WKB method (contd.)

$$\lambdabar = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2m(E - V(x))}}$$

Then
$$\left|\frac{d}{dx}\lambdabar\right| = \frac{1}{2}\,\frac{\hbar \times 2m\,|dV/dx|}{(2m(E-V(x)))^{3/2}}$$

$$= \lambdabar\,\frac{|dV/dx|}{2\cancel{m}(E - V(x))}$$

i.e.
$$\left|\frac{1}{\lambdabar}\frac{d\lambdabar}{dx}\right| = \frac{|dV/dx|}{2\,|E - V(x)|} \sim \frac{|dV/dx|}{\text{Kin. Energy}}$$

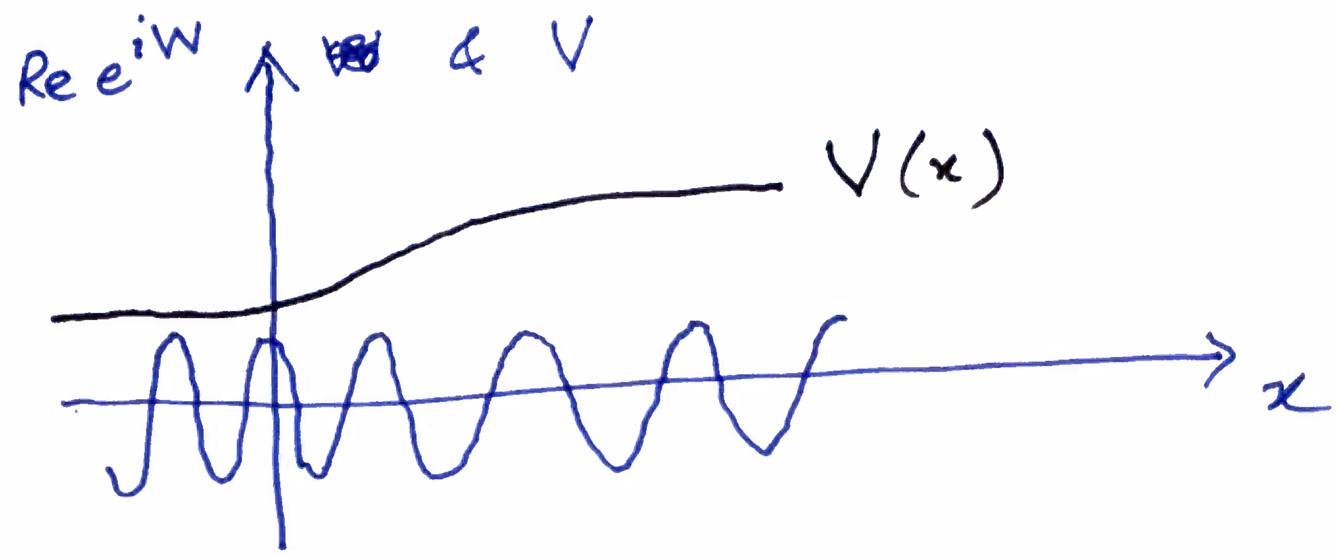
Thus another interpretation is that

for consistency we must have

$$\left| d\chi/dx \right| \ll 1 \,,$$

i.e. from above formula,

$$\lambda \left| dV/dx \right| \ll (\text{kin. Energy})$$



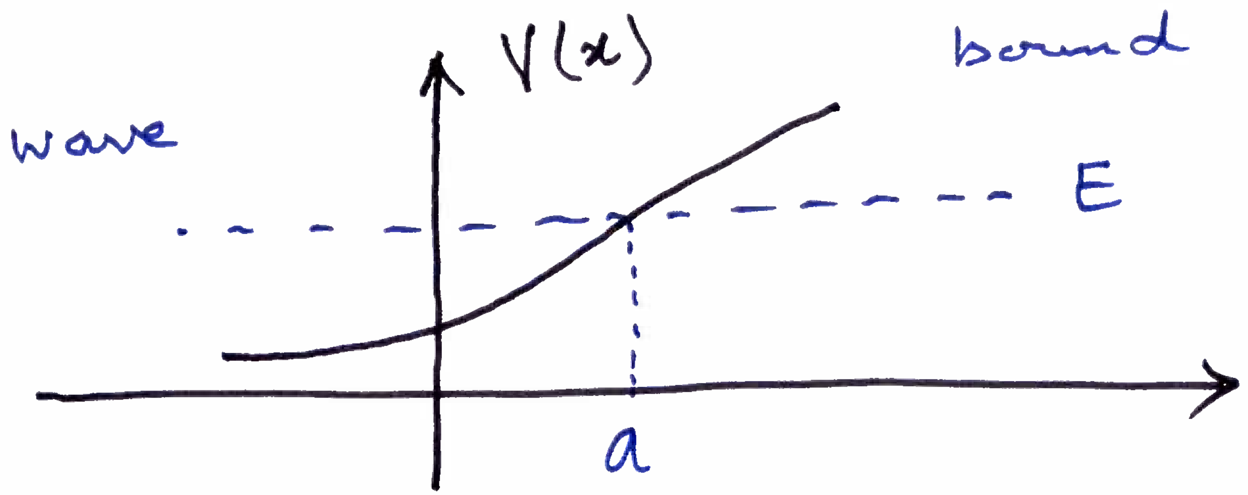
We extend the method from wave packets also to bound states where $E < V(x)$ and $W(x)$ is imaginary. i.e we propose the ansatz

$$\psi(x,t) \sim \frac{\text{const.}}{[V(x) - E]^{1/4}} \exp\left\{ \pm \frac{1}{\hbar} \int^{x} dx' \sqrt{2m(V(x') - E)} - \frac{iEt}{\hbar} \right\}$$

for the case of bound state

again with the assumption $\frac{\hbar}{\sqrt{2m(V(x)-E)}} \ll \frac{2(V(x)-E)}{dV/dx}$

Application to bound states



We note the classical turning point a (kin. energy = 0) separates two regions. Far away, $x \ll a$ or $x \gg a$ WKB approx. may be valid. But fails exactly at a.

Strategy : Assume that near a, $V(x)$ can be approximated to be linear. i.e. $\lambda V'' \ll V'$ and similarly $\lambda^{n-1} V^{(n)} \ll V'$ — left or right region

Let $V(x) - E = g(x-a)$ near $x=a$

with $g = V'(x=a) > 0$

Then $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + g(x-a)\psi = 0$$

The solutions to this are Bessel functions of order 1/3, called Airy functions

Define variable $z = \left(\frac{2mg}{\hbar^2}\right)^{1/3}(x-a)$

~~Let~~ $k^2 = \left(\frac{2mg}{\hbar^2}\right)^{2/3} z$

Can check WKB approx. amounts to

$$|z|^{3/2} >> \frac{1}{2}$$