i=1}^{\infty} E_i : E_i \in \mathcal{A}\ \forall i \in \mathbb{N} \right)$$

$A \subset \Omega$ is **π-measurable** if $\forall E \in \Omega : \pi(E) = \pi(E \cap A) + \pi(E \cap A^c)$
Set of all such A forms the measurable space $\mathcal{M}$

1.2.2 Caratheodary Proof

Having defined $\mathcal{A}, \mathcal{M}$ and π as above, we have:

- $\mathcal{M}$ **is a σ-algebra**: complement ofc; countable union by algebra and disjoint F_i
- The outer measure π satisfies disjoint finite additivity
- $\pi|_{\mathcal{M}}$ **is a measure**: $\geq$ holds through finite disjoint additivity followed by limit
- $\mathcal{A} \subset \mathcal{M}$: $\leq$ by countable subaddivity, $\geq$ by $\sum \mu(E_i) < \pi(E) + \epsilon$ coz inf
- $\pi|_{\mathcal{A}} = \mu$: use definition of inf and additivity

The uniqueness of Caratheodary's extension follows by taking $E_n \uparrow \Omega$, considering $B_n = \{E \in \mathcal{F}(\mathcal{A}) \mid \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\}$ and showing $B_n \supset \mathcal{A}$ is σ-algebra. This follows from Monotone Class Theorem on B_n.

$\mathcal{M}$ is the completion of $\mathcal{F}(\mathcal{A})$ with respect to measure $\pi|_{\mathcal{M}}$. This follows from $\mathcal{M}$ being complete and $\forall A \in \mathcal{M}\ \exists\ B, H \in \mathcal{F}(\mathcal{A}) : A = B \cup N, N \subset H, \mu(H) = 0$.
Regularity: for any $A \in \mathcal{M}$ there exists $B \in \mathcal{F}(\mathcal{A})$ such that $A \subset B$ and $\pi(A) = \pi(B)$.

1.2.3 Lebesgue Measure

For interval $(a, b] \subset \mathbb{R}$, let $S = \{(a, b] | -\infty \leq a \leq b \leq \infty\}$ be the set of all intervals
Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing and right-continuous function.
Extend it to $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ by letting $F(\infty) = \sup_{\mathbb{R}} F(x)$ and $F(-\infty) = \inf_{\mathbb{R}} F(x)$.
Define $\lambda_F : S \to [0, \infty]$ by $\lambda_F(a, b] = F(b) - F(a)$.

λ_F is a measure on semi-algebra S and is unique upto translation
Lebesgue measure is the unique Caratheodary extension of λ on $\mathbb{R}$ and the collection $\mathcal{L}$ of all λ^*-measurable sets is the **Lebesgue sigma-algebra**

1.2.4 Limits & Extremes

$\liminf A_n$ = elements that are in A_n for all n above some N (i.e., eventually always present, so like `eventually always`).
$\limsup A_n$ = elements that appear in infinitely many A_n (but not necessarily all after some point, so like `always eventually`).
Naturally $\liminf A_n \subset \limsup A_n$

1.2.5 Measurability

A set is measurable if it is in $\mathcal{F}$ (*i.e.* it is borel) or it satisfies outer measure condition.
A function is borel/measurable if it pulls back borel sets to borel/measurable sets.

Alternately, a set is a measurable if it is the pre-image of a measurable set under a measurable function.
Vitali's theorem: a subset of $\mathbb{R}$ is a Lebesgue null set if and only if all its subsets are Lebesgue measurable

1.3 Quiz 3

1.3.1 Integration

A function f is **simple function** if $f = \sum_1^n c_i \chi_{E_i}$ where $c_i \in \mathbb{R}$ and $E_i \in \mathcal{F}$ are disjoint
Every simple function is measurable and integrable
$f = \sum_1^n c_i \chi_{E_i} \implies \int_\Omega f d\mu = \sum_1^n c_i \mu(E_i)$
The integral is well-defined, which can be proven by using disjointness of support
For measurable f_n, their lim sup, lim inf and pointwise limit are also measurable

