Lecture 2

Wald conventions: (Ch. 3)

$$\nabla_b \omega_c = \partial_b \omega - \Gamma^d{}_{bc}\,\omega_d \qquad \omega \sim \binom{0}{1}$$

Curvature definition

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = \underbrace{R_{abc}{}^d}_{\text{Riemann}}\,\omega_d$$

[Compare e.g. Weinberg $\quad V^\mu{}_{;\nu;\rho} - V^\mu{}_{\rho;\nu} = R^\mu{}_{\sigma\nu\rho} V^\sigma$]

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, t^c = - R_{abd}{}^c\, t^d$$

In the coord index convention,

$$R_{\mu\nu\rho}{}^\sigma = \Gamma^\sigma_{\mu\rho,\nu} - \Gamma^\sigma_{\nu\rho,\mu} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\alpha\mu}$$

<u>Ricci</u> $R_{ac} = R_{abc}{}^{b}$

<u>Scalar</u> $R = R_a{}^a$

All the possible traces within R_{abcd}

Recall $R_{([ab][cd])}$ ← pairwise exch. symm

[ ] → antisymm.

<u>Weyl tensor</u>: (n-dim)

$$R_{abcd} = \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{2}{(n-1)(n-2)} R\, g_{a[c}g_{d]b} + \boxed{C_{abcd}}$$

antisymm. operation

Weyl tensor

Symmetries of Weyl are same as for Riemann

But all traces within Weyl vanish

Tetrad formalism

Recall $\frac{\partial}{\partial x^\mu}$ was a basis vector field

and the convention is

$$X = \sum_\mu X^\mu e_{(\mu)} \equiv \sum_{(\mu)} X^\mu \frac{\partial}{\partial x^{(\mu)}}$$

Hence action on a scalar function

$$Xf = X^\mu \frac{\partial}{\partial x^\mu} f \rightarrow \text{directional derivative of } f$$

This basis e_μ and its dual basis

$\omega^\mu \equiv dx^\mu$ (so that $\omega^\mu(e_\nu) \equiv \frac{\partial}{\partial x^\nu} x^\mu = \delta^\mu_\nu$)

are welded to the coord system x^μ

Instead we may introduce any <u>basis vector fields</u> (n vector fields in n dim) $(\boxed{e_\nu})_a$ s.t. || $(e_0)_a \; (e_1)_a \; \ldots$

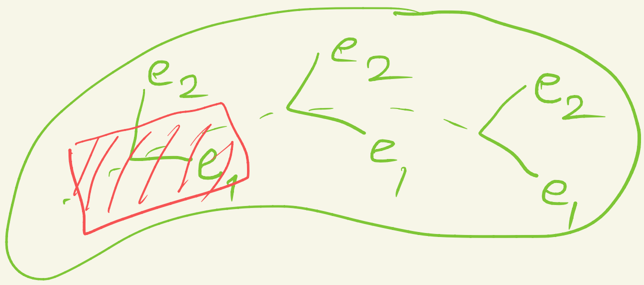
$$(e_\mu)^a (e_\nu)_a = \eta_{\mu\nu}$$ → Minkowski metric $\equiv (-1,1,1,1)$

Note our convention: $e_\nu \to e_{(\nu)}$

(NOT tensor index but enumeration index

Further,

$$\eta^{\mu\nu}(e_\mu)^a (e_\nu)_b = \delta^a_b$$



Should also have (?)

$$\eta^{\mu\nu}(e_\mu)_a (e_\nu)_b = g_{ab}$$

Tensor analysis in tetrad formalism

E. Cartan
repere mobile

Connection 1-form

→ "connexion" → cov. deriv.

$$\omega_{a\mu\nu} = (e_\mu)^b \nabla_a (e_\nu)_b$$

"1-form" → cov. (0,1) tensor field

Note μ, ν are enumeration labels.

