Lecture 3

Q from last time

$$(e_\mu)^{\boxed{a}}(e_\nu)_a = \eta_{\mu\nu}$$

enumeration index
not a tensor index

$\mu = 0, 1, 2, 3$ but not
$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

$\boxed{a}$ is a place holder to show
e is a $\binom{1}{0}$ tensor. abstract index

but $X^a Y_a$ it does mean
"contraction" operation

$$\boxed{X^a}\,\boxed{Y_a} \longrightarrow X^\mu Y_\mu$$

Wheeler $\nabla g = 0$

$\nabla_a g_{bc}$ Wald

$\binom{0}{1}$ operator $\rightarrow$ $\binom{0}{2}$ tensor

$$\nabla^a g_{ac} = \underbrace{g^{\mu\nu}\frac{\partial}{\partial x^\mu}}\, g_{\nu\sigma}$$

"contraction"

Contration becomes possible on tangent spaces of differential manifolds once the dual tangent space T_p^* at point p is introduced

$\omega \in T_p^*$ (same as V^* last time)
$v = T_p$ (" " V ")

Then $\omega(v) \in \mathbb{R}$

Secretly this is $\omega^T v$
$(\cdot \; - \; - \; -)\begin{pmatrix} \cdot \\ \vdots \\ \cdot \end{pmatrix} \to \mathbb{R}$

Given metric g we can choose
$w \in V$ s.t. $g(v, w)$
$= \omega(v)$

i.e. either $\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right)$

or $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

Today's themes :

Killing vectors

- Lie derivative

Conformal symmetry

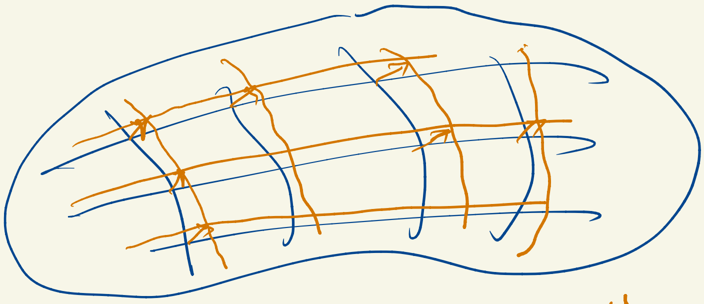
↳ Appendix D

Killing vectors

Given small coord. transfs

$$x'^{\mu} = x^{\mu} - \epsilon^{\mu}(x)$$

Lie derivative is essentially the changes induced in tensors by such a change



$\epsilon^\mu(x)$

Sometimes called "Lie dragging"

$$x^{\mu\prime} = x^\mu - \epsilon^\mu(x)$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - \frac{\partial \epsilon^\mu}{\partial x^\nu}$$

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu + \frac{\partial \epsilon^\nu}{\partial x^\mu} + O(\epsilon^2)$$

Consider

$$T'_{\mu\nu}(x) = T'_{\mu\nu}(x') + \frac{\partial T_{\mu\nu}}{\partial x^\lambda}\epsilon^\lambda + O(\epsilon^2)$$

We know

$$T'_{\mu\nu}(x) = T_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x^\mu}\frac{\partial x^\sigma}{\partial x^\nu}$$

$$= T_{\lambda\sigma}\left(\delta^\lambda_\mu + \epsilon^\lambda{}_{,\mu}\right)\left(\delta^\sigma_\nu + \epsilon^\sigma{}_{,\nu}\right)$$

$$= T_{\mu\nu} + T_{\lambda\nu}\epsilon^\lambda{}_{,\mu} + T_{\sigma\mu}\epsilon^\sigma{}_{,\nu} + T_{\mu\nu,\lambda}\epsilon^\lambda$$

Define $\Delta_\epsilon T_{\mu\nu} \equiv T'_{\mu\nu}(x') - T'_{\mu\nu}(x)$

We now show that

$$\Delta_\epsilon T_{\mu\nu} = T_{\mu\nu;\lambda}\epsilon^\lambda + T_{\sigma\nu}\epsilon^\sigma{}_{;\mu} + T_{\mu\sigma}\epsilon^\sigma{}_{;\nu}$$