For any measurable f, there is a sequence $\{f_n\}$ of non-negative simple functions converging pointwise to f with $f_{n+1}(w) \geq f_n(w)\ \forall w \in \Omega$ for each $n \in \mathbb{N}$
The integral of f is defined as $\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu$ for the above defined $\{f_n\}$
For measurable $f : \Omega \to \mathbb{R}^+$ we have $\int_\Omega f d\mu = \sup\{\int s d\mu \mid 0 \leq s \leq f\}$ for simple s

1.3.2 Functions

Measurable function f is **Lebesgue integrable** if $\int_\Omega f^+ d\mu \cdot \int_\Omega f^- d\mu < \infty$
Monotone Convergence Theorem: for a sequence $\{f_n\}$ of non-negative measurable functions such that $f_n(w) \uparrow f(w)\ \forall w \in \Omega$, we have $\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$
Proof proceeds by defining simple functions going up to each f_n then showing equality

Fatou's Lemma: for a sequence $\{f_n\}$ of non-negative measurable functions we have

$$\int_\Omega \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu \text{ and } \limsup_{n\to\infty} \int_\Omega f_n d\mu \leq \int_\Omega \limsup_{n\to\infty} f_n d\mu$$

1.3.3 Integration over Sets

For $A \in \mathcal{F} : \int_A f d\mu = \int_\Omega f \cdot \chi_A d\mu$. Integrability of f over A defined accordingly.
$\mu(A) = 0 \implies \int_A f d\mu = 0\ \forall f$ measurable. Proof by simple functions then limit.
If $f = g$ almost everywhere then f is measurable iff g is measurable.

If $f : \Omega \to \overline{\mathbb{R}}$ is integrable then it is finite almost everywhere: $\mu(\{w \mid |f(w)| = \infty\}) = 0$
The set of integrable functions forms a vector space over $\mathbb{R}^n$

A complex function is measurable/integrable if its real and imaginary parts are
For complex functions f we have $\int_\Omega f d\mu = \int_\Omega \text{Re}(f) d\mu + i \cdot \int_\Omega \text{Im}(f) d\mu$
If $f : \Omega \to \mathbb{C}$ is integrable then $|\int_\Omega f d\mu| \leq \int_\Omega |f| d\mu$

1.3.4 Product Measures

Dominated Convergence Theorem: for real measurable functions f_n such that $\lim_{n\to\infty} f_n(w) = f(w)$ *a.e.*, if there is an integrable function $g : \Omega \to \mathbb{R}^+$ such that $|f_n(w)| \leq g(w)\ \forall n \in \mathbb{N}$ *a.e.* then f is integrable and $\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0$ and $\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$

For σ-algebras $\mathcal{F}_1, \mathcal{F}_2$ we have $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ as the σ-algebra generated by $\mathcal{F}_1 \times \mathcal{F}_2$

For $E \subset \Omega_1 \times \Omega_2, x \in \Omega_1 : E_x = \{y \in \Omega_2 \mid (x, y) \in E\}$. Similarly we define section E^y
The x and y projections of a borel set in product measure are respectively borel in $\mathcal{F}_i$

Function $\mu : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathbb{R}^+$, $\mu(E \times F) = \mu_1(E)\mu_2(F)$ is σ-additive and σ-finite
The Caratheodary extension of this measure over $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the product measure

The projections of a measurable function are measurable in respective σ-algebras
For $E \in \mathcal{F}$ the function $x \mapsto \mu_2(E_x)$ is $\mathcal{F}_1$-measurable. Similarly $y \mapsto \mu_1(E^y)$ in $\mathcal{F}_1$
$\forall E \in \mathcal{F} : \int \mu_2(E_x) d\mu_1(x) = \mu(E) = \int \mu_1(E^y) d\mu_2(x)$

1.3.5 Iterated Integrals

Tonelli's theorem: for $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, if $f : \Omega \to [0, \infty)$ is $\mathcal{F}$-measurable then

$$\int_\Omega f d\mu = \int_{\Omega_1} \left(\int_{\Omega_2} f_x(y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_2} \left(\int_{\Omega_1} f^y(x) d\mu_1(x) \right) d\mu_2(y)$$

Fubini's theorem: same statement as above for integrable f but over all reals
Usual line of reasoning is to determine integrability of function by applying Tonelli on $|f|$ then use Fubini on f to compute integral
Polynomial functions such as $x - y, x \mapsto (x, x)$ are continuous hence measurable. They come handy in checking measurability of certain sets through their pre-images.

