General Topology (Munkres)

Reference: James Munkres, *Topology*

Topology is the study of shapes; it is fundamentally different from calculus as here we care about the nature of the functions over points in an interval, not the average behaviour of the function over the interval itself.
Topology is about continuous functions; calculus is about integrable functions.

A pair (X, τ) is called a **topological space** if: [σ-field / Borel set flashback]

- τ is a collection of subsets of X; $\phi, X \in \tau$
- τ is closed under arbitrary union and finite intersection

Structure theorem for Real number line: all open sets in $\mathbb{R}$ are disjoint unions of countably many open intervals (can separate and the separations must be open so intervals, and then countable coz there's a rational in each interval)

Examples of topology in X = $\mathbb{R}$: $\{\phi, \mathbb{R}\}$, all subsets of $\mathbb{R}$, all open sets in $\mathbb{R}$
We need the restriction of finite intersection and not arbitrary as arbitrary intersection of open sets need not be open.
Further examples: $X = \mathbb{R}^d, \tau =$ set of all open sets in $\mathbb{R}^d$; X, $A \subset X, \tau = \{\phi, X, A\}$
Given a topological space, you can append a subset of X to it and create a lot more topological spaces (including power set of X). Same applies to finitely many subsets.
Given X and two topologies τ_1, τ_2 on X, we say τ_1 is finer than τ_2 if $\tau_2 \subset \tau_1$

The graph of $d(x-0) + d(y-0) < 1$ is an open diagonal square for p=1, open unit circle for p=2 and open unit square for $p \to \infty$ where $d = (\sum |x_i - y_i|^p)^{\frac{1}{p}}$
$\tau_p =$ set of all open sets given by the metric on $\mathbb{R}^d$
Are there toplogical spaces where τ is not coming from a metric on X, i.e. τ is NOT the collection of open sets wrt some metric d on X? Yes, simply take $\{X, \phi\}$
X = [0,1] × [0,1] - vertical lines as open sets?

There is a simple necessary condition for metrisability: it should have property that given any two distinct members of τ there are disjoint members from the topology containing both of them
Hausdorff property?
τ_2 is the same as τ_∞ coz all squares can be expressed as union of circles and vice-versa.
Any two norms on any finite dimensional vector spaces are equivalent (upto scaling)

Norm is a map from fdvs to real positives, satisfying linearity,triangle inequality and positive definiteness
For example, given any norm d on $\mathbb{R}^2, \exists C \in \mathbb{R}^+$ such that $\forall x : d(x)/C \leq |x| \leq C \cdot d(x)$ where $|x|$ is the euclidean norm
Open set is defined as any member of a topology τ. Thus inevitably, X and ϕ are open sets. All previous properties like arbitrary union, finite intersection, complement closed still hold.
For non-empty X, β is a collection of subsets of X. We say β is a basis if $X = \bigcup_{B \in \beta} B$

It must be almost closed under intersection if $B_1, B_2 \in \beta$ and $p \in B_1 \cap B_2$ then $\exists B_3 \in \beta$ such that $p \in B_3 \subset B_1 \cap B_2$
For example, $X = \mathbb{R}^2, \tau =$ usual open set, $\beta =$ all balls of finite radius, then $\mathbb{R}^2 = \bigcup_{p\in\mathbb{R}^2} B(p, 1)$. This basis can be verified to be almost closed. You can shrink this by taking all balls of radius at most 1. Further simply take $\beta = B(X, 1/n) : x \in \mathbb{R}^2, n \in \mathbb{N}$
Excluding any finite number of points doesn't change shiz as we got enough choices.

