Lecture 10

Black holes.

Recap of Schwarzschild BH

History: Enigma of the apparent singularity of the metric at $r=2M$

Townsend (Les Houches 1997)

$$G = c = 1$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2$$

Collapse of a star to BH.

Set $r = R(t)$ watch surface

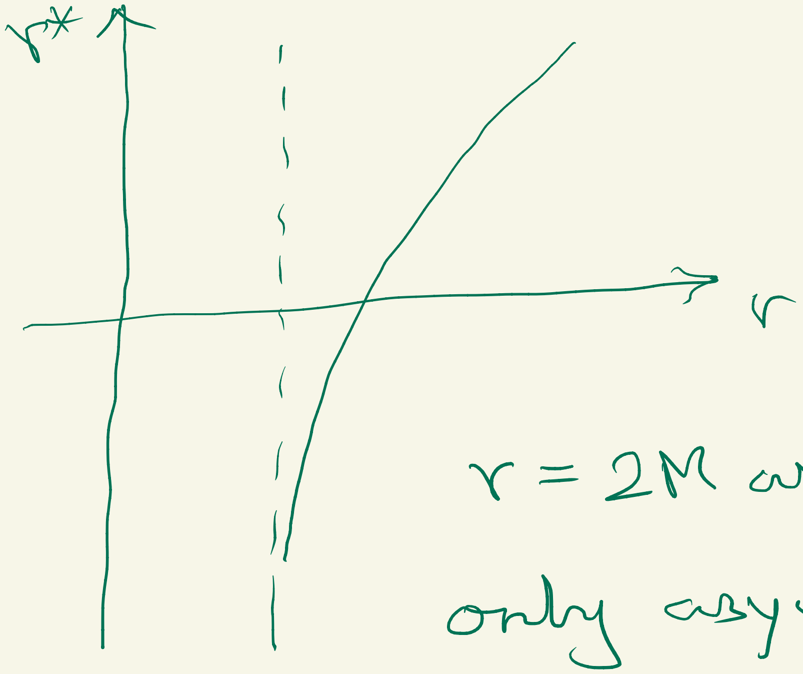
$R(t)$ evolve with $\dot{R} < 0$

Radial null geodesic

$$dt^2 = \frac{dr^2}{1-\frac{2M}{r}} \equiv dr^{*2}$$

Then $\quad r^* = r + 2M \ln \left| \dfrac{r-2M}{2M} \right|$

— Wheeler's tortoise coord.



$2M < r < \infty$

$-\infty < r^* < \infty$

$r = 2M$ approached only asymptotically in r^* coord.s

Eddington - Finkelstein coord

light cone coord.

$v = t + r^*$ ingoing lightlike coord.

$u = t - r^*$ $\longrightarrow$ outgoing

$$-\infty < v < \infty$$

For study of collapse adopt ingoing coord : (v, r, θ, ϕ)
(or r^*)

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2$$

Now $dt^2 = (dv - dr^*)^2$

$$= dv^2 + dr^{*2} - 2dv\,dr^*$$

$$-dt^2 + dr^{*2} = -dv^2 + 2dv\,dr^*$$

Thus

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

Inevitability of collapse beyond (inside) $r = 2M$:

$$2\,dv\,dr = -\left[-ds^2 + \left(\frac{2M}{r}-1\right)dv^2 + r^2 d\Omega^2\right]$$