1.4 Quiz 4

1.4.1 Definitions

A **Banach space** is a complete normed vector space. L^p spaces are Banach spaces.
For $p \in (0, \infty]$ the dual index to p is $q \in [1, \infty] : 1/p + 1/q = 1$.

For $a, b \in \mathbb{C}, p \geq 1 : |a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ by $\triangle$ and Jensen's
Young's inequality: for $a, b \in \mathbb{R}^+, p \geq 1 : ab \leq a^p/p + b^q/q$

1.4.2 Lp spaces

Measure space $(\Omega, \mathcal{F}, \mu)$ and measurable $f : \Omega \to \mathbb{C}$ then for $p \in \mathbb{R}^+$:

$$||f||_p = \left(\int_\Omega |f|^p d\mu \right)^{\frac{1}{p}}$$

Evidently $||f||_p = 0 \implies f = 0$ almost everywhere on Ω
$L^p(\Omega, \mathcal{F}, \mu) = \{f : \Omega \to \mathbb{C} \mid f \text{ measurable}, ||f||_p < \infty\}$
$l^p(\mathbb{C})$ is the vector space of p-summable sequences $i.e. \{x_i\} : \sum |x_i|^p < \infty$

L^p is vector space over $\mathbb{C}$ for all p but Hilbert only for $p = 2$ due to angle definition
Hölder's inequality: $\int_\Omega |f(w)g(w)| d\mu(w) \leq ||f||_p ||g||_q$ for dual indices p, q
Minkowski's inequality: for $f, g \in L^p, p \geq 1$ we have $||f + g||_p \leq ||f||_p + ||g||_p$

1.4.3 Metric Lp

$||\cdot||$ is a norm on $L^p(\Omega)$: definite, positive, homogeneous and $\triangle$ by Minkowski
L^p is complete: every Cauchy sequence in L^p is convergent
If a subsequence of Cauchy sequence $\{x_n\}$ converges to x_0 then $\lim_{n\to\infty} x_n \to x_0$
Completeness is shown by assuming Cauchy and constructing $g(w) = \sum_{n=1}^{\infty} |f_{n+1}(w) - f_n(w)| \in L^p$, then showing $\lim_{n\to\infty} f_n(w) = f \in L^p$ and $\lim_{n\to\infty} ||f - f_n||_p = 0$

Measurable f is essentially bounded if $\exists M > 0$ such that $|f| \leq M$ almost everywhere
$L^\infty(\Omega, \mathcal{F}, \mu) = \{f : \Omega \to \mathbb{C} \mid f \text{ is essentially bounded}\}$
For $f \in L^\infty(\Omega) : ||f||_\infty = \inf\{M \in \mathbb{R}^+ \mid \mu\{w \in \Omega \mid |f(w)| > M\} = 0\}$

1.4.4 Function spaces

For $f : \mathbb{R}^n \to \mathbb{C}$, support of f is defined as $supp(f) = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$
$\mathcal{C}_C(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is continuous with compact support}\}$
$\mathcal{C}_C(\mathbb{R}^n)$ is dense in $L^\infty(\mathbb{R}^n)$

Measure μ is **regular** if $\forall A \in \mathcal{F}, \epsilon \in \mathbb{R}^+, \exists$ open G, closed $E : E \subset A \subset G, \mu(G - E) < \epsilon$
Measure of E is also the supremum of measures taken over all closed or compact sets contained in E, and infimum over all open sets containing E
$\mu = \sum_1^\infty \delta_{1/n}$ is σ-finite but not regular

For any compact set contained in open, there's a continuous function with compact support contained in the open set and which is 1 over the compact set
Measures ν is absolutely continuous wrt μ if $\forall E \in \mathcal{F} : \mu(E) = 0 \implies \nu(E) = 0$