This signifies also a change in the functional form of coeff.s $T_{\mu\nu}(x)$

Thus when $T_{\mu\nu} \equiv g_{\mu\nu}$,

$$\Delta_\epsilon g_{\mu\nu} = +\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} + \cancelto{0}{g_{\mu\nu;\lambda}\epsilon^\lambda}$$

Killing vectors satisfy

$$\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} = 0$$

i.e. $\quad g_{\mu\nu}(x-\epsilon) = g_{\mu\nu}(x)$

i.e. metric coeffs remain the same functions of their arguments for special $\boxed{\epsilon^\mu(x)}$

Further, we can think of transformations

$$\tilde{g}_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x)$$

This is <u>not</u> a coord transf.

But we can define conformal Killing vector:

Proof of $\Delta_\epsilon T_{\mu\nu}$ formula:

$$\boxed{\Delta_\epsilon T_{\mu\nu}(x) = T_{\mu\nu,\lambda}\epsilon^\lambda + T_{\lambda\nu}\epsilon^\lambda{}_{,\mu} + T_{\lambda\mu}\epsilon^\lambda{}_{,\nu}}$$

want to show

$$= T^\lambda{}_\nu \epsilon_{\lambda;\mu} + T^\lambda{}_\mu \epsilon_{\lambda;\nu} + T_{\mu\nu;\lambda}\epsilon^\lambda$$

This is called the Lie derivative of $T_{\mu\nu}$

$$\epsilon^\lambda\left(T_{\mu\nu;\lambda} = T_{\mu\nu,\lambda} \boxed{- \Gamma^\sigma_{\mu\lambda} T_{\sigma\nu}} - \Gamma^\sigma_{\nu\lambda} T_{\mu\sigma}\right)$$

$$T^\lambda{}_\nu\left(\epsilon_{\lambda;\mu} = \epsilon_{\lambda,\mu} \boxed{- \Gamma^\rho_{\lambda\mu}\epsilon_\rho}\right) \rightarrow \text{?cancel?}$$

$$T^\lambda{}_\mu\left(\epsilon_{\lambda;\nu} = \epsilon_{\lambda,\nu} - \Gamma^\rho_{\lambda\nu}\epsilon_\rho\right)$$

$$\epsilon^\lambda T_{\sigma\nu}\Gamma^\sigma_{\mu\lambda} = \frac{1}{2}\epsilon^\lambda T_{\sigma\nu} g^{\sigma\alpha}\left(g_{\mu\alpha,\lambda} - g_{\mu\lambda,\alpha} + g_{\lambda\alpha,\mu}\right)$$

$$\underbrace{\epsilon_\beta g^{\beta\lambda} T^\rho{}_\nu \underbrace{g_{\sigma\rho} g^{\sigma\alpha}}_{\delta^\alpha_\rho}}_{\epsilon_\beta g^{\beta\lambda} T^\alpha{}_\nu}$$

$$= -\epsilon_\beta T^\alpha{}_\nu \Gamma^\beta_{\mu\lambda}$$

|| check

Thus Lie derivative is defined in a way that it has properties

1) Linearity
2) Leibnitz rule
3) Commutes with contractions
4) Reduces to ordinary derivative when acting on scalars

$\nabla(\omega^\mu \otimes v_\beta)$

$C(\omega, \beta) \to \mathbb{R}$

$C(\nabla(\omega^\mu \otimes v_\beta)) = \nabla(C(\omega^\mu \otimes v_\beta))$

Thus $\Delta_\epsilon S = S_{;\lambda}\epsilon^\lambda = S_{,\lambda}\epsilon^\lambda$

Then e.g.

$$\Delta_\epsilon T^\mu{}_\nu \equiv T^\mu{}_{\nu;\lambda}\epsilon^\lambda - T^\sigma{}_\nu \epsilon^\mu{}_{;\sigma} + T^\mu{}_\alpha \epsilon^\alpha{}_{;\nu}$$

etc. for up & down indices.