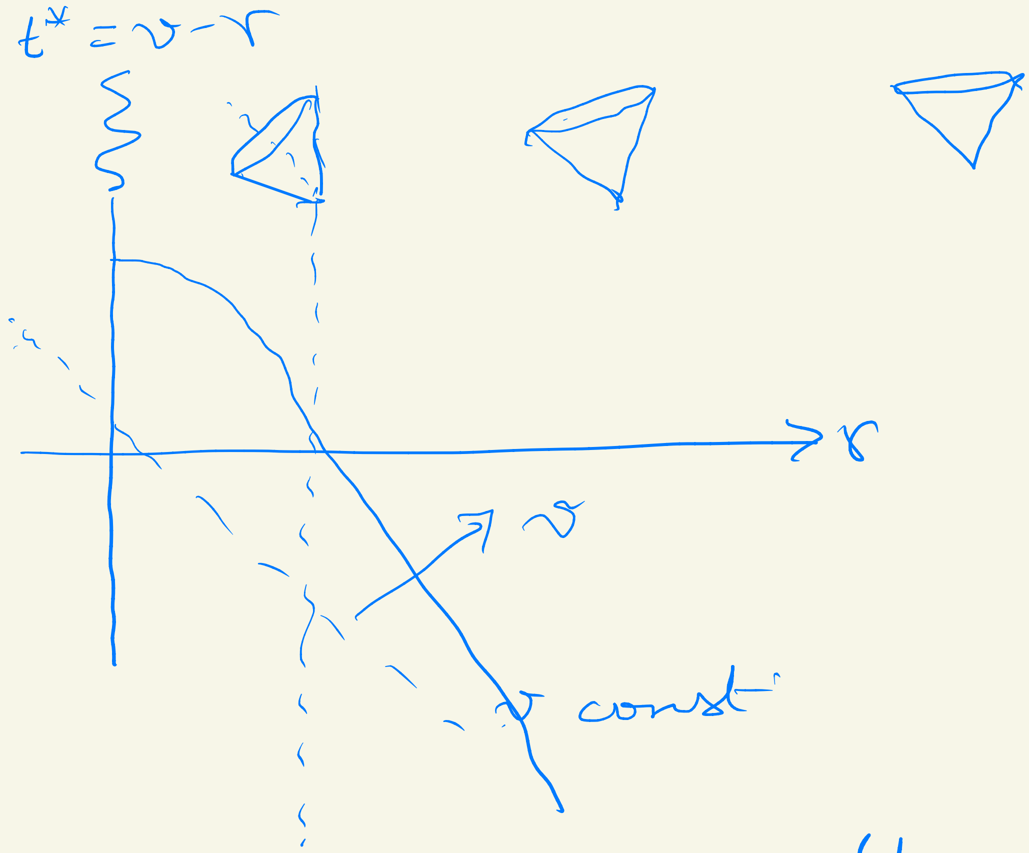
Thus for future directed, i.e. $ds^2 < 0$, $\Rightarrow dv\,dr < 0$ & $r < 2M$,

Thus ingoing $\equiv dv > 0$

$\Rightarrow dr < 0$

and the metric is not singular though has a zero at $r = 2M$

A picture of the collapse



Thus the v coord allows tracking through the $r = 2M$ singularity.

Q: Cauchy surface:

A: Any hypersurface on which legitimate initial value data can be provided for a hyperbolic PDE

$$\underbrace{\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)}_{\text{hyperbolic} \to \text{relative -ve sign}} \phi = f$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

NOT "initial" value but boundary value data

Elliptic PDE

Parabolic: $-\frac{\partial}{\partial t} + \nabla^2$

heat eqn. & Schrö. if $i\frac{\partial}{\partial t}$

Ref: Courant & Hilbert vol. II

Capturing extended Schw. manifold

Introduce both

$u = t - r^*$ $\quad v = t + r^*$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\,dv + r^2 d\Omega^2$$

[check : $du = dt - \dfrac{dr}{1 - \frac{2M}{r}}$ etc.]

However a zero persists at $r = 2M$

Kruskal - Szekeres coords :