When re-expressed entirely in coord. language we call them "Ricci rotation coefficients"

$$\omega_{\lambda\mu\nu} = (e_\lambda)^a (e_\mu)^b \nabla_a (e_\nu)_b$$

Property: $\omega_{a\mu\nu} = -\omega_{a\nu\mu}$

$$\omega_{a\mu\nu} = (e_\mu)^b \nabla_a (e_\nu)_b$$

but $\nabla_a (e_\mu)^b (e_\nu)_b = \nabla_a \eta_{\mu\nu} = 0$

$$= -(e_\nu)^b \nabla_a (e_\mu)_b$$

$$= -\omega_{a\nu\mu}$$

[metric compatibility]

Remark [Wald]: Need to use 3 facts in going from coord. language to tetrad language: (1) $\nabla_a g_{bc} = 0$ (2) ∇_a torsion free

(3) Definition of R: $\nabla_{[a}\nabla_{b]}\omega_c = R_{abc}{}^d \omega_d$

Thus for Riemann we get

$$R_{\rho\sigma\mu\nu} = R_{abcd}(e_\rho)^a (e_\sigma)^b (e_\mu)^c (e_\nu)^d$$

can be shown to be

$$= (e_\rho)^a (e_\sigma)^b (e_\mu)^c (\nabla_a \nabla_b - \nabla_b \nabla_a)(e_\nu)_c$$

which in terms of connection 1-forms is

$$= (e_\rho)^a (e_\sigma)^b \left(\nabla_a \omega_{b\mu\nu} - \nabla_b \omega_{a\mu\nu} \right)$$

Finally in terms of Ricci rot. coeff.

$$R_{\rho\sigma\mu\nu} = (e_\rho)^a \nabla_a \omega_{\sigma\mu\nu} - (e_\sigma)^a \nabla_a \omega_{\rho\mu\nu}$$

$$- \sum_{\alpha\beta} \eta^{\alpha\beta} \Big\{ \omega_{\rho\beta\mu} \omega_{\sigma\alpha\nu} - \omega_{\sigma\beta\mu} \omega_{\rho\alpha\nu}$$

$$+ \omega_{\rho\beta\sigma} \omega_{\alpha\mu\nu} - \omega_{\sigma\beta\rho} \omega_{\alpha\mu\nu} \Big\}$$

Ricci

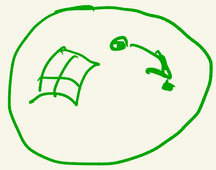
$$R_{\rho\mu} = \sum_{\sigma,\nu} \eta^{\sigma\nu} R_{\rho\sigma\mu\nu}$$

Finally we have conditions following from torsion-free ∇_a requirement

[detail to follow]

Killing vectors (Weinberg § 10.9)

If there is a vector field $\epsilon^\mu(x)$ s.t. $x^{\mu\prime} \to x^\mu - \epsilon^\mu$ leaves the metric $g_{\mu\nu}$ unchanged then it is called a Killing vector



on a 2-sphere
metric unchanged

Consider

$$x'^\mu = x^\mu - \epsilon^\mu(x)$$

and ϵ^μ are "small"

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - \epsilon^\mu_{\ ,\nu}$$

and $$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu + \epsilon^\nu_{\ ,\mu} + O(\epsilon^2)$$

Now consider the change in $g_{\mu\nu}$:

$$g'_{\mu\nu}(x) = g'_{\mu\nu}(x') + ??$$

$$= g_{\lambda\sigma}(\delta^\lambda_\mu + \epsilon^\lambda_{,\mu})(\delta^\sigma_\nu + \epsilon^\sigma_{,\nu})$$

$$= g_{\mu\nu}(x) + g_{\lambda\nu}\epsilon^\lambda_{,\mu} + g_{\sigma\mu}\epsilon^\sigma_{,\nu} + g_{\mu\nu,\lambda}\epsilon^\lambda$$

$$= g_{\mu\nu}(x) + \boxed{g_{\mu\nu;\lambda}}\epsilon^\lambda + \epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}$$

$$\to 0$$

$$g'_{\mu\nu}(x') = \frac{\partial x'^\rho \partial x'^\sigma}{\partial x^\mu \partial x^\nu} \times g_{\rho\sigma}(x(x'))$$

Thus if ϵ^μ satisfy

$$\boxed{\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} = 0}$$

Killing vector

then small shifts generated by this vector field leaves the functional form $g_{\mu\nu}(x)$ unchanged.