Lecture 7

Dynamical formulation of GR

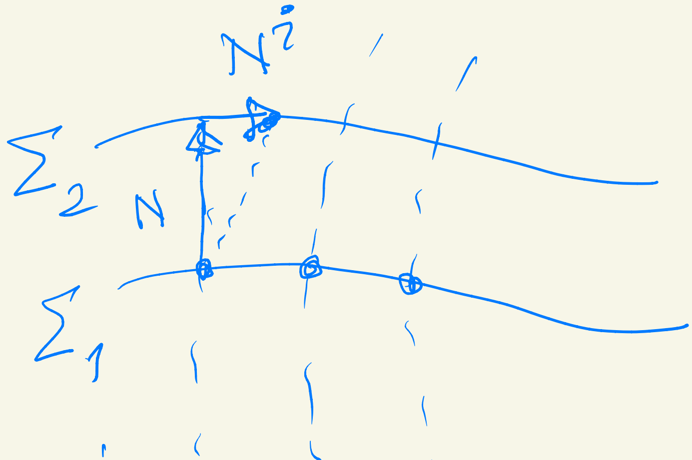
① "Slice" the 4-manifold into 3 dim hypersurfaces Σ

② Introduce ${}^{(3)}g_{ij}$ on the hypersurface

understood

③ "Lapse" and "shift" (three) functions which specify the evolution of g_{ij} as we move through the Σ's

[MTW]



"Shifts"

$$x^i_{\Sigma_2}(x^m) = x^i - N^i(t,x,y,z)\,dt$$

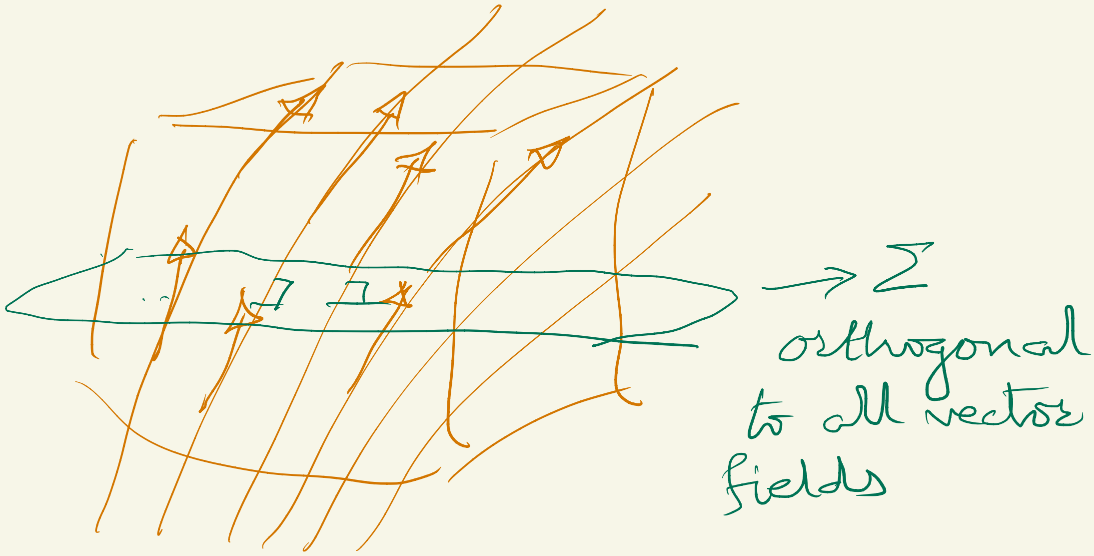
"Laps" of proper time

$$d\tau = N(t,x,y,z)\,dt$$

Note the Σ-foliation is chosen such that each Σ is orthogonal to a "congruence" of timelike geodesics.

Congruence: A family of curves in an open set of the manifold s.t. given any point only one curve from the family passes through it.