$U = -e^{-u/4M}$ $\quad V = e^{v/4M}$

$$ds^2 = -\frac{32 M^3}{r} e^{-r/2M} dU\,dV + r^2 d\Omega^2$$

To understand light cone structure, note,

$$UV = -e^{-r^*/2M}$$

$$= -\left(\frac{r-2M}{2M}\right)e^{r/2M}$$

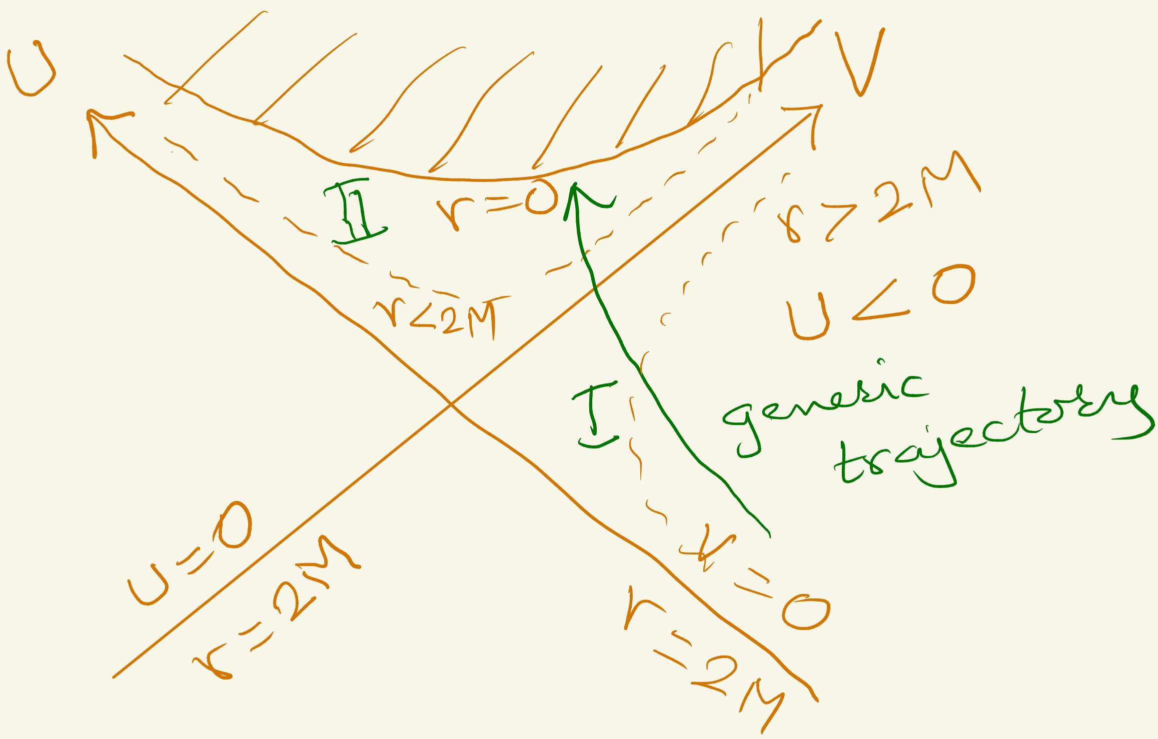
Thus note

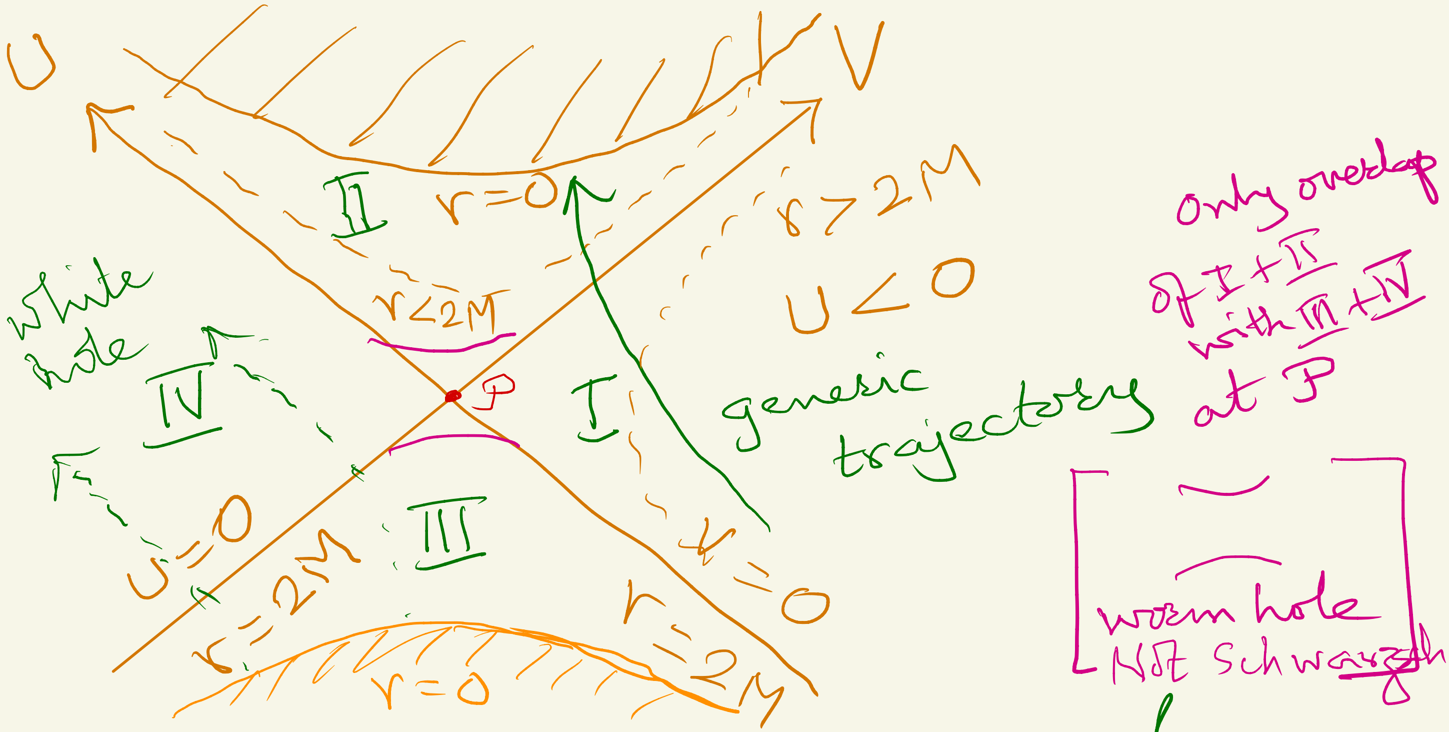
$r = 2M \longleftrightarrow UV = 0$

either $U = 0$

or $V = 0$

$r = 0 \longleftrightarrow UV = 1$





Regions I & II are the usual space-time sufficient for usual collapse

Additionally by analytic continuation U, V allow additional regions III & IV

Singularities : Use geodesics for characterising global structure

Identify "removable" singularity of metric coeffs → if allowed by choice of coordinates.

"Essential" or irremovable singularity where Riemann has divergences.

Extending geodesics maximally

Use affine param. for the geodesics :

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