Two topologies τ and τ' are equal if and only if they have the same basis set, i.e. there exists a set that forms the basis of them both.
A collection of subsets A is a subbase wrt τ if it is a base and β defined as all possible intersections of sets in A, has arbitrary unions of its elements as open sets in τ
Let $\{A_\alpha\}$ be a collection of sets A so that $\bigcup_\alpha A_\alpha = X$
$\mathcal{B} =$ all possible finite intersections of A_α's. By definition, closed under finite intersection
$\tau =$ all possible union of members of $\mathcal{B} \cup \{\phi\}$

Back to $X = \mathbb{R}^2$, let τ be all open sets and $\mathcal{B} =$ all balls of finite radius (*standard topology*). Let $Y \subset X$. Look at the sets $\tau' = \{V \cap Y : \text{V is open in X}\}$. Then (Y, τ') is called a subspace wrt the topology τ'
Supose $\mathcal{B}$ is a basis of (X, τ) and $Y \subset X$. Then $\mathcal{B}' = \{Y \cup V : V \in \mathcal{B}\}$ is a basis of the subspace topology τ'

In set X = {p}, the set {p} is open
$Y \subset X, A \subset Y$ is open in Y in subspace topology in Y if $\exists\ V$ open in X so that $V \cap Y = A$
Let $A \subset X$. We define closure(A) $= \bigcap_{A\subset B} B$ where all Bs are closed in X.
Closure(A) is a closed set in X and is in fact the smallest closed set containing A (NOT the definition by can be shown easily by minimality argument on intersection definition).

Theorem: **For $\mathbf{A \subset X, \bar{A} = A \cup}$ {limit points of A}**
Proof: we show two sided belongingness - $A \subset \bar{A}$ coz ofc, and each limit point must belong to $\bar{A}$ as $\bar{A}^c$ must be open so every point having an open ball entirely in the complement must lie there, but any limit point by definition doesn't satisfy this definition so it can't be in the complement. Thus all limit points $\in \bar{A}$.
Other side, note that we define $\bar{A}$ is defined to be the smallest set containing A and which is closed. So any such set must contain A, and we have shown that it must consequently contain any limit point of A too. Thus, our RHS satisfies both criteria, so by minimality $\bar{A}$ must lie inside the RHS and we done babyy
Alt Proof: first prove RHS is closed by showing that its complement is open, then prove that RHS is the smallest closed set containing A (see pics - 15 jan)
Examples: $X = \mathbb{R}, \tau =$ standard topology. $A = \mathbb{Q}, \bar{A} = \mathbb{R}$
$X = \mathbb{R}, \tau_2 =$ cofinite topology. $\tau_2 = \{A : A^c \text{ is finite}\} \cup \{\phi\}$: $A = \mathbb{Q}, A = \mathbb{R}$
$X = \mathbb{R}, \tau_3 =$ cocomplement topology $= \{A : A^c \text{ is countable}\}$. Here $A = \mathbb{Q}, \bar{A} = \mathbb{Q}$
$X = \mathbb{R}, \tau_4 =$ discrete topology $= \{A : A \subset \mathbb{R}\}$. $A = \mathbb{Q}, \bar{A} = \mathbb{Q}$ (every set is open in τ_4, no limit points)

Sequence converging to a point:$\{x_n\}, x_n \in X$. We say x_n converges to x if any open set V containing the point x, has an existing N such that $x_n \in V\ \forall\ n\ \geq N$.
An open collection $\mathcal{B} = \{(a, b) : 0 < a < b < 2\}$ contains (0,1) despite a not being permitted to

take the value 0 directly. This happens coz arbitrary unions of open sets is open and so just take $\bigcap_{n=1}^{\infty}(1/n, 1)$
Notice that in example 4 (pics), in the topology considered 0 and 2 are indistinguishable, which means $1/n$ converges to both 0 and 2.
Lemma: X a metric space and $\exists\ x_n \to x, y \in X$ then x = y.
You can establish convergence in an open set by doing so on each set of its basis (?)

