Lecture 8

Dynamics of gravity – conclusion

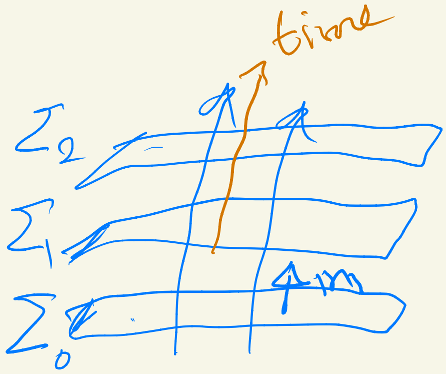
Overview:

①

$q_i \longleftarrow {}^{(3)}g_{ij}$

Σ

$p_j \longleftarrow K_{ij} \longrightarrow$ canonical momentum $\sim \partial_t g_{ij}$



②

Thus $\sqrt{g}\,R(g, \partial g)$

$$\longrightarrow {}^{(3)}R(g_{ij}, K_{ij}) + \underbrace{\left\{ N^i, N, \partial_k N^i, \partial_k N \right\}}$$

Recall EoM

G_{00} & G_{0i} are 4 constraints

[The choice of 4 coords further reduces the d.o.f. leaving 2 dyn. d.o.f.]

Then one finds

3 $\mathcal{L} = \sqrt{g}\, R$

$\xrightarrow{\text{ADM}} \underbrace{\partial_t g_{ij}\, \pi^{ij}} - N\mathcal{H} - N^i \mathcal{H}_i$

$\dot{\phi}\, \pi_\phi$ $\quad + \text{"}\partial_\mu S^\mu\text{"}$

$\hookrightarrow$ total derivative

with $\mathcal{H}, \mathcal{H} \sim g_{ij}, \pi^{ij} \approx K_{ij}$

[Comment: MTW introduce π^{ij} but do not formally transition to a Hamiltonian version]

Q: $G^0{}_0$, $G^0{}_i$ as constraints ??

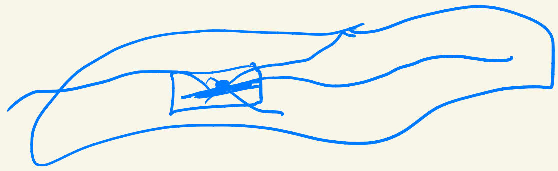
$G^0{}_0$ & $G^0{}_i(g, \partial_t g)$ & do not involve "acceleration" $\partial_t^2 g$

$$G_{\mu\nu} = 8\pi \underline{T_{\mu\nu}} \sim \partial_\mu \phi \frac{\delta \mathcal{L}}{\delta(\partial_\nu \phi)}$$

Noether currents $\sim \phi_a, \dot{\phi}_a$

4. The constraints $\mathcal{H}$, $\mathcal{H}^i$ can be solved → subject to the possibility of making simplifications for the given physical setting

eg. Σ → geodetically closed



Equivalently if Σ can be chosen to have $\mathrm{Tr}\, K = 0$

Then $\mathcal{H}^i$ → get satisfied

& $\mathcal{H}$ → Choquet-Bruhat & York

solve by fixing K. Kuchař

$$ {}^{(3)}g_{ij} = \underbrace{\psi^4(x)}_{\text{conformal transf.}} \, {}^{(3)}\hat{g}_{ij} $$

↳ any convenient choice

which yields a Laplacian type equation for ψ.

Electrodynamics analogy:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\rightarrow \quad \vec{E}^2 - \vec{B}^2$$

$$\vec{B} \rightarrow (\vec{\nabla} \times \vec{A})$$

$$\vec{E} \downarrow \quad \vec{\nabla} A^0 + \frac{\partial}{\partial t} \vec{A}$$

$$\downarrow$$

$$\frac{1}{2} |\dot{\vec{A}}|^2 \quad \text{but no } (\dot{A}^0)^2$$

(a) Thus A^0 is superfluous to dynamics

(b) $0 = \vec{\nabla} \cdot \vec{B}$ → no velocities or acceleration

↳ "solved" by choosing $\vec{B} = \vec{\nabla} \times \vec{A}$

(c) $\vec{\nabla} \cdot \vec{E} = \rho$ → constraint on how to choose $\vec{A}$ on Σ^0

Dirac's classification

$A^0 = 0$ → solvable and can be eliminated algebraically,
"Second class constraint"

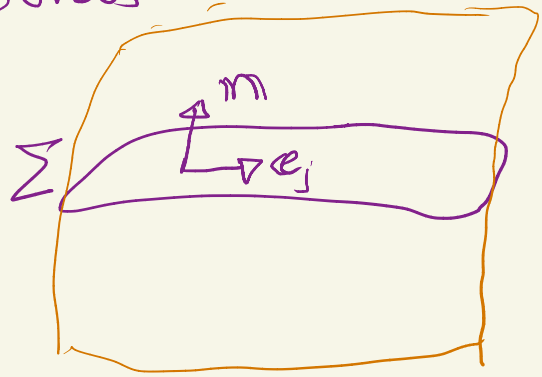
$$\vec{E} = \frac{\delta \mathcal{L}}{\delta \dot{\vec{A}}} = \vec{\pi} \rightarrow \vec{\nabla} \cdot \vec{\pi} = 0$$

Dynamics must preserve this constraint in time "First class constraint"

Some important formulae capturing above points:

① $dn = -K_i^{\,j} e_j$

$(\,!\,)$



Then for e_j within Σ

$$\nabla_i e_j = K_{ij} \frac{n}{n\cdot n} + {}^{(3)}\Gamma^h_{ji} e_h$$

Gauss-Weingarten eqn.s

Further in ADM,

$$K_{ik} = \frac{1}{2N}\Big[\underbrace{N_{i|k}}_{(N_{i;k}) \text{ projected to } \Sigma} + N_{k|i} - \frac{\partial}{\partial t} g_{ik}\Big]$$

Thus $K_{ik} \sim$ velocity $\sim$ canonical momentum

(2) & (3)

$$\mathcal{L}_{ADM} = +\pi^{ij}\frac{\partial}{\partial t}g_{ij} - N\mathcal{H} - N_i\mathcal{H}^i$$

$$\underbrace{+\frac{\partial}{\partial t}(\pi^{ij}g_{ij}) - 2\left[\pi^{ij}N_j \cdots\right]_{,i}}_{\text{equiv to boundary terms}}$$

with $\pi^{ij} = \sqrt{g}\,(g^{ij}\,\mathrm{Tr}\,\mathbb{K} - K^{ij})$

$g \to$ of Σ

and

$$\mathcal{H}(g_{ij}, \pi^{ij}) = \frac{1}{\sqrt{g}}\left(\mathrm{Tr}\,\pi^2 - \frac{1}{2}(\mathrm{Tr}\,\pi)^2\right) - \sqrt{g}R \quad \text{on } \Sigma$$

(4) Impose $\mathrm{Tr}\,\mathbb{K} = 0$

$$g_{ij} = \psi^4\delta_{ij}$$

Then

$$8\nabla^2_{\text{base}}\psi + \frac{1}{\psi^7}\sum_{ij}(K_{\text{base}\,ij})^2 = 0$$