s is affine. But in general the $\lambda(s)$ choice may not leave the eqn. in this form → such λ to be avoided

Maximal extension of geodesic →
the entire allowed range [usually $\mathbb{R}$]
of the affine parameter is used
and allowed.

Def: Non-singular spacetime
allows maximal extension of all
its geodesics. [& vice versa if
every possible geodesic can be
maximally extended ...] if necessary
by change of coord.s to avoid a
removable singularity.

Maximal extension of a manifold

Here the spacetime may have irremovable singularities

Then maximal extension of the manifold is s.t. every geodesic can be extended for all the values of its affine parameter or ends at an irremovable singularity.

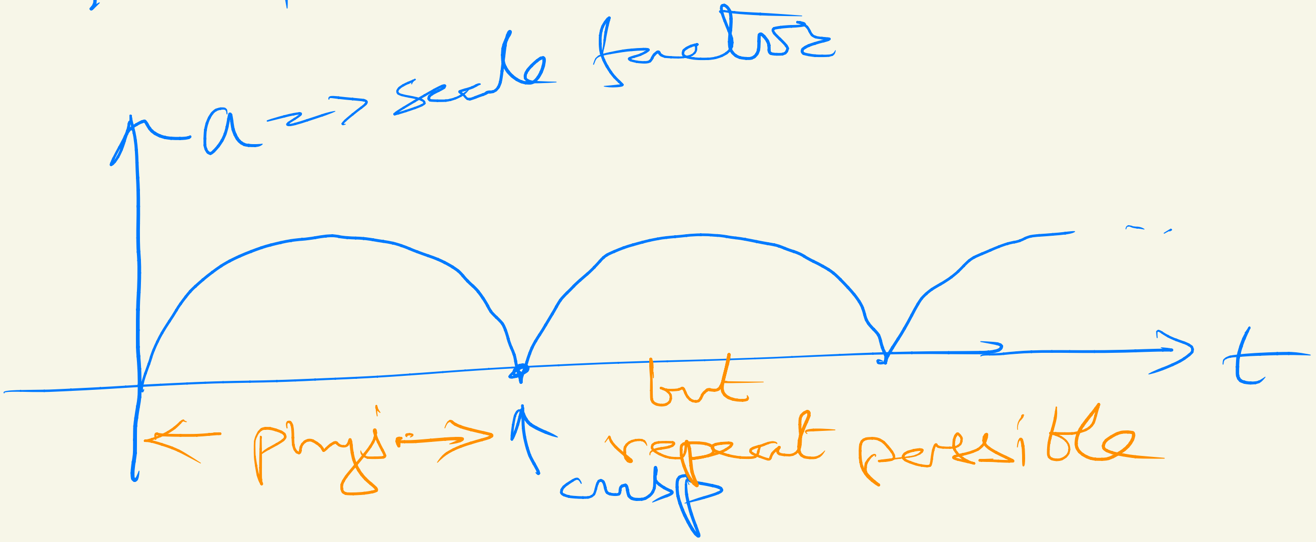
Thus K-Sz is a maximally extended version of Schwarsz.

[Also see historic timeline of BH solutions in Ashtekar's book draft]

Q&A Spacelike (BH) vs timelike (Big Bang) singularities.

Extension of Friedmann universe

$k = 1$



Likewise extension of de Sitter ... to be done