Hausdorff spaces

We say a set X is Hausdorff if $\forall\ p \neq q \in X, \exists\ V_p, W_q$ open in X such that $p \in V_p, q \in W_q$ and they're disjoint.
Examples: any metric spaces
Do converging sequences identify a topology? i.e. given a set X and topologies τ, τ' such that a sequence $x_n \to x$ in τ iff it converges in τ', then does this imply $\tau = \tau'$?
We know the answer is yes for metric spaces. Proof:
Let B(x,1) be an open ball (of radius 1) in τ centered at x. We wish to prove that B(x,1) is open in τ' as well (note that the metrics are different for both topologies else they'd be identical)
Sequences in Hausdorff spaces can converge to multiple points

Question 1. Suppose for (X, τ) we know sequences $\{x_n\}$ have $x_n \to x$ for some $x \in X$. Does this identify the topology? In other words, the Q above.
Answer is yes due to PBC, convergence definitions in both topologies. For subbase topologies we can take a ; complement of countable sets is open?
Cofinite topology has any open set defined as the complement of a finite set. For convergence, we need every term of the sequence to belong to an open ball for any radius, barring a finite head.

Lemma. *The only topology over X for which all sequences converge to all points is the trivial topology $\tau = \{\phi, X\}$*

Proof: Suppose U is open, and $U \neq \phi, X$. Then there are points p, q such that U contains p but not q. Consider the sequence $\{p, q, p, q, \ldots\}$ - note that it converges to q coz the only open set containing it is X.

Lemma. *There is no such pair (X, τ) and (X, τ') such that each x_n converges to a unique x in both topologies but $\tau \neq \tau'$.*

Question 2. $X = [0, 1], x_n = 1/n$. Is it possible to define τ on $[0, 1]$ so that x_n (and all its subsequences) are the only convergent sequences in τ?
NO (maybe)

Summary

Well-ordering principle: every nonempty subset of a well-ordered set has a (unique) minimum element, for example $\mathbb{N}$
Set-theoretic rules on union, intersection, complementation and exclusion
A relation between two sets is a subset of their Cartesian product
A function from A to B is a relation in which *every* input has *exactly one* image
A countable set is one which has a bijection from $\mathbb{N}$ to itself

Topological Spaces

Topology is the study of geometric object and their physical properties, characterised by sets often known as *topological spaces*.
A topology on set X is a collection of its subsets $\mathcal{T}$ satisfying:

- $\phi, X \in \mathcal{T}$ (null set and universe are always open)
- $\mathcal{T}$ is closed under *arbitrary* union
- $\mathcal{T}$ is closed under *finite* intersection

If a set $S \in \mathcal{T}$ then S is known as an *open set* (in $\mathcal{T}$, usually omitted)

$\mathcal{T} = \{\phi, X\}$: trivial topology; $\mathcal{T} = 2^X$: discrete topology
$\mathcal{T}_f$ cofinite topology: openness $\iff$ finiteness of complement
$\mathcal{T}_c$ cocomplement topology: openness $\iff$ countability of complement
$\mathcal{T} \subset \mathcal{T}' \Rightarrow \mathcal{T}'$ is finer than $\mathcal{T}$ (strictly)

Basis and Subbasis

A basis on set X is a collection of its subsets $\mathcal{B}$ satisfying:

- $\mathcal{B}$ spans X: $\forall\ x \in X, \exists\ B \in \mathcal{B} : x \in B$
- $\forall\ B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2 \Rightarrow \exists\ B_3 \subset B_1 \cap B_2 : x \in B_3$

Topology $\mathcal{T}$ generated by basis $\mathcal{B}$ is as follows:
A set U is open in $\mathcal{T}$ iff $\forall\ x \in U, \exists\ B \in \mathcal{B} : x \in B \subset U$

$\mathcal{T}$ can also be understood as the collection of *all possible unions of all basis elements*
To obtain the basis of a given topology $\mathcal{T}$, find a collection of open sets $\mathcal{C}$ such that $\forall U \in \mathcal{T}, \forall x \in U, \exists C \in C : x \in C \subset U$. This $\mathcal{C}$ is a basis of $\mathcal{T}$.