Thus, let ξ^a be the tangent vector field of the congruence



"integral curves $\Longleftrightarrow$ 1-param vector field"

can be used to
congruence $\Longleftrightarrow$ vector field

Here we demand $\xi^a \xi_a = -1$

Orthogonality of ξ^a to Σ means

$$ {}^{(3)}g_{ab} = g_{ab} + \xi_a \xi_b $$

Note $\xi^{c}\,{}^{(3)}g_{cb} = \xi^{c} g_{cb} + \xi^{c}\xi_{c}\xi_{b}$
$$= \xi_{b} - \xi_{b} = 0$$
[Wald]

Thus returning to MTW,

$$ds^2 = -(N\,dt)^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

<u>Goal</u> : Express Einstein equations in these variables (g_{ij}, N^i, N)

<u>Extrinsic curvature</u>

MTW : $n \equiv \xi^a$ Wald

$${}^{(4)}\nabla_i n = -K^j_{\ i}\, e_j$$

cov. deriv of n within Σ

[i.e. ${}^{(4)}\nabla_i \xi^a = -K^a_{\ i}$]

Note since $n \cdot n = -1$,

$$n \cdot \left({}^{(4)}\nabla_i n\right) = 0$$

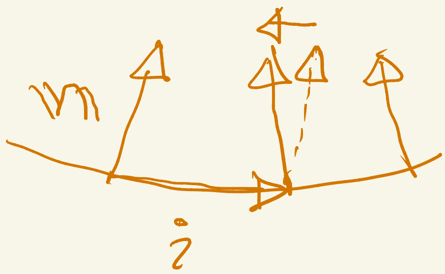
$\Rightarrow$ ${}^{(4)}\nabla n$ can be decomposed only using spacelike e_j

Then
$$\boxed{K_{im} = K_i^{\ j} g_{jm}}$$

We can show
$$K_{im} = K_{mi}$$
// check

This tensor defined on every hypersurface Σ is a measure of how Σ is "curved" within the 4-d manifold

$\delta n = -K(\delta x^i)$



We can show using

$$N_{i|k} \equiv N_{i;k}\Big|_{\text{restricted to } \Sigma}$$

we can obtain

$$\boxed{K_{ik} = \frac{1}{2N}\left(N_{i|k} + N_{k|i} - \frac{\partial g_{ik}}{\partial t}\right) = \frac{1}{2N}\left(\frac{\partial N_i}{\partial x^k} + \frac{\partial N_k}{\partial x^i} - \frac{\partial g_{ik}}{\partial t} - 2\Gamma_{pik}N^p\right)}$$

where ${}^{(3)}\Gamma_{mhi} \equiv \Gamma_{mhi} = e_m \cdot \nabla_i e_h$

i.e. $A_{h|i} = e_h \, {}^{(3)}\nabla_{e_i} A$

$$= \frac{\partial A_h}{\partial x^i} - A^m \Gamma_{mhi}$$

Then using this tensor we have

$$\boxed{\begin{aligned} -G^0{}_0 &= {}^{(4)}R^{12}{}_{12} + {}^{(4)}R^{23}{}_{23} + {}^{(4)}R^{31}{}_{31} \\ &= \frac{1}{2}\,{}^{(3)}R - \frac{1}{2}\frac{1}{n\cdot n}\left[(\mathrm{Tr}\,K)^2 - \mathrm{Tr}(K^2)\right] \end{aligned}}$$

$$\mathrm{Tr}\,K = g^{ij} K_{ij}$$

$$\mathrm{Tr}\,K^2 = K_j{}^m K_m{}^j$$

$\left\{ {}^{(3)}g_{ij} \text{ embedded in both} \right.$

Recall, of the 10 equations,

$G^0{}_0$ and $G^0{}_i$

involve only $g_{\mu\nu}$ and $\dot{g}_{\mu\nu}$

but no $\ddot{g}_{\mu\nu}$

Hence "constraint"s

$$\boxed{G^n{}_i = -(n\cdot n)^{-1}\left[K^m{}_{i|m} - (\mathrm{Tr}\,K)_{|i}\right]}$$

$G^i{}_j \rightarrow$ dynamical but complicated

Hamiltonian formulation

Start with Lagrangian $L(q, \dot{q})$

Define $p = \dfrac{\partial L}{\partial \dot{q}}$

Invert the above : $\dot{q}(p)$

$$H = p\,\dot{q}(p) - L(q, \dot{q}(p))$$

$\equiv H(q, p) \rightarrow$ defined on the phase space

The procedure fails for

(1) Systems with L first degree homogeneous in $\dot{q}$

(2) more generally somehow

$$p_a \equiv \frac{\partial L}{\partial \dot{q}_a}$$ are not invertible

Relativistic particle:

$$L = \frac{m}{2}\underbrace{\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}}_{u^\mu u_\mu = -1} \quad ?$$

$$p^\mu = \frac{\partial L}{\partial u_\mu} \quad \text{then} \quad p^\mu p_\mu = -m^2$$

$$E^2 = |\vec{p}|^2 + m^2c^4$$

Next time:

Lichnerowitz-Choquet-Bruhat + York data

$$^{(3)}g / \Omega^2$$