Ordinary Differential Equations

Reference: Notes by Prof. Saikat Mazumdar, *Ordinary Differential Equations*

This course is about finding explicit solutions to ODEs in the beginning, but more importantly we are incapable of doing that for a vast array of DEs. So we wish to convert stuff to first order ODE which can be solved, or at least approximate them to first order. Lastly if we're unable to do anything concrete at all what we can always do is qualitiative analysis of the behaviour of ODEs, such as aymptotes, bounds, etc.
Speaking from a pure mathematics pov, we'll look into existence and uniqueness theorems - will a DE always have a solution, and will it be unique. More specifically, we'll study Fixed Point Theorems. These will be used later in analysis. Moving further we'll also look at short time analysis of ODE solutions because often we can grapple with 'em on restricted domains. Post midsem we'll mostly cover discrete assorted topics independently, such as:

- Systems of (linear) ODEs
- Matrix Exponential
- Asymptotic behaviour of solutions
- Equilibrium points & Stability
- Boundary Value Problem
- Sturm Liouville Problem

First Order notion: $\gamma'(t) = f(t, \gamma(t)); \gamma(0) = \gamma_0$
Uniqueness of second order ODE solution is equivalent to that of first order but with vectors (so linear equations effectively).
Picard-Lindelöf gives that if $y = f(x)$ has $y' = 0$ at $x = x_0$ then x must be constant in a closed neighbourhood around x_0

Ways of solving basic ODEs:
- Substitution
- Linear first-order
- Homogeneous $du/dt = f(u/t)$
- Linear/linear $\to$ homg
- Exact ODE (IF)

Picard-Lindelöf:
Let D be an open rectangle in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be a function $f(t, y)$ continuous in t and Lipschitz continuous in y (slope bounded) at t, then $\exists \epsilon \in \mathbb{R}^+$ such that the IVP $y'(t) = f(t, y(t))$ has a unique solution in $[t - \epsilon, t + \epsilon]$

Linear constant coefficient ODEs $\mathcal{L}u = 0$ have unique solutions $u \in \mathcal{C}^k$
If $p(\mathcal{D})$ has a root α with multiplicity m, then the functions $\{e^{\alpha t}, t \cdot e^{\alpha t}, \ldots, t^{m-1}e^{\alpha t}\}$ are m linearly independent solutions of $\mathcal{L}u = 0$
For the case of complex root $\alpha = \text{Re}(\alpha) + i\,\text{Im}(\alpha)$ with m, we get solutions $\{e^{\text{Re}(\alpha)t}\sin(\text{Im}(\alpha)t)$, $e^{\text{Re}(\alpha)t}\cos(\text{Im}(\alpha)t), \ldots, t^{m-1}e^{\text{Re}(\alpha)t}\sin(\text{Im}(\alpha)t), t^{m-1}e^{\text{Re}(\alpha)t}\cos(\text{Im}(\alpha)t)\}$
This is coz complex roots occur in conjugate pairs since our LDE is real

Variation of Parameters

If we got linear non-homogeneous ODE $\mathcal{L}u = b$ where $\mathcal{L} = D^k + +a_1 D^{k-1} + \ldots + a_k I$, then we can get rid of b by applying any two solutions and taking their difference - if ψ_p is the particular solution of $\mathcal{L}u = b$ then $\phi - \psi_p$ is a solution to homogeneous $\mathcal{L}u = 0$.
By above discussion, $\mathcal{L}u = 0$ has k linearly independent solutions $\psi_1, \psi_2, \ldots, \psi_k$. Thus a solution of $\mathcal{L}u = b$ looks like $\phi = \psi_p + c_1\psi_1 + c_2\psi_2 + \ldots + c_k\psi_k$.

To obtain a particular solution we allow c_i to become $u_i(t)$ - real functions of t instead of constants.
We impose orthogonality conditions on $u_i'(t)$ to obtain a particular solution to $\mathcal{L}u = b$:
$\psi_p = u_1\psi_1 + u_2\psi_2 + \ldots + u_k\psi_k \implies \psi_p' = (u_1'\psi_1 + \ldots + u_k'\psi_k) + (u_1\psi_1' + \ldots + u_k\psi_k')$
We assume $u_1'\psi_1 + \ldots + u_k'\psi_k = 0$ and proceed:
$\psi_p'' = (u_1'\psi_1' + \ldots + u_k'\psi_k') + (u_1\psi_1'' + \ldots + u_k\psi_k'')$
We again assume $u_1'\psi_1' + \ldots + u_k'\psi_k' = 0$.

