Real Analysis

Reference: Walter Rudin, *Principles of Mathematical Analysis* (aka Baby Rudin)

Sets and Metric Spaces

- A is finite $\iff \exists\ n : A \sim J_n = \{1, 2, 3, \ldots, n\}$
- Bijection between sets is an equivalence relation
- Countable union of countable sets is countable
- Set of countably many countably large tuples is countable
  (implying algebraic numbers are countable)
- Cantor's diagonalisation process $\Rightarrow$ Real numbers are uncountable
- Proving A is uncountable: show that every countable subset is proper
  ($\exists$ an element outside it $\in A$)
- All metrics must satisfy triangle inequality and that forms basis of most contadiction proofs
- p : limit point of set E if $\forall N_r(p)\ \exists\ q \in N_r(p)$ such that $q \in E$
- E closed if every limit point of E is a point of E;
  E open if every point of E is an interior point of E
- The set $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is neither open nor closed (by definition)

Open and Closed Sets

- If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E
- E is dense in X if every point of X is a limit point of E, or a point of E (or both)
- A set E is open if and only if its complement is closed and vice-versa (proof by definition, consider points)
- Arbitrary union and finite intersection of open sets is open (reverse for closed sets by complementation) Arbitrary intersection of open sets need NOT be open: $G_i = (-\frac{1}{n}, \frac{1}{n})$ is open but $H = \cap_{i=1}^{\infty} G_i$ isn't
- Closure $\bar{E}$ of set E is the smallest closed set containing E
- Let $E \subset Y \subset X$ for a metric space X, then E is open relative to Y if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$
- If $Y \subset X$ then a subset E of Y is open relative to Y **iff** $E = Y \cap G$ for some open subset G of X.

Basic Topology

- A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover (Cover is collection of open sets the union of which contains the said set)
- **Compact subsets of metric spaces are closed**
- **Closed subsets of compact sets are compact**
- If $\{K_a\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_a\}$ is nonempty, then $\cap K_a$ is nonempty.
- If E is an infinite subset of a compact set K, then E has a limit point in K
- **Nested Interval Theorem**: let $k \in \mathbb{N}$. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1} \ \forall \ n \in \mathbb{N}$, then $\cap_1^\infty I_n$ is not empty
- **Every k-cell is compact**
- **Heine-Borel theorem**: in R^k, E is compact $\iff$ E is closed and bounded
- Every open set in R^1 is a countable union of disjoint segments (open intervals)
- Weierstrass Theorem: Every bounded infinite subset of R^k has a limit point in R^k

Perfect and Connected Sets

- Every non-empty perfect set in R^k is uncountable
- Cantor set (remnant points after repeated middle third removal) is perfect
- A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated sets
- A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E, y \in E$, and $x < z < y$, then $z \in E$

Functions and Continuity

- Epsilon-delta definition of limit: lying in delta-neighbourhood $\Rightarrow$ lying in epsilon-neighbourhood
- Continuous functions need f(p) to be defined, so p must be in E
- **Continuous $\iff$ pulls back open sets to open sets** (V open $\Rightarrow f^{-1}(V)$ open)
- A functional vector is continuous iff each component function of its is continuous
- **Continuous functions map compact sets to compact sets**
- By Heine-Borel, real continuous functions map compact sets to closed and bounded sets Consequently, Extreme Value Theorem holds true: continuous functions attain sup and inf
- Inverse mapping of bijective continuous functions on a compact set is also continuous

- Uniform continuity is continuity over a set where same delta function works for all points
- **Continuous mapping on a compact set is uniformly continuous**
- **Continuous functions map connected sets to connected sets**
- For monotonic functions f, $f(x^-)$ and $f(x^+)$ exist for all points x in the domain
- **Monotonic functions can have at most countably many discontinuities, all jump**

A **homeomorphism** is a bijective function from metric space (X, d) to (Y, ρ) which is continuous and its inverse is also continuous. TFAE for $f : X \to Y$

- f is a homeomorphism
- $G \subset X$ is open if and only if $f(G) \subset Y$ is open
- $F \subset X$ is closed if and only if $f(F) \subset Y$ is closed

Sequences and series of functions

- If X is a compact metric space and $\{p_n\}$ a Cauchy sequence in X, then it converges to some point of X
- **Pointwise convergence** of $\{f_n\}$: $\lim_{n\to\infty} f_n(x) = f(x)$ for each fixed x.
- **Uniform convergence** of $\{f_n\} \to f$: $\sup_{x\in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.
  (Same N works for all x).
- **Uniformly convergent sequence of continuous functions is continuous**.
- Uniformly convergent series of continuous functions can be integrated term-by-term over a compact interval. ($\int(\sum f_n) = \sum \int f_n$).
- For a sequence of differentiable functions $\{f_n\}$, if $\sum f_n$ converges at one point and $\sum f'_n$ converges uniformly, then $\sum f_n$ converges uniformly and is differentiable term-by-term.
- **Weierstrass M-Test**: If $|f_n(x)| \leq M_n$ for all $x \in E$ and $\sum M_n$ converges, then $\sum f_n(x)$ converges uniformly on E.
- **Equicontinuity**: family of functions $\mathcal{F}$ is equicontinuous if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$ whenever $d(x, y) < \delta$. (Same δ works for all)
- **Arzela-Ascoli Theorem**: If $\{f_n\}$ is a sequence of functions on a compact metric space K that is uniformly bounded and equicontinuous, then $\{f_n\}$ has a uniformly convergent subsequence.

Proof techniques

Complement of open is closed and vice-versa so try to reframe problem for openness
Compactness implies finite cover, implies a set of neighbourhoods having a defined minimum, so helps in finding a new interval for proving/contradicting openness

All open sets in R^1 can be written as a countable union of disjoint segments Suppose $\mathbf{f}$ is a continuous mapping of $[a,b]$ into R^k and $\mathbf{f}$ is differentiable in (a,b). Then $\exists\ x \in (a,b)$ such that $|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)|\mathbf{f}'(x)|$

A continuous function extends from (a,b) to [a,b] iff f is uniformly continuous
A set is F_σ if it is the countable union of closed sets
A set is G_δ if it is the countable intersection of open sets

Baire Category Theorem states that for a complete metric space X, the intersection every countable collection of dense open subsets of X is dense in X

The Lebesgue covering lemma states that for every open cover of a compact metric space, there exists a real number $\delta > 0$ such that every open ball in the space of radius δ is contained in some element of the cover. This number δ is called a Lebesgue number for the covering. The lemma is used to prove that sequentially compact metric spaces are equivalently compact metric spaces.

Topics' Summary

Metric spaces
Open closed subsets
Hausdorff property
Limit points
Closure,interior,boundary
Continuity
Uniform continuity
Intermediate value theorem
Homeomorphism
Compactness
Heine-Borel theorem
Bolzano-Weierstrass property
Sequential compactness
Lebesgue covering lemma
Fundamental theorem of algebra
F-sigma, G-delta sets
Differentiability
Rolle's theorem
CMVT, LMVT
IVT for derivatives
L'hopital's rule
Taylor's theorem
Convex, concave functions
AM-GM proof (using convexity)
Riemann integration
Partitions,refinement,tagged partition
Cantor set
Fundamental theorem of calculus (2 forms)
Rectifiable curves
Improper integral
Integral test, convergence/existence
Gamma function
Convergence of functions
Uniform and pointwise convergence