Hamilton - Jacobi Method

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PH 422 L 11 / Slide 1

Hamilton's Principal Function

Action $S = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t), t)$

We treat $S[q(t), \dot{q}(t), t]$ as a "functional" of functions $q(t)$, $\dot{q}(t)$

Requiring $\delta S = 0$ subject to $q(t_1) = q_1$, $q(t_2) = q_2$ held fixed, gives Euler-Lagrange eqn.s

$$-\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}}\right) + \frac{\delta L}{\delta q} = 0$$

Let

$q_{EL}(t)$ be the solution of E-L eqn.s

And let

$$S(q_f, t_f) = \int_{t_1}^{t_f} L(q_{EL}(t'), \dot{q}_{EL}(t'), t')\, dt'$$

This object is called Hamilton's Principal function and is an ordinary function of endpoint value q_f at time t_f

Question? What is $\partial S/\partial q_f$

$$\delta S = \int_{t_1}^{t_f} dt \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right)$$

$$\therefore \delta S = \frac{\delta L}{\delta \dot q}\,\delta q \Big|_{t_1}^{t_f} + \int_{t_1}^{t_f} dt \underbrace{\left(-\frac{d}{dt}\left(\frac{\delta L}{\delta \dot q}\right) + \frac{\delta L}{\delta q}\right)\delta q}_{=0}$$

Now set $q = q_{EL}$

$$\delta \mathcal{S} = \frac{\delta L}{\delta \dot q}\Big|_{t_f} \delta q_f \qquad \text{ie} \qquad \frac{\partial \mathcal{S}}{\partial q} = p$$

$\mathcal{S}$ is different from $S[q(t), \dot q(t)]$

What is $\dfrac{\partial \mathcal{S}}{\partial t}$? Note

$$L = \frac{d\mathcal{S}}{dt} = \frac{\partial \mathcal{S}}{\partial q}\,\dot q + \frac{\partial \mathcal{S}}{\partial t}\,\cancel{\delta t}$$

Thus

$$\frac{\partial S}{\partial t} = L - \frac{\partial S}{\partial q}\dot{q} = L - p\dot{q}$$

$$= -H(q_f, p_f, t_f)$$

$$\boxed{\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0}$$

Hamilton - Jacobi equation

For H indep. of t, energy is conserved, and in fact, $H = E$

In this case we can write

$$S = W(q) - Et$$

or, $H\left(q, \frac{\partial W}{\partial q}\right) = E$

Example, $H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 q^2$

Then H-J eqn. is

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}\omega^2 q^2 = E$$

An understanding of How this can be used for explicit solution of the dynamical requires understanding canonical transformations

Canonical transformations

These are transformations

$$Q_i = Q_i(q_j, p_j, t)$$

$$P_i = P_i(q_j, p_j, t)$$

s.t. new variables Q_i, P_i solve also the canonical equations with corresponding transformed Hamiltonian K

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \;;\; \frac{\partial H}{\partial p_i} = \dot{q}_i \quad \Bigg\| \quad \frac{\partial K}{\partial Q_j} = -\dot{P}_j \;;\; \frac{\partial K}{\partial P_j} = \dot{Q}_j$$

Since both must satisfy dynamics i.e. Hamilton's Principle of least action,

$$\delta\int\left\{\sum P_i \dot{Q}_i - K(Q_i\, P_i\, t)\right\}dt = 0$$

$$= \delta\int\left\{\sum p_i \dot{q}_i - H(q_i\, p_i\, t)\right\}dt$$

i.e. $\sum P_i\, dQ_i - K dt = \sum p_i\, dq_i - H dt - dF_1$

i.e. the differentials differ at most by an exact differential dF_1 (Note $\int_i^f \frac{dF_1}{dt}dt = F_f - F_i$ $\therefore \delta\int \frac{dF}{dt}dt = 0$)

Thus, we observe,

$$dF_1 = \sum p_i \, dq_i - \sum P_i \, dQ_i + (-H + K)\,dt$$

compare

$$dF_1 = \sum \frac{\partial F_1}{\partial q_i} dq_i + \sum \frac{\partial F_1}{\partial Q_i} dQ_i + \frac{\partial F_1}{\partial t}$$

$$\therefore \quad p_i = \frac{\partial F_1}{\partial q_i}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$K = \frac{\partial F_1}{\partial t} + H$$

If we ~~to~~ solve for F_1 from these we have the "generator" of the required transf.

The last eqn. is suspiciously similar to H-J eqn. if we demand $K = 0$

But $F_1(q_i\; Q_i)$ whereas $S(q_i\; p_i)$

Thus actually we need a generator which is a function of $q_i\; P_i$

Let $F_2(q_i\; P_i) = F_1(q_i\; Q_i) + \sum Q_i P_i$

Then $dF_2 = \sum_i \left\{ \frac{\partial F_1}{\partial q_i} dq_i + \underbrace{\frac{\partial F_1}{\partial Q_i} dQ_i + P_i dQ_i}_{=0} + Q_i dP_i \right\}$

Then we find

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}$$

Thus we can argue that H-J eqn. can be solved by a canonical transf. of type F_2

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0$$

Suppose we find ~~s~~ a solution

$$S = f(t, q_1, \ldots q_n; \alpha_1 \ldots \alpha_n) + A$$

Treat f as if it is F_2 with the integration constants α_i to be P_i

Then we should have

$$p_i = \frac{\partial f}{\partial q_i} \quad \& \quad \beta_i = \frac{\partial f}{\partial \alpha_i}$$

... β_i are "new co-ord.s"

$$K = H + \frac{\partial f}{\partial t} = 0$$ ... since f solves H-J eqn.

Thus, we use n first order eqn.s

$$\frac{\partial f}{\partial \alpha_i}(q_j, \alpha_j) = \beta_i$$

Inverting the solutions we know $q_i(\alpha_j \, \beta_j \, t)$

And we know $p_i = \frac{\partial f}{\partial q_i}$

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Conclusions:

1. H-J equation is a first order equation with a strong resemblence to Schrödinger eqn.
2. Its solution $S(q,t)$ is actually a canonical transf. of type F_2.

   2 a) S is generator of time evolution "transf."
3. Thus in QM $e^{i\Delta S/\hbar}$ is the time-evolution

unitary transf. for small ΔS

4. We learnt that for arbitrary t_i, t_f,

$$ {}_D\langle q_f t_f | q_i t_i \rangle_D = \sum_{\text{paths}} e^{i S(\text{path})/\hbar} $$

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