Continuing this process, if $u_1', u_2', \ldots, u_k'$ satisfy the equations $u_1'\psi_1^{(n)} + \ldots + u_k'\psi_k^{(n)} = 0$ for all $0 \le n \le k-2$ and $u_1'\psi_1^{(k-1)} + \ldots + u_k'\psi_k^{(k-1)} = b$ then we get that:
$\mathcal{L}(\psi_p) = u_1\mathcal{L}(\psi_1) + \ldots + u_k\mathcal{L}(\psi_k) + b = b$ and we thus have our particular solution ψ_p
The said family of equations has a unique solution since the Wronskian is non-zero as ψ_i are all linearly independent

Planar Equations

We wish to solve the linear system $X' = AX$ where $X = (X_1(t)\ X_2(t))^T$.
First $X' = (0\ 0)^T$ only has the solution $(0\ 0)^T$ for invertible A, and more for singular A.
Thereafter if λ is an eigenvalue of A with eigenvector v then $X(t) = e^{\lambda t}v$ is a solution.
The general solution of $X' = AX$ is $X(t) = c_1 e^{\lambda_1 t}v_1 + \ldots + c_n e^{\lambda_n t}v_n$ for reals c_i.

When A has distinct eigenvalues its eigenvectors are linearly independent so we get an $\mathbb{R}^2$ spanner.
It gets particularly interesting when their signs are opposite:
$X(t) = \alpha e^{\lambda_1 t}(1\ 0) + \beta e^{\lambda_2 t}(0\ 1)$ lies across $\mathbb{R}^2$ - first solution becomes 0 while other explodes for large t. This gives rise to a saddle shape on the real planar function.

Phase portraits for the following cases in 2×2 matrix A's eigenvalues:
One positive one negative: **saddle** shape
Both positive: **source**; both negative: **sink**

Grönwall's Inequality: if real ψ satisfies $\psi(t) < A + \int_0^t (B\psi(s) + c)ds\ \forall t \in [0, T]$, where $B > 0$, then $\psi(t) \le Ae^{Bt} + c/B \cdot (e^{Bt} - 1)\ \forall t \in [0, T]$
For proving, consider $G(t) = e^{-Bt}\int_0^t \psi(s)ds$ to get $G'(t) \le (A + ct)e^{-Bt}$ then integrate

The Wronskian of n differentiable functions is the determinant formed with the functions and their derivatives up to order n–1.

Existence and Uniqueness

A Banach space is a complete normed vector space., *i.e.* X equipped with a norm $||\cdot||$ such that every Cauchy sequence in X converges to a limit within X.
A domain is an open, connected subset of the given set

$$\text{IVP: } \frac{dx}{dt} = f(t,x);\ x(t_0) = x_0 \implies x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

This can be formulated as $x = Tx$ for a suitable Banach Space X and $T : C \to X$ where $C \subset X$ is closed. We wish to find the fixed points of this mapping operator.
If we iteratively define a nested sequence $x, Tx, T^2x, \ldots$ where $x_{m+1} = Tx_m$, then if the limit exists it must lie in C due to closure and thus the limit point would be a fixed point as $x_\infty = Tx_\infty$

Fixed Point Theorem

Map $T : C \to X$ satisfies $||Tx - Ty|| \leq \theta||x - y||\ \forall x, y \in C \subset X$ with $\theta < 1$, then T is a **contraction map**. Here $||\cdot||$ denotes the norm over the Banach Space X.
Banach's Theorem: Let $C \subset X$ be closed and $T : C \to X$ be a contraction map. Then T has a *unique fixed point* $\bar{x} \in C$.
We first obtain a fixed point using the limit of a contraction map then obtain an interval using Banach's theorem that shows its uniqueness.