A subbasis S for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis S is defined to be the collection T of all unions of finite intersections of elements of S.

Order and Product

The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.
Pasting Lemma: Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

Continuity and Uniform

Continuous functions pull back open sets to open sets.
Homeomorphism: bijective continuous map with a continuous inverse. A bijective continuous is a homeomorphism iff it is open (or closed, one implies the other).

Quotient Topology

Connectedness

Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.
A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.
If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.
A space X is locally connected if every neighbourhood of a point contains a connected subset containing the point. Analogous for path-connectedness.

- If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.
- The union of a collection of connected subspaces of X that have a point in common is connected; continuous functions on finite connected topologies are constant.
- Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.
- The image of a connected space under a continuous map is connected; a finite cartesian product of connected spaces is connected.
- X is locally connected iff for every open U, each component of U is open in X.
- Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered sets. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Compactness

A set is compact iff every cover has a finite subcover.
A set is compact iff every sequence has a convergent subsequence.
Continuous functions map compact sets to compact sets.
Continuity over compact sets implies uniform continuity.
If $f : X \to Y$ is a bijective continuous map, it is a homeomorphism iff X is compact and Y is Hausdorff.
Closed subsets of compact subspaces are compact.
Compact subsets of Hausdorff spaces are closed.
Finite product of compact spaces is compact. (For metric spaces through subsequence definition, else general through direct multiplication)

Tychonoff's Theorem

Product topology is the smallest one on the product which makes all projection maps continuous (thus $\prod_{i \neq j} X_i U_j$ open for all open U_j, which forms a subbasis)
Tube Lemma: if Y is compact and N is an open set in $X \times Y$ containing line $a \times Y$ for some $a \in X$, then it contains strip $W \times Y$ for some open W containing a.

Lebesgue Covering Lemma: if C is an open covering of a compact metric space X then there exists $\delta \in \mathbb{R}^+$ such that $\forall S \in X : diam(S) < \delta \Rightarrow \exists U \supset S \in C$.
The metric $d[(x_1, x_2, \ldots), (y_1, y_2, \ldots)] = \sup_n |x_n - y_n|/n$ generates the product topology on $[0, 1]^{\mathbb{N}}$

Compactification & Localisation

X is locally compact if for every point there's an open set containing it whose closure is compact.
A subset M of a locally-compact (Hausdorff) space X is itself locally compact iff it is of the form $U \cap F$ for some open $U \subset X$ and some closed $F \subset X$.
All locally compact Hausdorff spaces have a one-point compactification which yields compact Hausdorff spaces containing them.
To define the new topology, we add all open sets of the form $U = \{\infty\} \cup (X \setminus K)$ where K is a compact set in X.
$[0, 1]^{\mathbb{R}}$ is compact Hausdorff but NOT metrisable as any any metric space must have a countable local basis.

Normal Spaces and Lemmas

X is normal if it is Hausdorff and any 2 disjoint closed sets are separated by open sets.
Urysohn's Lemma: in a **normal** space X any two disjoint closed sets can be separated by a **continuous** function taking one set to 0 other to 1.
Every metric space is normal.
Every compact Hausdorff space is normal.

To prove Urysohn's Lemma we construct balls $U_r : 0 < r \leq 1$ such that $A \subset U_0, B \subset U_1$ and $\overline{U_r} \subset U_s \; \forall s > r$.
The countable collection of balls (indexed by rational numbers) is created by taking $U_1 = X \setminus B$ followed by repeated normality application on $U_i - U_1$.

Tietze Extension Theorem: a continuous function on a closed subset of a normal space can be continuouly extended to the whole normal space.
Stone-Cech Compactification: a continuous on a normal space can be extended continuously to its compact closure.
Proof proceeds by finding a continuous injective open map φ from X to $[0, 1]^J$ so that $\overline{\varphi(X)}$ is compact and its projection to $[0, 1]$ is continuous.