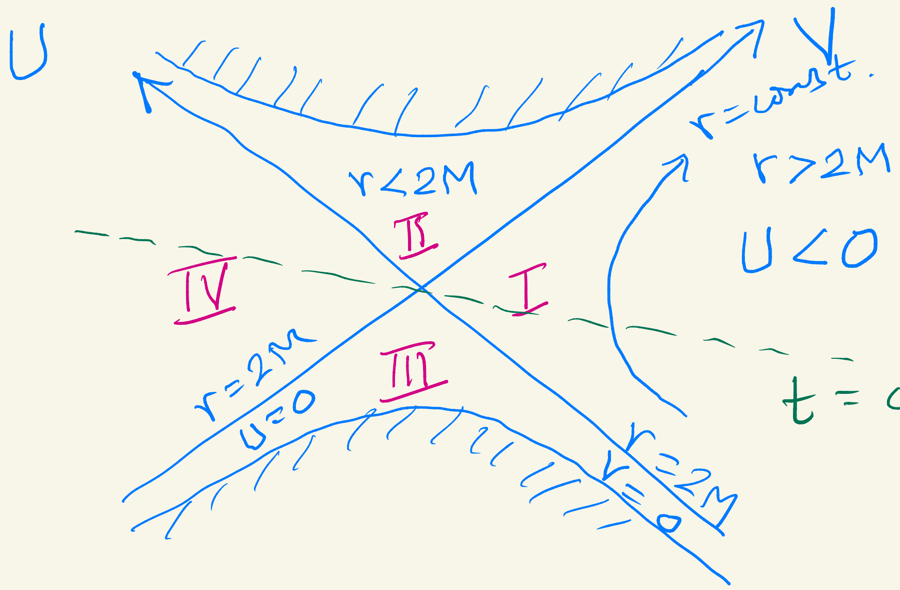
L 11 Einstein-Rosen bridge & Energy conditions

[Townsend ; Gauntlett (Cambridge) notes]

Schwarzschild - fully extended in K-Sz coord.s



$\frac{U}{V}$ = const $\Rightarrow t = \text{const}$

Recall $\quad u = t - r^* \qquad v = t + r^*$

$$U = -e^{-u/4M} \qquad V = e^{v/4M}$$

$$ds^2 = -\frac{32}{r} e^{-r/2M} dU\,dV + r^2 d\Omega^2$$

$$r = 2M \quad \longleftrightarrow \quad UV = 0$$

$$r = 0 \quad \longleftrightarrow \quad UV = 1$$

Can one move along $t = \text{const.}$ surface from $I \longleftrightarrow IV$?

Answer : No b.c.s at $U = V = 0$, there is an S^2 but $dr = 0$

Einstein-Rosen bridge :

Patch the sequence of S^2 from I with a sequence from IV

Recall we can set (for $dt = 0$)

$$ds^2 = dr_*^2 + r^2(r_*)\, d\Omega^2$$

with $r_* = r - 2M \ln\left|1 - \frac{r}{2M}\right|$

$$dr_* = \frac{dr}{1 - \frac{2M}{r}}$$

$$r_*(r \to 0) = r - 2M\left(-\frac{r}{2M}\right) = 0$$

$$\therefore ds^2(r = 2M) = 4M^2 d\Omega^2$$

"Isotropic coordinates"

$$r = \left(1+\frac{M}{2\rho}\right)^2 \rho$$

Then $$\frac{2M}{r} = \frac{2M}{\rho} \times \frac{4\rho^2}{(2\rho+M)^2}$$

$$\therefore 1-\frac{2M}{r} = 1-\frac{8M\rho}{(2\rho+M)^2} = \left(\frac{1-\frac{M}{2\rho}}{1+\frac{M}{2\rho}}\right)^2$$

Next
$$dr = d\rho\left(1+\frac{M}{2\rho}\right)\left(1-\frac{M}{2\rho}\right)$$
$$= d\rho\left(1-\frac{2M}{r}\right)^{1/2}\left(1+\frac{M}{2\rho}\right)^2$$

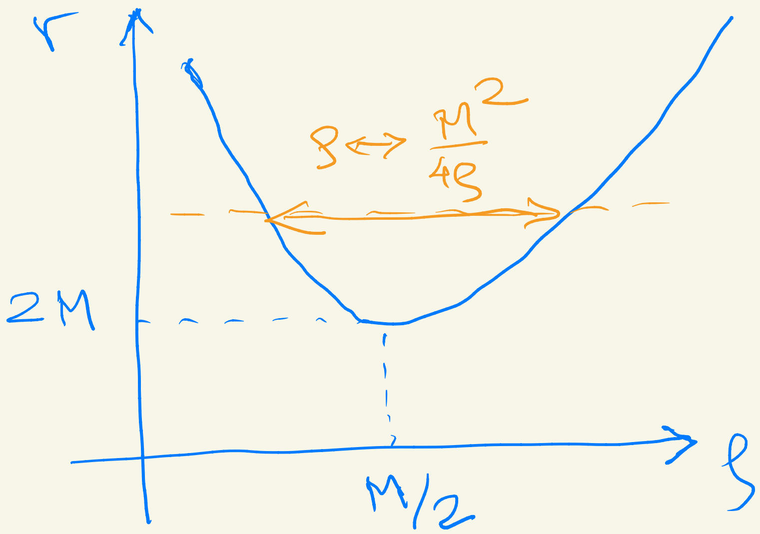
$$\therefore ds^2 = \left(\frac{1-M/2\rho}{1+M/2\rho}\right)^2 dt^2 - \left(1+\frac{M}{2\rho}\right)^4 \left\{ d\rho^2 + \rho^2 d\Omega^2 \right\}$$

No singularity as a function of ρ

And $dt = 0$ surfaces are conformal to Euclidean 3-space.

However ρ covers only $r > 2M$

Plot of r vs ρ



Note self-similarity under

$$\rho \to \frac{M^2}{4\rho}$$

$$r \to \left(1 + \frac{M}{2 \times \frac{M^2}{4\rho}}\right)^2 \frac{M^2}{4\rho} \to \left(\frac{M}{2\rho} + 1\right)^2 \rho = r$$

and $\rho = M/2$ goes into itself "fixed point"

However this mapping corresponds to $(U, V) \longleftrightarrow (-U, -V)$ check

Thus ρ covers only I & IV with $\rho = \frac{M}{2}$ or $r = 2M$ as the crossing point.