Time dependent perturbation theory

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It is possible to split (as before)

$$H = H_0 + V$$

... but V is $V(t)$, time dependent

(i) Explicit t-dependence → (a) transient for a short time (b) oscillatory/periodic time dep.

(ii) Situation stationary; V is actually time indep, but system sees V for a limited time → "scattering"

Interaction picture :

A hybrid between Schrödinger & Heisenberg pictures. Pioneered by Dirac

Idea: We know: for H_0, let

$$H_0 |n\rangle = E_n |n\rangle$$

$H_0 \to$ no explicit t-dep.

Then for a state $|\alpha\rangle$ in Schrödinger picture,

$$|\alpha\; t=0\rangle = \sum_n c_n |n\rangle$$

Then $|\alpha\; t\rangle_S = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle$

idea of interaction picture : build in the time dependence due to $V(t)$ in the coeff.s c_n :

$$|\alpha\, t\rangle_I \sim \sum_n c_n(t)\, e^{-iE_n t/\hbar}\, |n\rangle$$

Check : for H_0 case :

$$|\alpha\ t=0\rangle_S = \sum_n c_n\, |n\rangle$$

$$c_n = \langle n|\alpha\ t=0\rangle_S$$ ... consider making $c_n \to c_n(t)$

$$i\hbar \frac{\partial}{\partial t}|\alpha\, t\rangle_S = i\hbar\frac{\partial}{\partial t}\sum_n c_n^{(0)}(t)\,|n\rangle$$

$$\text{should} = H_0|\alpha\, t\rangle_S = \sum_n c_n^{(0)}(t)\, E_n\, |n\rangle$$

i.e. $\quad i\hbar \dfrac{d}{dt} c_n^{(0)}(t) = E_n c_n^{(0)}(t)$

$$\therefore \quad c_n^{(0)} = c_n^{(0)}(t=0)\, e^{-iE_n t/\hbar}$$

Thus propose <u>interaction picture</u> :

$$|\alpha\, t\rangle_I = e^{iH_0 t/\hbar}\, |\alpha\, t\rangle_S$$

.... with the split
$H = H_0 + V(t)$

Compare $\quad |\alpha\rangle_H = e^{iHt/\hbar}\, |\alpha\, t\rangle_S$

& for operator A, $\quad A_H(t) = e^{iHt/\hbar} A_S\, e^{-iHt/\hbar}$

$$A_I(t) = e^{iH_0 t/\hbar} A_S\, e^{-iH_0 t/\hbar}$$

Consider diff. eqn. for $|\alpha\rangle_I$:

$$i\hbar \frac{\partial}{\partial t}|\alpha\, t\rangle_I = i\hbar \frac{\partial}{\partial t}\left(e^{iH_0 t/\hbar}\,|\alpha\, t\rangle_S\right)$$

$$= -H_0\, e^{iH_0 t/\hbar}|\alpha\, t\rangle_S + e^{iH_0 t/\hbar}(H_0+V)|\alpha\, t\rangle_S$$

$$= e^{iH_0 t/\hbar}\, V\, \underbrace{e^{-iH_0 t/\hbar}\, e^{iH_0 t/\hbar}}_{\text{insert}}\, |\alpha\, t\rangle_S$$

regroup

$$= V_I(t)\,|\alpha\, t\rangle_I$$

We can now see that this matches the expectation we had, $|\alpha\, t\rangle_I \sim \sum c_n(t)\, e^{-iE_n t/\hbar}|n\rangle$

and we try to check, $i\hbar \frac{\partial c_n}{\partial t} \sim V_I\, c_n$

Let $$c_n(t) = \langle n | \alpha\, t \rangle_I$$

$$i\hbar \frac{\partial}{\partial t} c_n(t) = \langle n | \, i\hbar \frac{\partial}{\partial t} | \alpha\, t \rangle_I$$

$$= \langle n | \, V_I(t) \, | \alpha\, t \rangle_I$$

$$\sum_m |m\rangle\langle m|$$

$$= \sum_m \langle n | V_I(t) | m \rangle \, c_m(t)$$

Now $$\langle n | e^{iH_0 t/\hbar} V_s e^{-iH_0 t/\hbar} | m \rangle = e^{i(E_n - E_m)t/\hbar} \langle n | V_s | m \rangle$$

$$\equiv e^{i\omega_{nm} t} \, V_{nm}$$

$$\omega_{nm} \equiv (E_n - E_m)/\hbar$$

Thus

$$\boxed{i\hbar \frac{\partial}{\partial t} c_n(t) = \sum_m e^{i\omega_{mn} t} V_{nm} c_m(t)}$$

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... typically any one level (say $n=0$ i.e. ground state) may be occupied to begin with, and above equation tells how other levels m connected to n by matrix elements V_{nm} begin to get occupied.