Solution to Cauchy's Problem

$x'(t) = f(x(t)) \implies x(t) = x_0 + \int_{t_0}^{t} f(x(s))ds$, so T is defined over the set of all continuous functions as $T : \gamma \to x_0 + \int_0^t f(\gamma(s))ds$ where γ is a continuous curve
For T to be a contraction map: $|Tx(t) - Ty(t)| = |\int_0^t f(x(s)) - f(y(s))ds| \leq$
$\int_0^t |f(x(s)) - f(y(s))| \leq K\int_0^t |x(s) - y(s)|ds \leq K\int_0^t ||x - y||_\infty ds$ assuming Lipschitz

Choose a $\delta < 1/K$ then $K\int_0^t ||x - y||_\infty ds \leq K\delta||x - y||_\infty < ||x - y||_\infty$
Thus, $T : \mathcal{C}[-\delta, \delta] \to \mathcal{C}[-\delta, \delta]$ is a contraction map with sup norm of Banach Space, provided f is Lipschitz. Boundedness is inevitable as continuity over closed interval.

Matrix Solutions for Linear Systems

First-order autonomous linear system of ODEs: matrix form $\mathbf{x}' = A\mathbf{x}$
Here $\mathbf{x}(t) \in \mathbb{R}^n$ is a vector of functions and A is an $n \times n$ matrix of constant coefficients.
Solution exponential of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$: λ scalar, $\mathbf{v}$ constant vector.

$$\lambda e^{\lambda t}\mathbf{v} = A(e^{\lambda t}\mathbf{v}) \implies \lambda\mathbf{v} = A\mathbf{v}$$

This reduces the problem of solving the differential equation to the algebraic problem of finding the **eigenvalues** λ and corresponding **eigenvectors v** of the matrix A.

General Solution. The general solution depends on the nature of the eigenvalues.

- **Real, Distinct Eigenvalues:** If A has distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, the general solution is a linear combination of the individual solutions:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

- **Complex Eigenvalues:** If $\lambda_1 = \alpha + i\beta$ is a complex eigenvalue with eigenvector $\mathbf{v}_1 = \mathbf{u} + i\mathbf{w}$, then its conjugate $\lambda_2 = \alpha - i\beta$ is also an eigenvalue with eigenvector $\mathbf{v}_2 = \mathbf{u} - i\mathbf{w}$. Two real-valued, linearly independent solutions are given by the real and imaginary parts of $e^{\lambda_1 t}\mathbf{v}_1$:

$$\mathbf{x}_1(t) = e^{\alpha t}(\mathbf{u}\cos(\beta t) - \mathbf{w}\sin(\beta t))$$

$$\mathbf{x}_2(t) = e^{\alpha t}(\mathbf{u}\sin(\beta t) + \mathbf{w}\cos(\beta t))$$

Classification of Fixed Points in 2D

For a 2D system $\mathbf{x}' = A\mathbf{x}$, the origin $\mathbf{x} = \mathbf{0}$ is always a fixed point (or equilibrium point). The behavior of solutions near the origin is determined by the eigenvalues λ_1, λ_2 of the 2×2 matrix A.

Criteria for Classification of the Origin. Let λ_1 and λ_2 be the eigenvalues of A.

- **Sink (Stable Node):** The eigenvalues are real, distinct, and negative ($\lambda_1 < \lambda_2 < 0$). All trajectories approach the origin as $t \to \infty$.
- **Source (Unstable Node):** The eigenvalues are real, distinct, and positive ($0 < \lambda_1 < \lambda_2$). All trajectories move away from the origin as $t \to \infty$.
- **Saddle:** The eigenvalues are real and of opposite sign ($\lambda_1 < 0 < \lambda_2$). Trajectories approach the origin along the eigenvector for λ_1 (the *stable manifold*) and move away along the eigenvector for λ_2 (the *unstable manifold*). The origin is *unstable*.
- **Spiral Sink (Stable Spiral):** The eigenvalues are a complex conjugate pair $\lambda = \alpha \pm i\beta$ with negative real part ($\alpha < 0$). Trajectories spiral into the origin.
- **Spiral Source (Unstable Spiral):** The eigenvalues are a complex conjugate pair $\lambda = \alpha \pm i\beta$ with positive real part ($\alpha > 0$). Trajectories spiral away from the origin.
- **Center:** The eigenvalues are a purely imaginary conjugate pair ($\lambda = \pm i\beta$, so $\alpha = 0$). Trajectories form closed elliptical orbits around the origin.

  A center is *stable but not asymptotically stable.*

This classification can also be summarized using the trace $\tau = \text{tr}(A) = \lambda_1 + \lambda_2$ and determinant $\Delta = \det(A) = \lambda_1\lambda_2$.